Analyzing the Stability of Switched Systems Using Common Zeroing-Output Systems

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***Abstract*—This paper introduces the notion of common zeroing-output systems (CZOS) to analyze the stability of switched systems. The concept of CZOS allows one to verify weak zero-state detectability. It characterizes a common behavior of any individual subsystem when the output signal for each subsystem is "approaching" zero. Heuristically speaking, it removes the effect of switching behavior, and thus enables one to analyze stability properties in systems with complex switching signals. With the help of CZOS, the Krasovskii–LaSalle theorem can be extended to switched nonlinear time-varying systems with both arbitrary switching and more general restricted switching cases. For switched nonlinear time-invariant systems, the needed detectability condition is further simplified, leading to several new stability results. Particularly, when a switched linear time-invariant system is considered, it is possible to generate a recursive method, which combines a Krasovskii–LaSalle result and a nested Matrosov result, to find a CZOS if it exists. The power of the proposed CZOS is demonstrated by consensus problems in literature to obtain a stronger convergence result with weaker conditions.**

***Index Terms*—Arbitrary switching, asymptotic stability, common zeroing-output systems, switched nonlinear time-varying systems, weak zero-state detectability.**

I. Introduction

SWITCHED systems have gained more and more attention in various engineering applications as they appear more frequently due to increasing complexity of engineering systems [12], [23], [28], [36], [38], [39]. On one hand, the introduction of switching can enhance the performance of the overall systems. On the other hand, the existence of switching makes it quite challenging in stability analysis, in particular in checking uniform global asymptotic stability (UGAS), for instance a switched linear time-invariant (LTI) system turns to be a time-varying (switching-dependent) system. This paper proposes a novel concept of common zeroing-output system (CZOS), which can be used to remove the effect of switching, leading to a simplification of stability analysis.

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A. Motivation

Checking stability properties of switched systems under arbitrary switching is quite challenging. The method of a common Lyapunov function is a widely used technique [23], [24], [33]–[35], [40]. This technique uses a "common" Lyapunov function for all subsystems (modes) to conclude UGAS. This common Lyapunov function has a uniformly *negative definite* derivative along any trajectory of any subsystem.

A common Lyapunov function is hard to find as shown by the following simple switched LTI system with two subsystems:

$$\Sigma_1 : \begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ -1 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix},$$
$$\Sigma_2 : \begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}-1 & 1\\ -1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} \qquad (1)$$

where $x_i, i = 1, 2$, is the state. Although both subsystems are uniformly globally exponentially stable with the same eigenvalues, there is no common quadratic Lyapunov function for such a switched system as the following system:

$$\Sigma_1^{-1} \oplus \Sigma_2 : \begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \left(\begin{bmatrix}0 & 1\\ -1 & -1\end{bmatrix}^{-1} + \begin{bmatrix}-1 & 1\\ -1 & 0\end{bmatrix}\right)\cdot\begin{bmatrix}x_1\\ x_2\end{bmatrix} = \begin{bmatrix}-2 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

is not asymptotically stable [23], [33], [34]. Finding a non-quadratic common Lyapunov function is not easy, though it may exist [6]. On the other hand, it is clear that $V = (x_1^2 + x_2^2)/2$ is such that

$$\dot{V}\,|_{\Sigma_1} = -x_2^2 \le 0 \text{ and } \dot{V}\,|_{\Sigma_2} = -x_1^2 \le 0. \qquad (2)$$

This Lyapunov function candidate is a common, quadratic "weak" Lyapunov function. "Weak" in the sense that its time derivative along trajectories (of each subsystem) is *negative semidefinite*. Nevertheless, this does not suffice to conclude UGAS in the presence of (arbitrary) switching.

More generally, as many engineering systems have dissipative models, energy functions are often good candidates for weak Lyapunov functions [14]. The question then arises, "Given a common, weak Lyapunov function, under what additional conditions can UGAS be inferred?"

When a weak Lyapunov function exists for (the trivial solution of) a nonlinear time-invariant (NLTI) system, the classic Krasovskii–LaSalle theorem [14], [15], [17] can be used to infer UGAS. However, due to complex behaviors arising from arbitrary switching, the classic Krasovskii–LaSalle theorem cannot be applied, as shown by the following example:

$$\Sigma_1 : \begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ -1 & -2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix},$$

$$\Sigma_2 : \begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & -1\\ 1 & -2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} \tag{3}$$

where $x_i, i = 1, 2,$ is the state. This system has a common *weak* Lyapunov function $V = (x_1^2 + x_2^2)/2$, with negative semidefinite derivative along the trajectories: $\zeta \in \{1, 2\}$, $\dot{V}|_{\Sigma_\zeta} = -2x_2^2 \leq 0$. In this case, the origin is uniformly globally stable (UGS) [18]. Because

$$\dot{V}\left|_{\Sigma_\zeta}\right. = -2x_2^2 \equiv 0 \Rightarrow x_1 = x_2 \equiv 0 \ \forall \zeta \in \{1, 2\}$$

it is easy to deduce by appealing to the Krasovskii–LaSalle theorem that the origin is UGAS, whenever the switching signals satisfy some mild dwell time conditions [22]. However, when arbitrary switching is allowed, UGAS does not hold, as the following system:

$$\Sigma_1 \oplus \Sigma_2 : \begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \left(\begin{bmatrix}0 & 1\\ -1 & -2\end{bmatrix} + \begin{bmatrix}0 & -1\\ 1 & -2\end{bmatrix}\right)\cdot\begin{bmatrix}x_1\\ x_2\end{bmatrix} = \begin{bmatrix}0 & 0\\ 0 & -4\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

is indeed not asymptotically stable [6], [23]. In fact, for a switching signal chosen as

$$\lambda(t) = \begin{cases} 1, \ \text{if}\, t \in [t_{ni}, t_{ni} + 1/n^2) \\ 2, \ \text{if}\, t \in [t_{ni} + 1/n^2, t_{ni} + 1/n^2 + 1/(n^2+2)) \end{cases}$$

the trajectory starting from $(x_1(0), x_2(0)) = (1, 0)$ will not converge to the origin, but exhibit an accumulation point, as is illustrated in Fig. 1. Notice that for any positive integer *n* and any $1 \leq i \leq n$,

$$t_{11} = 0, t_{n(i+1)} = t_{ni} + 1/n^2 + 1/(n^2+2), t_{(n+1)1} = t_{n(n+1)}.$$

This example illustrates that the switched behavior, due to interaction between switching signals and system dynamics, may become very rich, and hence difficult to characterize. It also illustrates that as any switched system is time-varying, the limiting behavior of a switched system needs to be explored very carefully.

This work aims at demonstrating UGAS for a large class of switched systems, allowing for arbitrary switching signals, when it is known that the trivial solution is UGS. The concept of CZOS is introduced with the aim of removing the influence of switching signals. CZOS can be interpreted as an extension of the classic concept of limiting equation [1] used to great effect in nonlinear time-varying (NLTV) systems. Based on this notion of CZOS, this paper presents a new approach to conclude UGAS of switched NLTV systems.

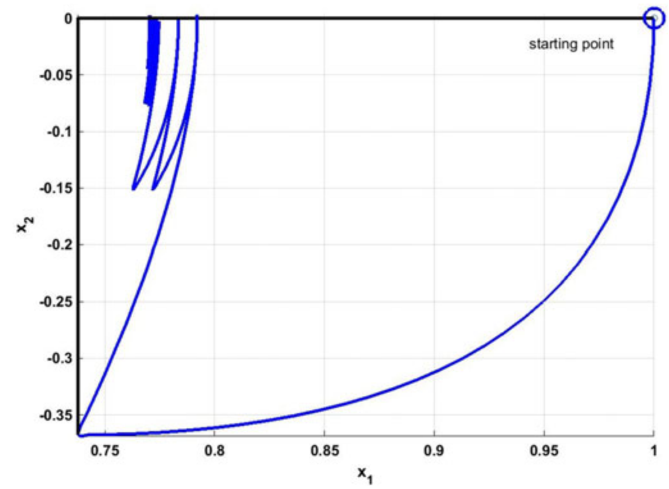

Fig. 1. A trajectory of (3) using the switching signal λ.

B. Literature Review

As indicated in [25], *"a switching system is a dynamical system that consists of a finite number of subsystems and a logical rule that orchestrates switching between these subsystems"*. In general, there are two major types of switched systems:

1) *Restricted switching* where switching signals are subject to limitations such as dwell time conditions.
2) *Arbitrary switching* where switching signals are essentially unconstrained.

The majority of the literature deals with the stability properties for switched systems under restricted switching [2], [8], [10], [26], [31], [32], [41]. If the switching signal is one of the design or control variables, or one of the subsystems is not stable, a restricted switching is a very natural choice in order to ensure the asymptotic stability of the switched system.

Also, whenever the switching law is expressed through state-dependent relationships, arbitrary switching has to be considered. Although introducing arbitrary switching signals or state-dependent switching does provide more flexibility in design, which is attractive from a performance point of view, it makes the (stability) analysis of switched systems much more challenging as shown in (1) and (3) above.

Using a common Lyapunov function is the dominant analysis/design technique. Nevertheless, as explained above, in this context even when considering switching of LTI systems only, it is usually hard to find a common Lyapunov function. When the dimension and the number of the subsystems are both small, it is possible to build on knowledge of the vector fields to explore the behavior and check UGAS without using common Lyapunov functions [3], [6]. However, for general switched systems with large orders or many switching subsystems, such a method is very hard to be applied.

If a common weak Lyapunov function is available, as pointed out in [16], [17], and [27], the observability-type and/or persistently exciting conditions can be called upon to perform a complete stability analysis of the trivial solution in switched systems. The output persistently exciting (OPE) condition has been proposed for NLTV systems [17] and switched NLTV systems [18], [20] combined with appropriate analysis tools. Equivalent to OPE condition, weak zero-state detectability (WZSD) can be used to check UGAS of NLTV systems [16] and switched NLTV systems [22]. Although WZSD is more easy to use than OPE, checking either OPE or WZSD remains challenging due to the need for dealing with complex switching behaviors. It is, therefore, important to provide tools that can be used to simplify the analysis further. This paper develops tools in this direction.

C. Contributions

The focus is to check UGAS of (the trivial trajectory) in the behavior of a switched NLTV system allowing for arbitrary switching under the restriction that the trivial trajectory is known to be UGS. Even for NLTV systems, working with Krasovskii–LaSalle results, Matrosov results, persistent excitation condition and detectability is not trivial. Each method in the literature has its particular advantages and can be shown to be very useful in special classes of systems. Unfortunately, there appears to be no obvious set of guidelines to select which method can be used to advantage for a particular given NLTV system. These methods are really parts of a tool-kit for the analysis of UGAS. Here we add to this tool-kit, and extend the theoretical analysis to simplify the use of (some of these) tools for a large class of systems.

Along the line of WZSD, the concept of CZOS characterizes a special case of limiting process by using a switching-independent behavior coming from any individual subsystem (mode) when the output signal of each subsystem is "equal to" or "approaching" zero. It is worthwhile to highlight that in the analysis of CZOS, switching-dependent output signals are used to keep switching information visible in output or some selected virtual output. More precisely, for any given output function $h_\varsigma(t, x)$, a particular switching-dependent output is selected as

$$y_c(t) = \lim_{n\to\infty} h_{\lambda_n(t+t_n)}(t + t_n, x(t)) \tag{4}$$

where $t_n \to \infty$ is a time sequence, x is a solution related to the CZOS, and $\{\lambda_n\}$ is a sequence of switching signals. By selecting an appropriate output to generate the CZOS, the influence of switching in checking WZSD will be greatly simplified as switching only affects the output. It is this observation that underscores why the ideas can be applied even when trajectories are generated by complex switching signals.

The mere existence of CZOS provides for a simpler environment to check detectability and ultimately UGAS. Admittedly, establishing a CZOS for a general switched NLTV system remains not trivial. In the context of switched LTI systems, it will be shown how to systematically find a CZOS if there exists. This technique can be used in conjunction with the OPE condition to provide even more flexibility in checking UGAS of switched LTI systems.

The contribution of this paper is summarized as follows:

1) A new concept of CZOS is introduced. If a CZOS exists, the verification of WZSD for switched NLTV systems will be greatly simplified (Theorem 1). This is captured in a generalized Krasovskii–LaSalle theorem, which can be applied to both arbitrary switching signals and less restricted switching signals (Theorem 2).
2) These results are then further refined in the context of NLTI systems (Theorem 3). For arbitrary switching, a WZSD condition is proposed for the CZOS together with the (nonswitched) common output function that is defined as the product of the output functions of all subsystems (Theorem 4). Under an analytic condition, it is reduced to a (necessary) WZSD for each subsystem (Theorem 5). For a less restricted class of switching signals, a "joint" WZSD condition is proposed for the CZOS together with the (switching-dependent) output functions of some subsystems (Theorem 6). A special case, which has a zero CZOS function, is also discussed (Corollary 1). A simple example shows that the proposed results can guarantee UGAS under a less restricted switching that might be necessary to guarantee UGAS.
3) For the class of switched LTI systems, the generalized Krasovskii–LaSalle result can be reduced to the verification of algebraic conditions expressed in the system matrices. In this manner, in combination with the OPE condition (along the line of the well-known nested Matrosov result), a recursive method is derived to systematically find a CZOS (Theorem 7).
4) To show the effectiveness of the proposed results, two consensus problems presented in [36] are revisited. It was shown many consensus problems can be rewritten as switched LTI systems, even allowing for variable communication topologies among the agents seeking the consensus (Theorem 8). By using the results presented in this paper, stronger stability results can be derived, such as uniform global exponential stability, under weaker conditions than presently reported in the literature.

The paper is organized as follows. For completeness sake, and ease of reading, Section II revisits some of the important concepts and tools in UGAS such as the OPE condition, WZSD, almost bounded output energy, various stability criteria, and the concept of limiting functions. Section III presents a simplified detectability condition for WZSD based on the concept of CZOS. Next in Section IV, switched NLTI systems are considered. Section V deals with arbitrarily switched LTI systems, where uniform global exponential stability can be inferred by verifying some key matrix algebraic conditions. Section V also illustrates the power of results in the context of some well-known consensus problems. Conclusion remark is afforded in Section VI.

Notations

1) $\Re$ is the set of all real numbers, $\Re_+ = [0, \infty)$, $\aleph = \{1, 2, 3, \ldots\}$.
2) $|t|$ is the absolute value of a real number t and $\|v\|$ denotes the Euclidean norm of a vector $v \in \Re^p$.

3) For a closed set $X \subseteq \Re^p$, $g : \Re_+ \times X \to \Re^q$ is continuous in $u \in X$, uniformly in $t \in \Re_+$, if for any $\varepsilon > 0$ and any $u \in X$, there exists a $\delta(\varepsilon, u) > 0$ such that

$$\|g(t,v) - g(t,u)\| < \varepsilon,\ \forall t \geq 0,\ \forall v \in X \text{ with } \|v - u\| < \delta.$$

4) For a closed set $X \subseteq \Re^p$, $g : \Re_+ \times X \to \Re^q$ is said to be almost uniformly bounded if, there is a measure zero set S of $\Re_+$ such that for any $0 < \varepsilon < 1$, there exists $\eta(\varepsilon) > 0$ such that $\|g(t,u)\| \leq \eta, \forall t \in \Re_+ \backslash S, \forall u \in X$ with $\varepsilon \leq \|u\| \leq 1/\varepsilon$.
5) For a measurable set $S \subseteq \Re^p$, let $m(S)$ denote the Lebesgue measure of S.
6) For a finite set $S = \{u_1, \ldots, u_n\}$ contained in $\Re^p$, let $\text{Span}(S) = \{\, \alpha_1 u_1 + \cdots + \alpha_n u_n \mid \alpha_i \in \Re, 1 \leq i \leq n\}$.
7) For a set $S \subseteq \Re^p$, let $S^\perp = \{\, u \in \Re^p \mid u^T v = 0, \forall v \in S\}$.
8) For a $q \times p$ matrix C, let $Ker(C) = \{\, u \in \Re^p \mid Cu = 0\}$.
9) For a continuously differentiable function $V : \Re^p \to \Re$, ∇V denotes the gradient function of V, i.e., for all $u \in \Re^p$, $\nabla V(u) = (\partial V/\partial x_1(u), \partial V/\partial x_2(u), \ldots, \partial V/\partial x_p(u))$.
10) I_p is the $p \times p$ identity matrix, $\otimes$ denotes the Kronecker product and $1_p = [1\, 1 \cdots 1]^T \in \Re^p$.

II. Preliminaries

A. Basic Concepts for Switched NLTV Systems

For a finite index set Λ and a nonempty closed set $X \subseteq \Re^p$, consider the following switched NLTV system:

$$\dot{x} = f(t, x, \lambda) \tag{5}$$

$$y = h(t, x, \lambda) \tag{6}$$

where $t \in \Re_+$, $x \in X$ is the state and λ is a Λ-valued switching signal; $f : \Re_+ \times X \times \Lambda \to \Re^p$ is a system function and $h : \Re_+ \times X \times \Lambda \to \Re^q$ is an output function. In switched systems, a switching signal is required to be piecewise constant, right-continuous and has finite discontinuous points over any finite time intervals [10], [23]. Notice that the results proposed in this paper could be applied to a more general set $\chi \subseteq \Re^p \times \Lambda$ [18]. To simplify the whole discussion, we only consider the case $\chi = X \times \Lambda$.

It is assumed throughout that for each $\zeta \in \Lambda$, $f(\cdot,\cdot,\zeta)$ has the local Caratheodory property [9]. This condition provides a sufficient condition for the existence of solutions.

Let (x, λ) be any pair with $\lambda : [t_0, \infty) \to \Lambda$ being a switching signal and $x : [t_0, \infty) \to X$ being a complete solution of (5) w. r. t. λ. A notion $t_0(x) = t_0$ is used and (x, λ) is referred as a solution pair.

If $\lim_{s \to t^-} \lambda(s) \neq \lambda(t)$, t is said to be a jumping point of λ. Let Φ be a set of solution pairs. Denote $\Phi^{sw} = \{\lambda \mid (x, \lambda) \in \Phi \text{ for some } x\}$. For any function $g : \Re_+ \times X \times \Lambda \to \Re^q$ and any $\zeta \in \Lambda$, it is denoted that $g_\zeta(t,u) = g(t,u,\zeta), \forall t \geq 0, \forall u \in X$.

The main focus of this paper is to study UGAS of system (5). Precise definitions for various stability properties for switched systems can be found in [14], [18].

Assumption 1 ensures that the output signal is measurable.

Assumption 1: For each $\zeta \in \Lambda$, $h(\cdot, x(\cdot), \zeta)$ is measurable on $[a, b]$ for any $0 \leq a < b$ and any continuous function $x : [a, b] \to X$.

Assumption 1 is very weak as it holds for a large class of functions including continuous functions and those functions satisfying the local Caratheodory condition [9].

Both the OPE condition and WZSD, as necessary conditions, play important roles in checking UGAS for switched systems [18], [20].

Definition 1: The pair (h, f) is *output-persistently exciting* (OPE) w. r. t. Φ if, for any $0 < \delta < 1$ there exist $T(\delta) > 0$ and $\varepsilon(\delta) > 0$ such that for any $(x, \lambda) \in \Phi$ and for any $t \geq t_0(x)$, the following implication holds:

$$\delta \leq \|x(\tau)\| \leq 1/\delta\ \forall t \leq \tau \leq t + T \Rightarrow$$
$$\int_t^{t+T} \|h(\tau, x(\tau), \lambda(\tau))\|^2 d\tau \geq \varepsilon. \tag{7}$$

When Φ is the set of all solution pairs, it is said that (h, f) is OPE for simplicity.

Definition 2: The pair (h, f) is WZSD w. r. t. Φ if, for any $0 < \varepsilon < 1$ there are *no sequences* $\{(x_n, \lambda_n)\} \subseteq \Phi$ and $\{t_n\} \subseteq \Re_+$ such that for each $n \in \Re$, the following hold:
1) $t_n \geq t_0(x_n) + 2n$.
2) $\varepsilon \leq \|x_n(t + t_n)\| \leq 1/\varepsilon, \forall -n \leq t \leq n$.
3) For almost all t in $\Re$,

$$\lim_{n \to \infty} h(t + t_n, x_n(t + t_n), \lambda_n(t + t_n)) = 0. \tag{8}$$

The following lemma reveals the relationship between OPE and WZSD [22]:

Lemma 1: Consider the switched system (5) and (6), if the pair (h, f) is WZSD w.r.t. Φ, then it is OPE w.r.t. Φ. ■

Remark 1: In [20], the concept of PE pairs was introduced to provide flexibility in checking OPE using tools such as the well-known Matrosov results. Roughly speaking, two systems are PE pair if the OPE condition of one system can be used to obtain the OPE condition of the other system. Based on the concept of PE pairs, new systems (either with a reduced complexity state or a richer output) may be generated. By checking the OPE of the new system, the PE pair result infers the OPE condition for the original system. By combining the concept of PE pairs with the relation between WZSD and OPE (Lemma 1), it is possible to generate a sequence of PE pairs so that at the end the WZSD is nearly trivial. For an example of such reasoning, refer to the proof of Theorem 7. ■

With OPE or WZSD, UGS combined with a generalized convergence condition of the output ensures UGAS. This convergence condition is stated in Assumption 2.

Assumption 2: For any $0 < \varepsilon < 1$, there exists a positive constant $M(\varepsilon)$ such that for any $(x, \lambda) \in \Phi$, defined on $[t_0, \infty)$, and for any pair (s, t), with $t_0 \leq s \leq t$ and $\varepsilon \leq \|x(\tau)\| \leq 1/\varepsilon$,

$\forall s \leq \tau \leq t$, the following integral inequality holds:

$$\int_s^t \|h(\tau, x(\tau), \lambda(\tau))\|^2 d\tau \leq M + \varepsilon(t-s). \tag{9}$$

Remark 2: Roughly speaking, Assumption 2 indicates that the output signals gradually converge to zero [17], [18]. ■

Proposition 1 is established in [18] and [20].

Proposition 1: Let $X \subseteq \Re^p$ be a nonempty closed set and Λ be a finite set. Consider the switched system (5) and (6). Let Φ denote a set of solution pairs. If the origin is UGS w. r. t. Φ, the pair (h, f) is OPE w. r. t. Φ and Assumptions 1 and 2 hold, then the origin becomes UGAS w. r. t. Φ. ■

B. Limiting Functions

As mentioned in [1], in order to verify UGAS of any time-varying dynamic system with respect to any initial time instant, the limiting behavior of such a system should be investigated. Generally speaking, the limiting systems or the limiting equations of a NLTV system represent the limiting behavior of a family of trajectories as initial time instants approach to infinity. Properties of limiting systems can be characterized by limiting functions. This section revisits the concept of limiting functions for a NLTV system [16], [17].

Definition 3 describes what limiting functions are. A related concept of an asymptotically almost periodic function is also introduced.

Definition 3: Let $X \subseteq \Re^p$ be a nonempty closed set and $g : \Re_+ \times X \to \Re^n$ be a continuous function.

1) A continuous function $\bar{g} : \Re \times X \to \Re^n$ is said to be a limiting function of g w. r. t. a sequence $t_n \to \infty$ if $\{g(\cdot + t_n, \cdot)\}$ converges uniformly to $\bar{g}$ on every compact subset of $R \times X$.
2) The function g is said to be an asymptotically almost periodic function if, for any unbounded sequence $\{t_n\}$ in $\Re_+$ there exists a subsequence $\{t_{n_k}\}$ so that $\{g(\cdot + t_{n_k}, \cdot)\}$ converges uniformly to a continuous function $\bar{g} : \Re \times X \to \Re^n$ on every compact subset of $\Re \times X$.

Remark 3: In Definition 3, 1) defines the limiting functions, and 2) identifies those functions "g" for which limiting functions are easy to find. ■

Remark 4: Limiting functions were first introduced for NLTV systems in [1] and many functions are asymptotically almost periodic functions as explained in [17]. ■

III. Common Zeroing-Output System and Weak Zero-State Detectability

This section first introduces the concept of CZOS to characterize the limiting behavior of switched systems. It is then used to propose a simplified detectability condition to check WZSD, followed by a new Krasovskii–LaSalle theorem based on a switching-dependent output. An example is presented to show how to use a CZOS to check UGAS.

Definition 5 introduces CZOS with the help of zeroing pair.

Definition 4: For a closed set $X \subseteq \Re^p$, let $g : \Re_+ \times X \to \Re^q$ and $\hat{g} : \Re_+ \times X \to \Re^{\hat{q}}$ be two functions. The pair $(g, \hat{g})$ is said to be a zeroing pair if for any time sequence $t_n \to \infty$, any constant $0 < \varepsilon < 1$ and any sequence $\{u_n\} \subset \Re^p$, with $\varepsilon \leq \|u_n\| \leq 1/\varepsilon, \forall n \in \aleph$, the following holds:

$$\lim_{n\to\infty} g(t_n, u_n) = 0 \Rightarrow \lim_{n\to\infty} \hat{g}(t_n, u_n) = 0. \tag{10}$$

Definition 5: A system $\dot{x} = f_c(t, x)$ is said to be a CZOS for the switched NLTV system (5), (6) if, for each $\zeta \in \Lambda$ $(h_\zeta, f_\zeta - f_c)$ is a zeroing pair where $f_c : \Re_+ \times X \to \Re^p$. The function f_c is called a CZOS function for simplicity.

Remark 5: Intuitively, a CZOS function implies that the switched NLTV system (5) converges to a common nonswitching system $\dot{x} = f_c(t, x)$ as the output (6) of each subsystem converges to zero. This concept is similar to the concept of limiting equation [1] and akin to output-injection conditions discussed in [17]. The system $\dot{x} = f_c(t, x)$ is much simpler than (5), as switching is absent from the state equation (the effect of switching though is incorporated in the definition of the CZOS). Together with the appropriate detectability condition in relation to (6), the CZOS can be used to simplify WZSD in the UGAS analysis even when the original system (5) may be subject to arbitrary switching. ■

Assumption 3 assumes the existence of CZOS for a switched system.

Assumption 3: There is an asymptotically almost periodic function $f_c : \Re_+ \times X \to \Re^p$ which is a CZOS function for the switched NLTV system (5), (6).

Once a CZOS is obtained, checking WZSD of a switched NLTV system can be simplified by using this nonswitched system as shown in Assumption 4.

Assumption 4: For any limiting function $\bar{f}_c$ of f_c w. r. t. a sequence $t_n \to \infty$ and any bounded solution $\bar{x} : \Re \to X$ of $\dot{\bar{x}} = \bar{f}_c(t, \bar{x})$, if

$$\lim_{n\to\infty} h(t + t_n, \bar{x}(t), \lambda_n(t + t_n)) = 0 \tag{11}$$

for almost all t in $\Re$ and some $\{\lambda_n : [s_n, \infty) \to \Lambda\} \subseteq \Phi^{sw}$, with $0 \leq s_n \leq t_n, \forall n \in \aleph$, and $(t_n - s_n) \to \infty$, then

$$\inf_{t\in\Re} \|\bar{x}(t)\| = 0. \tag{12}$$

Remark 6: An output $y_c(t) = \lim_{n\to\infty} h(t + t_n, \bar{x}(t), \lambda_n(t + t_n))$ is used to check WZSD for the limiting behavior of nonswitched CZOS. Different from switching independent CZOS, this output still keeps the information of switching. Dependent on the properties of the switching signals, the output can further be simplified. For example, when restricted switching is considered, a weaker detectability condition could be used to check WZSD as illustrated in Example 3 and the consensus problems in Section V. ■

Remark 7: As clearly pointed out in the proof of Theorem 1 (Appendix A), Assumption 4 implies WZSD for the switched NLTV system (5) and (6) [16], [22]. Roughly speaking, Assumption 4 says that a bounded state trajectory, constrained in the zero locus of the output function, equals to zero at some finite time instant $t = t_0$, or when the time approaches $\pm\infty$. The same assumption is needed for NLTV systems. For example, it was used as in [17, (H2)]. ■

Remark 8: For a given switched NLTV system [(5) and (6)], more than one CZOS might exist. Assume that there is one CZOS function f_c satisfying Assumptions 3 and 4 while another CZOS function f_d only satisfies Assumption 3, that is, equality (12) does not hold. For any limiting function $\bar{f}_d$ of f_d w. r. t. one sequence $t_n \to \infty$, we may assume that there is a limiting function $\bar{f}_c$ of f_c w. r. t. $\{t_n\}$ without loss of generality (replacing $\{t_n\}$ by a subsequence of it). It can be shown that $\bar{f}_c(t, \bar{x}(t)) = \bar{f}_d(t, \bar{x}(t))$ for almost all *t* in $\Re$ based on Assumption 3 where $\bar{x} : \Re \to X$ is any bounded solution of $\dot{\bar{x}} = \bar{f}_d(t, \bar{x})$ satisfying (11) (see more details in the proof of Theorem 1). Since $\bar{f}_c$ and $\bar{f}_d$ are both continuous, $\bar{f}_c(t, \bar{x}(t)) = \bar{f}_d(t, \bar{x}(t)), \forall t \in \Re$ [29]. Thus, $\bar{x}$ is also a bounded solution of $\dot{\bar{x}} = \bar{f}_c(t, \bar{x})$. Hence (12) holds, a contradiction was reached. Therefore even though two CZOSs can be obtained for the same switched system, they have the same features, and lead to the same conclusions. ■

The first main result is presented in Theorem 1 with its proof provided in Appendix A.

Theorem 1: Let $X \subseteq \Re^p$ be a nonempty closed set and Λ be a finite index set. Consider the switched NLTV system (5), (6) where for each $\zeta \in \Lambda$, the system function f_ζ is almost uniformly bounded and $h_\zeta(t, x)$ is continuous in *x*, uniformly in *t*. Let Φ denote a set of solutions pairs. Suppose Assumptions 1, 3, and 4 hold. Then, (h, f) is WZSD. ■

Remark 9: Theorem 1 shows that checking WZSD can be greatly simplified if a CZOS exists. When finding a CZOS is hard for the given output function, the tools from [20, Ths. 5 and 6] can be employed to change the output function, see in particular Lemma 4. Finding a CZOS for a new output, which usually has a higher dimension than the original output, may be easier, this observation is illustrated in Example 4 in Section V. ■

Applying Lemma 1, Proposition 1, and Theorem 1 to switched NLTV systems leads to Theorem 2. This result can be viewed as a generalization of the classic Krasovskii–LaSalle theorem [17].

Theorem 2: Let $X \subseteq \Re^p$ be a nonempty closed set and Λ be a finite index set. Consider the switched NLTV system (5) and (6) where for each $\zeta \in \Lambda$, the system function f_ζ is almost uniformly bounded and $h_\zeta(t, x)$ is continuous in *x*, uniformly in *t*. Let Φ denote a set of solution pairs. Suppose the origin is UGS w. r. t. Φ. Under Assumptions 1–4, the origin becomes UGAS w.r.t. Φ. ■

Remark 10: Assumption 3 plays a key role in Theorem 2 since the UGS property and Assumption 2 can be checked using common weak Lyapunov functions [18]. As a CZOS is a standard NLTV system, then the techniques used in [16] and [17] could be applied. Particularly, WZSD of the CZOS can be checked with some output signals in the form of $y_c(t) = \lim_{n\to\infty} h(t + t_n, \bar{x}(t), \lambda_n(t + t_n))$. ■

Finding a CZOS function is now key. In the context of switched LTI systems, a constructive method to identify CZOS functions systematically, see Section V, is available. Observe also that a CZOS function may be found while common Lyapunov functions remain elusive, as alluded to in the Introduction, see also Example 1.

Example 1: [20] Consider a switched NLTV system under arbitrary switching as follows:

$$\Sigma_1 : \begin{array}{l} \dot{x}_1 = -\eta_1(t, x_1, x_2)x_2 \\ \dot{x}_2 = \eta_1(t, x_1, x_2)x_1 - x_2 \end{array} \quad \Sigma_2 : \begin{array}{l} \dot{x}_1 = \eta_2(t, x_1, x_2)x_2 - x_1 \\ \dot{x}_2 = -\eta_2(t, x_1, x_2)x_1 \end{array} \tag{13}$$

where for each $\zeta \in \{1, 2\}$, $x_\zeta \in \Re$ is a state and $\eta_\zeta : \Re_+ \times \Re^2 \to \Re$ is an asymptotically almost periodic function. Moreover, the following persistently exciting condition, used in [16]–[18], [20], is satisfied.

(PE) For any constant $0 < \delta < 1$, there exist $T(\delta) > 0$ and $\varepsilon(\delta) > 0$ such that for any $\theta \in \Re$,

$$\delta \le |\theta| \le 1/\delta \Rightarrow \int_t^{t+T} \left|\eta_\zeta(\tau, \theta\, e_\zeta^T)\right|^{r_\zeta} d\tau \ge \varepsilon\ \forall t \in \Re_+ \tag{14}$$

for some $r_\zeta > 0$ where $e_1 = (1, 0)^T$ and $e_2 = (0, 1)^T$.

It is not hard to find a Lyapunov candidate *V* as $V = (x_1^2 + x_2^2)/2$ such that

$$\dot{V}|_{\Sigma_1} = -x_2^2 \le 0 \text{ and } \dot{V}|_{\Sigma_2} = -x_1^2 \le 0. \tag{15}$$

To guarantee UGAS of (13), the following steps are used:

Step 1 (Checking regularity): By definition, it is easy to see that for each $\zeta \in \{1, 2\}$, the output function $h_\zeta = x_{3-\zeta}$ is continuous in *x*, uniformly in *t*. Moreover, Assumption 1 holds naturally. Noting that $\eta_\zeta, \zeta \in \{1, 2\}$, are asymptotically almost periodic functions, they are uniformly bounded [17]. Then, it is not difficult to show that, for each $\zeta \in \{1, 2\}$, the system function f_ζ is almost uniformly bounded and satisfies the local Caratheodory condition.

Step 2 (Checking UGS and Assumption 2): Consider the set Φ of all solution pairs. According to (15) and [18, Proposition 1], the origin is UGS w.r.t. Φ. Moreover, Assumption 2 holds by integrating two sides of (15).

Step 3 (Checking Assumption 3 by finding a CZOS function): Choose a (CZOS) function $f_c : \Re_+ \times \Re^2 \to \Re^2$ as $f_c = (\eta_2 x_2, \eta_1 x_1)^T$. It is an asymptotically almost periodic function and the following inequality holds:

$$\|f_\zeta(t, x_1, x_2) - f_c(t, x_1, x_2)\| \le |h_\zeta(x_1, x_2)|\ \|\varphi_\zeta(t, x_1, x_2)\|$$

$\forall t \in \Re_+, \forall x_1, x_2 \in \Re,\ \forall \zeta \in \{1, 2\}$, where $h_\zeta = x_{3-\zeta}$, and $\varphi_1 = (-\eta_1 - \eta_2, -1)$ and $\varphi_2 = (-1, -\eta_1 - \eta_2)$ are both uniformly bounded. This results that for each $\zeta \in \{1, 2\}$, $(h_\zeta, f_\zeta - f_c)$ is a zeroing pair. Assumption 3 holds.

Step 4 (Checking detectability-Assumption 4): Consider the nonswitched CZOS: $\dot{x} = f_c(t, x)$ obtained from the function f_c with its limiting equations written as

$$\bar{\Sigma} : \begin{array}{l} \dot{\bar{x}}_1 = \bar{\eta}_2(t, \bar{x}_1, \bar{x}_2)\bar{x}_2 \\ \dot{\bar{x}}_2 = \bar{\eta}_1(t, \bar{x}_1, \bar{x}_2)\bar{x}_1 \end{array} \tag{16}$$

where for each $\zeta \in \{1, 2\}$, $\bar{\eta}_\zeta$ is a limiting function of η_ζ w. r. t. some sequence $t_n \to \infty$. If

$$\lim_{n\to\infty} \bar{x}_{3-\lambda_n(t+t_n)}(t) = \lim_{n\to\infty} h_{\lambda_n(t+t_n)}(\bar{x}_1(t), \bar{x}_2(t)) = 0$$

for some $\{\lambda_n\} \subseteq \Phi$, it has $\bar{x}_1(t)\bar{x}_2(t) = 0$. Since $\bar{x}_1$ and $\bar{x}_2$ are both continuous, $\bar{x}_1(t)\bar{x}_2(t) = 0$ for all t in $\Re$ [30]. Noting (16), it follows that $(\bar{x}_1^2)' = 2\bar{\eta}_2\bar{x}_1\bar{x}_2 \equiv 0$ and $(\bar{x}_2^2)' = 2\bar{\eta}_1\bar{x}_1\bar{x}_2 \equiv 0$. The continuity of $\bar{x}_1$ and $\bar{x}_2$ implies that the pair $(\bar{x}_1, \bar{x}_2)$ is constant. Thus, either $\bar{x}_1 \equiv 0$ or $\bar{x}_2 \equiv 0$.

We will show $\bar{x}_1 = \bar{x}_2 \equiv 0$ by Contradiction. Without loss generality, assume that $\bar{x}_2 \equiv a \neq 0$. Then, $\bar{x}_1 \equiv 0$. Using the first equation in (16), it results that $0 = \dot{\bar{x}}_1 = \bar{\eta}_2(t, 0, a)a$ and consequently $\bar{\eta}_2(\cdot, 0, a) \equiv 0$. Choosing a small constant $\delta > 0$ such that $\delta \leq |a| \leq 1/\delta$. Employing (PE) (see (14)) and the Lebesgue Dominance Theorem [30], it follows that

$$\begin{aligned}\varepsilon &\leq \lim_{n\to\infty} \int_{t_n}^{t_n+T} |\eta_2(\tau, 0, a)\,|^{r_2}\, d\tau \\ &= \int_0^T \lim\nolimits_{n\to\infty} |\eta_2(t_n+\tau, 0, a)|^{r_2}\, d\tau \\ &= \int_0^T |\bar{\eta}_2(\tau, 0, a)|^{r_2}\, d\tau = 0\end{aligned}$$

for some $\varepsilon > 0, T > 0$ as the sequence $\{\eta_2(t_n + \cdot,\, 0,\, a)\}$ converges to $\bar{\eta}_2(\cdot, 0, a)$, leading to a contradiction. It follows that $\bar{x}_1 = \bar{x}_2 \equiv 0$ and $\inf_{t\in\Re}\|(\bar{x}_1(t), \bar{x}_2(t))\| = 0$. Assumption 4 holds. Theorem 2 now concludes UGAS.

Remark 11: As highlighted already in the Introduction for the linear system (1), which is a special case of (13), there is no quadratic common Lyapunov function for such a system. Other than using common Lyapunov functions, in [20] and [37], additional Lyapunov-like functions were constructed to show the UGAS property of systems like (13). There are also other methods to guarantee UGAS for small orders switched LTI systems as discussed in [3], [6]. When constructing additional Lyapunov-like function appears cumbersome, Theorem 2 provides an alternative tool to check UGAS for a large class of switched NLTV systems. The key idea here is to check a simplified detectability condition using a switching-independent CZOS. Thus, Theorem 2 is in the spirit of classic Krasovskii–LaSalle theorem and LaSalle invariance principle [14], [15], [17]. ■

Remark 12: It is noted that the regularity condition, related to η_ζ and used in [20] and [37], has been relaxed by assuming that η_ζ is asymptotically almost periodic. It only requires certain continuity property [17]. This example clearly shows the potential of Theorem 2 as it is able to confirm UGAS of the system (13) under a weaker condition. ■

IV. Switched NLTI Systems

As a special case of switched NLTV systems, switched NLTI systems are considered to simplify the conditions needed in Theorem 2, though the results obtained here hold for switched NLTV systems. If a CZOS exists, Assumption 4 is most interesting condition to check in Theorem 2 as the switching information is still in the "limiting output equation" (11). This section aims at further simplifying this limiting output by using the knowledge of switching signals.

More precisely, we show that the origin is UGAS under the following conditions:

1) A time-invariant common weak Lyapunov function exists.
2) A time-invariant CZOS exists.
3) This CZOS is weakly zero-state detectable w.r.t. some suitable output functions.

Two types of switching signals are considered. One is arbitrary switching while the other is a generalized restricted switching. Here a generalized restricted switching indicates that switching signal can be fast (as discussed in Remark 15). For arbitrary switching, the output function of the CZOS is chosen as the product of the output functions of all subsystems. When an analytic condition holds, the needed detectability condition becomes a necessary requirement for each subsystem. For the case of generalized restricted switching, the output functions of the CZOS are obtained by summing the absolute values of the output functions of some subsystems.

In the remainder of this paper, with two continuous functions $g_1 : X \to R^p$ and $g_2 : X \to R^q$, the pair (g_2, g_1) is said to be WZSD if for any bounded solution $x : \Re \to X$ of $\dot{x} = g_1(x)$ satisfying $g_2(x(t)) = 0, \forall t \in \Re$, we have $\inf_{t\in\Re}\|x(t)\| = 0$. This definition is equivalent to the one given in Definition 2 for non-switched NLTI systems (Λ is a singleton) [22].

A. Further Simplified Conditions

In this section, Theorem 2 is applied to the following switched NLTI systems:

$$\dot{x} = f_\lambda(x) \tag{17}$$

where Λ is a finite index set, $X \subseteq \Re^p$ is a closed set, λ is a Λ-valued switching signal, $x \in X$ is the state vector, and for each $\zeta \in \Lambda$, $f_\zeta : X \to \Re^p$ is continuous. Since the function f_ζ is time-independent and continuous, it is almost uniformly bounded.

Let Φ be a set of solution pairs (x, λ) with $t_0(x) = 0$. Since time-invariant systems are considered, Assumptions 1 and 2 and UGS in Theorem 2 can be replaced by the existence of a time-invariant common weak Lyapunov function:

(I1) There exist a function $V : \Re^p \to \Re_+$ and two class-K_∞ functions α and β such that V is continuously differentiable on X and for any $(x, \lambda) \in \Phi$, the following inequalities hold:

$$\alpha(\|x(t)\|) \leq V(x(t)) \leq \beta(\|x(t)\|)\ \forall t \in \Re_+ \tag{18}$$

$$\dot{V}_{\lambda(t)}(x(t)) = \nabla V(x(t)) f_{\lambda(t)}(x(t)) \leq 0\ \forall t \in \Re_+. \tag{19}$$

Under (I1), the origin is UGS, see [18]. A virtual output is then obtained as

$$h_\zeta = \sqrt{|\nabla V(x) f_\zeta(x)|}\,. \tag{20}$$

This time-independent, but switching-dependent function h_ζ is continuous on X. Hence h_ζ is continuous in x, uniformly in t. Moreover, Assumption 1 holds. Assumption 2 also holds by (I1) [18]. Accordingly, Assumption 3 is simplified as follows:

(I2) There is a continuous function $f_c : X \to \Re^p$ such that for any $\zeta \in \Lambda$ and any $u \in X - \{0\}$, the following implication holds:

$$\nabla V(u) f_\zeta(u) = 0 \Rightarrow f_\zeta(u) = f_c(u). \tag{21}$$

The following lemma shows that (I2) implies Assumption 3, see [19] for a proof.

Lemma 2: Let $X \subseteq \Re^p$ be a nonempty closed set and $g : X \to \Re^q$ and $\hat{g} : X \to \Re^{\hat{q}}$ be two continuous functions. Suppose that for any $u \in X - \{0\}$, $g(u) = 0$ implies $\hat{g}(u) = 0$. Then, the pair $(g, \hat{g})$ forms a zeroing pair. ■

For switched NLTI systems, Assumption 4 is reduced to the following condition:

(I3) For any bounded solution $x : \Re \to X$ of $\dot{x} = f_c(x)$ satisfying

$$\lim_{n\to\infty} \nabla V(x(t)) f_{\lambda_n(t+t_n)}(x(t)) = 0 \tag{22}$$

for almost all *t* in $\Re$, for some $\{\lambda_n\} \subseteq \Phi^{sw}$ and some sequence $t_n \to \infty$ with $t_n \geq 0, \forall n \in \aleph$, it has $\inf_{t\in\Re}\|x(t)\| = 0$.

Consequently a simplified result of Theorem 2 for switched NLTI systems is obtained.

Theorem 3: Let $X \subseteq \Re^p$ be a nonempty closed set. Consider the switched NLTI system (17) where Λ is finite and f_ζ is continuous. Let Φ denote a set of solution pairs (x, λ) with $t_0(x) = 0$. Suppose there exist $V : \Re^p \to \Re_+$ and $f_c : X \to \Re^p$ such that (I1)–(I3) holds. Then, the origin becomes UGAS w. r. t. Φ. ■

Two different cases will be explored to check conditions (I1)–(I3) in the following sections.

B. Arbitrary Switching

Let $X = \Re^p$ and consider the switched NLTI system (17) where for each $\zeta \in \Lambda$, $f_\zeta : \Re^p \to \Re^p$ is continuous. Let Φ denote the set of all solution pairs (x, λ) with $t_0(x) = 0$, which indicates the case of arbitrary switching.

Under (I1), a common weak Lyapunov function $V : \Re^p \to \Re_+$ w. r. t. Φ already exists. It thus leads to the following continuous "common" output function:

$$h_*^V = \prod\nolimits_{\zeta\in\Lambda} \nabla V(x) f_\zeta(x). \tag{23}$$

Observe that for any $t \in \Re$, any $t_n \to \infty$ and any $\{\lambda_n\} \subseteq \Phi^{sw}$,

$$\aleph = \bigcup\nolimits_{\zeta\in\Lambda} \{\, n \in \aleph \mid \lambda_n(t + t_n) = \zeta \,\}. \tag{24}$$

Since Λ is finite, there exist infinitely many $n \in \aleph$ such that $\lambda_n(t + t_n) = \zeta_0$ for some $\zeta_0 \in \Lambda$. If equality (22) in (I3) holds, it results in the following equality:

$$\nabla V(x(t)) f_{\zeta_0}(x(t)) = \lim_{n\to\infty} \nabla V(x(t)) f_{\lambda_n(t+t_n)}(x(t)) = 0. \tag{25}$$

Hence $h_*^V(x(t)) = 0$ for almost all *t* in $\Re$. Since *x* and h_*^V are both continuous, $h_*^V(x(t)) = 0$ for all *t* in $\Re$ [29], [30]. Therefore, (I3) is implied by WZSD of (h_*^V, f_c).

Theorem 4 is then obtained as a special case of Theorem 3.

Theorem 4: Let $X = \Re^p$. Consider the switched NLTI system (17) where Λ is finite and *f* is continuous. Let Φ denote the set of all solution pairs (x, λ) with $t_0(x) = 0$. Suppose (I1), (I2), and (h_*^V, f_c) is WZSD. Then, the origin becomes UGAS w. r. t. Φ. ■

Example 2 from [2] and [8] shows the usefulness of Theorem 4.

Example 2: Consider the following switched NLTI system:

$$\Sigma_1 : \begin{array}{l} \dot{x}_1 = -x_2 - x_1 \\ \dot{x}_2 = x_1 \end{array} \quad \Sigma_2 : \begin{array}{l} \dot{x}_1 = -x_2 - a(x_1)x_1 \\ \dot{x}_2 = x_1 \end{array} \tag{26}$$

where $x_\zeta \in \Re$, $\zeta = 1, 2$, is a state and

$$a(s) = \begin{cases} 1, s < 0 \\ 0, s \geq 0. \end{cases} \tag{27}$$

Let Φ be the set of all solution pairs. By selecting a Lyapunov candidate $V = (x_1^2 + x_2^2)/2$, it has

$$\dot{V}|_{\Sigma_1} = -x_1^2 \leq 0 \text{ and } \dot{V}|_{\Sigma_2} = -a(x_1)x_1^2 \leq 0. \tag{28}$$

So $V : \Re^2 \to \Re_+$ is a common weak Lyapunov function w. r. t.Φ, i.e., (I1) holds. Choose a (CZOS) function $f_c : \Re^2 \to \Re^2$ as $f_c = (-x_2, x_1)^T$. The following inequality

$$\|f_\zeta(x_1, x_2) - f_c(x_1, x_2)\| \leq \sqrt{|\nabla V(x_1, x_2) f_\zeta(x_1, x_2)|}$$

holds for all $x_1, x_2 \in \Re$, $\zeta \in \{1, 2\}$ where $f_1 = (-x_2 - x_1, x_1)^T$ and $f_2 = (-x_2 - a(x_1)x_1, x_1)^T$. Thus, (I2) is true. Let $x = (x_1, x_2)^T : \Re \to \Re^2$ be a bounded solution of $\dot{x} = f_c(x)$ and it can be explicitly solved as

$$x_1 = b\cos(t + c),\ x_2 = b\sin(t + c) \tag{29}$$

for some $b, c \in \Re$. The common output function can be described as $h_*^V = [\nabla V(x) f_1(x)][\nabla V(x) f_2(x)] = a(x_1)x_1^4$. Observe that

$$a(s) = 0 \text{ if and only if } s \geq 0.$$

Thus, $h_*^V(x(t)) \equiv 0$ forces that $x_1(t) \geq 0, \forall t \in \Re$. Hence $b = 0$ and $x_1 = x_2 \equiv 0$. Therefore (h_*^V, f_c) is WZSD and the system (26) is UGAS (Theorem 4).

Although the second system Σ_2 is not linear, Φ is scaling invariant, i.e.,

$$(x, \lambda) \in \Phi \Rightarrow (rx, \lambda) \in \Phi, \forall r > 0.$$

This is due to $a(rs) = a(s), \forall s \in \Re, \forall r > 0$. Applying [18, Lemma 1], the system (26) is also uniformly globally exponentially stable.

Remark 13: It is worthwhile to point out that the result proposed in [2] cannot be applied to the system (26) to conclude UGAS. The result in [8] needs some dwell time conditions for switching signals. Furthermore, the result stated in [18, Th. 2.3] cannot be used directly either since the system does not satisfy the so-called zero small-time distinguishable property. Applying Theorem 4, the system (26) is shown to be uniformly globally exponentially stable under arbitrary switching. This example again demonstrates the practicality of the obtained results. ■

When the system function and the output function are both analytic, WZSD of (h_*^V, f_c) can further be simplified. To this end, the definition of analytic functions is recalled.

Definition 6: [30] A function $\rho : \Re \rightarrow \Re$ is said to be an analytic function if, for any $s \in \Re$ there is a $\delta > 0$ such that

$$\rho(t) = \sum_{n=0}^{\infty} a_n (t-s)^n \ \forall t \in \Re \text{ with } |t - s| < \delta \tag{30}$$

where $a_n \in \Re, \forall n \in \aleph$. Simultaneously, a vector valued function $\rho : \Re \rightarrow \Re^q$ is said to be analytic if its each component function ρ_i, $1 \leq i \leq q$, is analytic.

Many functions are analytic functions. Lemma 3 (the principle of permanence) is a fundamental result to characterize analytic functions.

Lemma 3: [30, Theorem 8.5] Suppose $\rho : \Re \rightarrow \Re^q$ is an analytic function. If there is a sequence $\{\tau_n\} \subseteq \Re - \{\tau_0\}$ that converges to $\tau_0 \in \Re$ and satisfies $g(\tau_n) = 0, \forall n \in \mathrm{N}$, then ρ is the zero function. ■

Next condition (I4) is presented to ensure that needed functions are analytic. This condition holds when the system function (17), the CZOS function f_c and the common weak Lyapunov function are all analytic [30].

(I4) For any $\zeta \in \Lambda$, $\nabla V(x(t)) f_\zeta(x(t))$ is an analytic function on $\Re$ for any bounded solution $x : \Re \rightarrow \Re^p$ of $\dot{x} = f_c(x)$.

Consider the common output function (23). We would like to show that (h_*^V, f_c) is WZSD under (I4) and a necessary condition.

Suppose $h_*^V(x(\cdot)) \equiv 0$ for some bounded solution $x : \Re \rightarrow \Re^p$ of $\dot{x} = f_c(x)$. If $\inf_{t \in \Re} \|x(t)\| \neq 0$, we will find a contradiction. We first claim that there exist $\zeta_0 \in \Lambda$ and a sequence $t_n \rightarrow 0$, with $t_n \neq 0$ and such that

$$\nabla V(x(t_n)) f_{\zeta_0}(x(t_n)) = 0 \ \forall n \in \aleph. \tag{31}$$

If the claim is false, there exists $\delta > 0$ such that

$$\nabla V(x(t)) f_\zeta(x(t)) \neq 0 \ \forall \zeta \in \Lambda \ \forall t \text{ with } 0 < |t| < \delta.$$

This implies $h_*^V(x(t)) \neq 0, \forall t$ with $0 < |t| < \delta$, and leads to a Contradiction. Hence the claim is true.

According to the claim, (I4) and Lemma 3, $\nabla V(x(t)) f_{\zeta_0}(x(t))$ is then the zero function. Under (I2), it can be seen that x is also a solution of $\dot{x} = f_c(x) = f_{\zeta_0}(x)$. If $(\nabla V(x) f_{\zeta_0}, f_{\zeta_0})$ is WZSD, a contradiction is found. Therefore $\inf_{t \in \Re} \|x(t)\| = 0$ and (h_*^V, f_c) is WZSD under (I2), (I4), and the following necessary condition:

(I5) $(\nabla V(x) f_\zeta, f_\zeta) \ \forall \zeta \in \Lambda$, is WZSD.

Theorem 5 summarizes the above discussion.

Theorem 5: Let $X = \Re^p$. Consider the switched NLTI system (17) where Λ is finite and f is continuous. Let Φ denote the set of all solution pairs (x, λ) with $t_0(x) = 0$. Suppose (I1), (I2), (I4), and (I5) hold. Then, the origin is UGAS w. r. t. Φ. ■

C. Less Restricted Switching

In this section, the following class of switching signals is considered:

Definition 7: For any $\Lambda' \subseteq \Lambda$, any $T_0 > 0$, any $\tau_0 > 0$, and any $t \geq T_0$, let $\Gamma(\Lambda', T_0, \tau_0, t)$ be the set of switching signals $\lambda : \Re_+ \rightarrow \Lambda$ that satisfies

$$m(\{s \in [t - T_0, t + T_0] \mid \lambda(s) = \zeta\}) \geq \tau_0 \ \forall \zeta \in \Lambda'. \tag{32}$$

Remark 14: Suppose λ has a dwell time $\tau_D > 0$, i.e., any distinct jumping points t and s have the distance larger than or equal to τ_D, and the following ergodicity property holds [5]:

(E) There exist $T > 0$ and $\Lambda' \subseteq \Lambda$ such that for any $s \geq 0$, $\Lambda' \subseteq \lambda([s, s+T])$.

Then, $\lambda \in \Gamma(\Lambda', T + \tau_D, \tau_D, t)$ for any $t \geq T + \tau_D$. ■

Remark 15: Roughly speaking, inequality (32) means that the accumulated (total) time of staying in each mode $\zeta \in \Lambda'$ in the time interval $[t - T_0, t + T_0]$ is larger than or equal to τ_0. Thus, it may contain a fast switching signal. For example, with $\Lambda = \{1, 2\}$, the following switching signal

$$\lambda(s) = \begin{cases} 1, n + (2i)/2^{n+1} \leq s < n + (2i+1)/2^{n+1}, 0 \leq i < 2^n \\ 2, \ n + (2i-1)/2^{n+1} \leq s < n + (2i)/2^{n+1}, 1 \leq i \leq 2^n \end{cases}$$

is in $\Gamma(\{1,2\}, 2, 1, t), \forall t \geq 2$, where n is the greatest integer less than or equal to s. This switching signal switches fast enough so that existing dwell time conditions (see [10], [23]) are not satisfied. However, it is still in the set $\Gamma(\{1,2\}, 2, 1, t)$. Thus, $\Gamma(\Lambda', T_0, \tau_0, t)$ describes a class of switching signals that may contain some fast switching signals. ■

Under such a less restricted switching, (I3) can be deduced by the following "joint" detectability condition.

(I6) There exist $T_0 > 0$ and $\tau_0 > 0$ such that for any $\lambda \in \Phi^{sw}$ and any $s \geq T_0$, there exists $\Lambda' \subseteq \Lambda$ (depending on λ and s) such that $\lambda \in \Gamma(\Lambda', T_0, \tau_0, s)$ and $(h_{\Lambda'}^V, f_c)$ is WZSD where $h_{\Lambda'}^V$ is a continuous "common" output function defined as follows:

$$h_{\Lambda'}^V = \sum_{\zeta \in \Lambda'} |\nabla V(x) f_\zeta(x)|. \tag{33}$$

For less restricted switching signals, Theorem 6 is obtained. The proof is provided in Appendix B.

Theorem 6: Let X be a nonempty closed subset of $\Re^p$. Consider the switched NLTI system (17) where Λ is finite and f is continuous. Let Φ denote a set of solution pairs (x, λ) with $t_0(x) = 0$. Suppose (I1), (I2), (I4), and (I6) hold. Then, the origin becomes UGAS w. r. t. Φ. ■

An interesting and important case is $f_c = 0$. In this case, (I4) naturally holds because $\nabla V(x(t)) f_\zeta(x(t)), \forall \zeta \in \Lambda$, is a constant function with x being a solution of $\dot{x} = f_c(x) = 0$. (I2) and (I6) can also be simplified accordingly.

(I2*) The following implication holds: For any $\zeta \in \Lambda$ and any $u \in X - \{0\}$,

$$\nabla V(u) f_\zeta(u) = 0 \Rightarrow f_\zeta(u) = 0. \tag{34}$$

(I6*) There exist $T_0 > 0$ and $\tau_0 > 0$ such that for any $\lambda \in \Phi^{sw}$ and any $t \geq T_0$, there exists $\Lambda' \subseteq \Lambda$ satisfying $\lambda \in \Gamma(\Lambda', T_0, \tau_0, t)$ and the following implication: For any $u \in X$,

$$\nabla V(u) f_\zeta(u) = 0, \ \forall \zeta \in \Lambda' \Rightarrow u = 0. \tag{35}$$

Corollary 1 is a direct outcome from Theorem 6.

Corollary 1: Let X be a nonempty closed subset of $\Re^p$. Consider the switched NLTI system (17) where Λ is finite and f is continuous. Let Φ denote a set of solution pairs. Suppose (I1), (I2*), and (I6*) hold. Then, the origin becomes UGAS w. r. t. Φ. ■

Example 3 illustrates the effectiveness of Corollary 1.

Example 3: Consider the following switched NLTI system:

$$\Sigma_1 : \begin{array}{l} \dot{x}_1 = -\eta(x_1) \\ \dot{x}_2 = 0 \end{array} \quad \Sigma_2 : \begin{array}{l} \dot{x}_1 = 0 \\ \dot{x}_2 = -\eta(x_2) \end{array} \tag{36}$$

where $x_i \in \Re$, $i = 1, 2$, and η: $\Re \to \Re$ is a continuous function with $\eta(0) = 0$ and $\eta(s)s > 0, \forall s \neq 0$.

Since each subsystem is not asymptotically stable, the origin is not UGAS under arbitrary switching. Let Φ be a set of solution pairs $(x, \lambda) : \Re_+ \to \Re^2$. By selecting a Lyapunov candidate $V = (x_1^2 + x_2^2)/2$,

$$\dot{V}\left|_{\Sigma_\zeta}\right. = -x_\zeta \eta(x_\zeta) \leq 0 \; \forall \zeta \in \{1, 2\}.$$

Hence (I1) holds. Notice that $\dot{V}|_{\Sigma_\zeta} = 0$ if and only if $x_\zeta = 0, \forall \zeta \in \{1, 2\}$. Thus, (I2*) also holds. Suppose that for each $\zeta \in \{1, 2\}$, there exist $T_\zeta > 0$ and $\tau_\zeta > 0$ such that for any $\lambda \in \Phi^{sw}$ and any $t \geq T_\zeta$

$$m(\{s \in [t - T_\zeta, t + T_\zeta] \mid \lambda(s) = \zeta\}) \geq \tau_\zeta. \tag{37}$$

Then, (I6*) holds with

$$\Lambda' = \Lambda = \{1, 2\}, \; T_0 = \max(T_1, T_2), \tau_0 = \min(\tau_1, \tau_2).$$

So the origin is UGAS w. r. t. Φ according to Corollary 1. It is worthwhile to highlight that inequality (37) is a persistently "showing up" condition for each ζ in $\{1, 2\}$. Without this condition, the origin may not be UGAS w. r. t. Φ.

V. Switched LTI Systems

Many switched systems have LTI subsystems. The simplicity of switched LTI systems makes it easy to apply the obtained results, leading to a systematic procedure to generate CZOS employing relatively simple algebraic conditions. Moreover, the achieved criterion (Theorem 6) can be used in consensus problems under less restricted switching to guarantee stronger results with weaker conditions. These applications validate the novelty of the proposed results.

A. Generalized Algebraic Conditions for Arbitrarily Switched LTI Systems

This section studies a class of switched LTI systems as follows:

$$\dot{x} = A_\zeta x \tag{38}$$

where $x \in \Re^p$ is the state and $A_\zeta, \zeta \in \Lambda$, is a $p \times p$ system matrix. Let Φ denote the set of all solution pairs (x, λ) with $t_0(x) = 0$, which indicates that the system (38) is arbitrarily switched.

The following assumption guarantees the existence of a quadratic common weak Lyapunov function.

(S1) There is a positive definite matrix P such that $PA_\zeta + A_\zeta^T P \leq 0, \forall \zeta \in \Lambda$.

Condition (S1) indicates Condition (I1). In order to check (I2), one natural choice is to select $C_\zeta = PA_\zeta + A_\zeta^T P$ as an output matrix. However, sometimes, there is no CZOS for this given output matrix (as shown in Example 4), it is possible to use the concept of OPE to generate other output signals, which might have CZOS. This idea was motivated from the nested Matrosov theorem [37] as discussed in [20]. Lemma 4 is a tool to generate new output. Its proof is provided in Appendix C.

Lemma 4: Consider the switched LTI system (38). For each $\zeta \in \Lambda$, let C be a $(Np) \times p$ matrix. Suppose there is a $p \times p$ (possibly not positive definite) symmetric matrix Q such that

$$u^T Q A_\zeta u \leq 0, \; \forall \zeta \in \Lambda, \forall u \in \Re^p \text{ with } (u^T \otimes I_N) C_\zeta u = 0. \tag{39}$$

Then, the OPE condition of $((x^T \otimes I_{(N+1)})\tilde{C}_\zeta x, Ax)$ implies the OPE condition of $((x^T \otimes I_N)C_\zeta x, Ax)$ where

$$\tilde{C}_\zeta = \begin{bmatrix} C_\zeta \\ QA_\zeta \end{bmatrix} \; \forall \zeta \in \Lambda. \quad \blacksquare$$

With Lemma 4, Condition (I2) can be modified as follows:

(S2) There exist a $p \times p$ matrix A_c and a finite sequence $\{P_1, \ldots, P_N\}$ of $p \times p$ symmetric matrices such that for each $\zeta \in \Lambda$,

$$u^T (P_j A_\zeta) u \leq 0 \; \forall 2 \leq j \leq N \; \forall u \in \Re^p \text{ with } h_\zeta^{j-1}(u) = 0 \tag{40}$$

$$A_c u = A_\zeta u \; \forall u \in \Re^p \text{ with } h_\zeta^N(u) = 0 \tag{41}$$

where for any $1 \leq i \leq N$,

$$h_\zeta^i(u) = \begin{bmatrix} u^T P_1 A_\zeta u & \cdots & u^T P_i A_\zeta u \end{bmatrix}^T \; \forall u \in \Re^p. \tag{42}$$

When a CZOS function takes the form as $f_c = A_c x$, the matrix A_c is called a CZOS matrix. The matrix A_c appeared in (S2) is then a CZOS matrix.

To check WZSD for switched LTI systems, (S3) is used. Notice that this assumption is a necessary condition for UGAS of arbitrarily switched LTI systems.

(S3) Each bounded solution $x : \Re_+ \to \Re^p$ of $\dot{x} = A_\zeta x$, with $h_\zeta^N(x(t)) \equiv 0$, satisfies $\lim_{t\to\infty} x(t) = 0$ where h_ζ^N is the function defined in (42) with $i = N$.

Proposition 1, Theorem 1, Lemma 1, and Lemma 4 are used to obtain Theorem 7 (a simpler and extended version of Theorem 5). The proof is presented in Appendix D.

Theorem 7: Under (S1)–(S3) with $P_1 = P$, the origin of the arbitrarily switched LTI system (38) is uniformly globally exponentially stable. ■

Remark 16: When $N = 1$ and $P_1 = P$, (S2) is reduced to the following assumption under (S1):

(S2*) There exists a $p \times p$ matrix A_c such that

$$A_c u = A_\zeta u, \forall \zeta \in \Lambda, \forall u \in Ker(PA_\zeta + A_\zeta^T P).$$

Moreover, (S3) is implied by the following assumption:

(S3*) $(PA_\zeta + A_\zeta^T P, A_\zeta), \forall \zeta \in \Lambda$, is detectable.

Theorem 7 can then be deduced by Theorem 5. However, as illustrated in Example 4, (S2*) might be not true. By extending output functions [20], perhaps a CZOS matrix can be found, see Example 4 and the proof of Theorem 7. ■

Remark 17: If a quadratic common Lyapunov function exists, i.e., there is a positive definite matrix P such that $PA_\zeta + A_\zeta^T P < 0$, (S1) and (S3*) hold with $N = 1$ and $P_1 = P$

as $(PA_\zeta + A_\zeta^T P, A_\zeta), \forall \zeta \in \Lambda$, is observable. Moreover, (S2*) also holds for any matrix A_c. This indicates that the stability result using quadratic common Lyapunov functions is a special case of Theorem 7 [23]. This demonstrates the novelty of the proposed result. ∎

The most difficult condition in Theorem 7 is (S2). Usually it is hard to find CZOS matrices for a switched LTI system with many subsystems. But, for the simple case of two subsystems, a necessary and sufficient condition can be used to find CZOS matrices, see Lemma 5 with the proof in Appendix E.

Lemma 5: Let W_1 and W_2 be two subspaces of $\Re^p$. Given two $p \times p$ matrices A_1 and A_2, there exists a $p \times p$ matrix A_c such that

$$A_c u = A_\zeta u, \forall u \in W_\zeta \ \forall \zeta \in \{1, 2\} \tag{43}$$

if and only if the following condition holds:

$$A_1 u = A_2 u \ \forall u \in W_1 \cap W_2. \tag{44}$$

∎

Remark 18: By means of Lemma 5, an Induction technique can be used to generate a recursive algorithm to check the existence of CZOS matrices for the general case of *n* subsystems. Due to space limitation, the detailed discussion is omitted. ∎

The following example shows the usefulness of Theorem 7 and Lemma 5.

Example 4: Consider a switched LTI system of the form (38) with $p = 4, \ \Lambda = \{1, 2\}$ and

$$A_1 = \begin{bmatrix} -1 & 1 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & -1 & 0 & 1 \\ 0 & 0 & -1 & -1 \end{bmatrix},$$
$$A_2 = \begin{bmatrix} 0 & 1 & -1 & 0 \\ -1 & -1 & -1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & -1 & -1 \end{bmatrix}. \tag{45}$$

For this system, it is easy to check that (S1) holds with $P = I_4$. Moreover,

$$\operatorname{Ker} C_1^0 = \operatorname{Span}\{e_2, e_3\} \text{ and } \operatorname{Ker} C_2^0 = \operatorname{Span}\{e_1, e_3\}$$

where $C_\zeta^0 = A_\zeta + A_\zeta^T, \ \zeta = 1, 2$, is a negative semidefinite matrix and

$$e_1 = (1,0,0,0)^T, e_2 = (0,1,0,0)^T, e_3 = (0,0,1,0)^T,$$
$$e_4 = (0,0,0,1)^T.$$

Since $A_1 e_3 \neq A_2 e_3$, there are no CZOS matrices for the output matrices $C_\zeta^0, \ \zeta = 1, 2$, by Lemma 5.

Choosing $P_2 = e_3 e_4^T + e_4 e_3^T$, (40) holds, i.e., $u^T(P_2 A_\zeta)u \le 0, \forall u \in Ker(C_\zeta^0), \forall \zeta \in \{1, 2\}$.

By direct computation, the following holds:

$$\left\{ u \in \Re^4 \ \middle| \ h_\zeta^2(u) = 0 \right\} = \operatorname{Span}\{e_{3-\zeta}\} = \operatorname{Ker}(C_\zeta) \ \forall \zeta \in \{1, 2\}$$

where h_ζ^2 is the function defined in (42) with $i = 2$ and

$$C_1 = [e_1 \ e_3 \ e_4]^T, \ C_2 = [e_2 \ e_3 \ e_4]^T.$$

Since $\operatorname{Ker}(C_1) \cap \operatorname{Ker}(C_2) = \{0\}$, (41) is true according to Lemma 5. Hence (S2) holds with $P_1 = P = I_4$, $P_2 = e_3 e_4^T + e_4 e_3^T$ and $N = 2$. Due to $(C_\zeta, A_\zeta), \forall \zeta \in \Lambda$, being observable, (S3) holds. Applying Theorem 7, the origin is then uniformly globally exponentially stable under arbitrary switching.

Although this system has no quadratic common Lyapunov functions, the obtained result provides a set of effective algebraic conditions to guarantee UGAS under arbitrary switching. This example shows that by combining Lemma 4, we can extend the output function so that CZOSs can be found more easily. This shows the flexibility of the proposed result.

B. Application to Consensus Problems

The potential of the proposed results is demonstrated by two consensus problems studied in [36]. Under appropriate assumptions on communication topology, the closed-loop systems of two consensus problems can be rewritten as switched LTI systems:

$$\dot{x} = (I_N \otimes A \ - L_{\lambda(t)} \otimes BB^T P) \, x \tag{46}$$

where N is the number of agents, k is the dimension of states of agents, $x \in \Re^{Nk}$ is the stacked error state, *A* and *P* are $k \times k$ matrices, B is a $k \times q$ matrix, $L_\zeta \ge 0$ is the graph Laplacian of a undirected graph ζ, and $\lambda : \Re_+ \to \Lambda$ is a switching signal with Λ denoting a set of possible undirected graphs having the node set $\{1, 2, \ldots, N\}$.

It is emphasized that reaching consensus among a group of agents is not a standard stabilization problem. It can be transformed into a stabilization problem under a constraint $x(t) \in X = S^\perp, \forall t \ge 0$, where roughly speaking, $S \subseteq \Re^{Nk}$ is closely related to the set to which each agent converges [28], [36]. By defining a proper stacked error state *x*, the control objective is to find sufficient conditions to ensure the convergence of x under an extra constraint $x(t) \in S^\perp, \forall t \ge 0$.

To this end, let Φ be a set of solution pairs $(x, \lambda) : \Re_+ \to S^\perp \times \Lambda$. Condition (C1) is related to the dynamic property of each agent.

(C1) (A, B) is stabilizable and *P* is positive definite with

$$PA + A^T P \le 0. \tag{47}$$

The condition related to the connectivity of graphs is also needed.

(C2) There exist $T_0 > 0$ and $\tau_0 > 0$ such that for any $\lambda \in \Phi^{sw}$ and any $t \ge T_0$, there exists $\Lambda' \subseteq \Lambda$ satisfying $\lambda \in \Gamma(\Lambda', T_0, \tau_0, t)$ and

$$(L_\zeta \otimes I_k) u = 0, \forall \zeta \in \Lambda' \Rightarrow u \in S. \tag{48}$$

Notice that $S^\perp$ is a closed subset of $\Re^{Nk}$ and is scaling invariant, i.e., $ru \in S^\perp, \forall r > 0, \forall u \in S^\perp$. Define an extension $\tilde{\Phi}$ of Φ as $\tilde{\Phi} = \{(rx, \lambda) \mid (x, \lambda) \in \Phi, r > 0\} \supseteq \Phi$, then, any $(x, \lambda) \in \tilde{\Phi}$ is a solution pair of (46). By definition, $\tilde{\Phi}$ is scaling invariant.

Remark 19: Condition (48) is related to the connectivity of communication topology for consensus. It is a generalization of some standard assumptions such as every node is reachable from node 0 in the union graph (Remark 21) and the union graph $\zeta_{\Lambda'}$ is connected (Remark 22). The requirement that $\lambda \in \Gamma(\Lambda', T_0, \tau_0)$ links to some "persistent existence" of the graph ζ over some time intervals $[t - T_0, t + T_0]$. It is weaker than the dwell time assumption used in [5], [36], and [41] as it allows fast switching among different modes (Remarks 14 and 15). ■

The following steps are used to show that the system (46) is uniformly globally exponentially stable w.r.t. $\tilde{\Phi}$ when (C1) and (C2) hold.

Step 1 [Verifying (I1)]: By (C1) and differentiating the Lyapunov candidate $V = x^T(I_N \otimes P)x$ along the trajectories of (46) yields

$$\dot{V} = x^T(I_N \otimes (PA + A^T P) - 2L_{\lambda(t)} \otimes PBB^T P)x$$
$$\leq -\|(I_N \otimes C)\, x\|^2 - x^T(L_{\lambda(t)} \otimes PBB^T P)x \leq 0 \quad (49)$$

where $C = \sqrt{-PA - A^{\mathrm{T}} P}$. This shows (19) in (I1) holds. With $\alpha = a_{\min} s$ and $\beta = a_{\max} s, \forall s \geq 0$, (18) also holds where $a_{\min}$ and $a_{\max}$ are the minimum and maximum eigenvalues of P, respectively. This shows that V is a quadratic common weak Lyapunov function w.r.t. $\tilde{\Phi}$. Particularly, (I1) holds. A virtual output function is thus defined as

$$h_\zeta(u) = \begin{pmatrix} I_N \otimes C \\ \sqrt{L_\zeta} \otimes \sqrt{PBB^T P} \end{pmatrix} u \; \forall \zeta \in \Lambda \; \forall u \in \Re^{N \times n}.$$

Step 2 [Verifying (I2)]: Choose $f_c = (I_N \otimes A\,)\, x$ as a CZOS function. Then,

$$\|f_\zeta(u) - f_c(u)\| =$$
$$\left\|(\sqrt{L_\zeta} \otimes P^{-1}\sqrt{PBB^T P})\, (\sqrt{L_\zeta} \otimes \sqrt{PBB^T P})\, u\right\|$$
$$\leq M\, \|h_\zeta\,(u)\| \; \forall \zeta \in \Lambda \; \forall u \in \Re^{N \times n} \quad (50)$$

where $M = \max_{\zeta \in \Lambda} \|(\sqrt{L_\zeta} \otimes P^{-1}\sqrt{PBB^T P})\|$ and

$$f_\zeta(u) = (I_N \otimes A\; - L_\zeta \otimes BB^T P)\, u.$$

In view of (49),

$$\nabla V(u) f_\zeta(u) = u^T(I_N \otimes (PA + A^T P)$$
$$- 2L_\zeta \otimes PBB^T P)u = 0$$

implies $h_\zeta\,(u) = 0, \forall \zeta \in \Lambda, \forall u \in R^{N \times n}$. By (50), (I2) holds.

Step 3 [Verifying (I4) and (I6)]: Since all involved functions are linear, they are analytic. Hence (I4) holds. Employing (C2), it remains to check that $(h^V_{\Lambda'}, f_c)$ is WZSD to show (I6). Let $x : \Re \to X = S^\perp$ be any bounded solution of $\dot{x} = f_c(x) = (I_N \otimes A)x$ that satisfies

$$\nabla V(x(t)) f_\zeta(x(t)) = 0, \forall t \in \Re, \forall \zeta \in \Lambda'.$$

Since

$$\nabla V(u) f_\zeta(u) = 0 \Rightarrow h_\zeta\,(u) = 0 \; \forall \zeta \in \Lambda \; \forall u \in R^{N \times n} \quad (51)$$

we have $(\sqrt{L_\zeta} \otimes \sqrt{PBB^T P})\, x(t) = 0, \forall t \in \Re, \forall \zeta \in \Lambda'$. Hence

$$(L_\zeta \otimes I_k)(I_N \otimes PBB^T P)\, x(t) = (L_\zeta \otimes PBB^T P)\, x(t)$$
$$= (\sqrt{L_\zeta} \otimes \sqrt{PBB^T P})\, (\sqrt{L_\zeta} \otimes \sqrt{PBB^T P}) x(t)$$
$$= 0, \forall t \in \Re, \forall \zeta \in \Lambda'.$$

According to (48) in (C2), $(I_N \otimes PBB^T P)\, x(t) \in S$. Due to $x(t) \in S^\perp$, this results in

$$\left\|(I_N \otimes B^T P)\, x(t)\right\|^2 = x^T(t)(I_N \otimes PBB^T P)x(t)$$
$$= 0, \forall t \in \Re.$$

Therefore $(I_N \otimes B^T P)x(t) = 0, \forall t \in \Re$. Again by (51), the following holds:

$$\begin{bmatrix} I_N \otimes C^T & I_N \otimes PB \end{bmatrix}^T x(t) = 0, \forall t \in \Re. \quad (52)$$

In Lemma 6 (see the proof in Appendix F), it will be shown that $((C^T, PB)^T, A)$ is observable when (C1) holds. It is then concluded that $x \equiv 0$ in view of (52) and $\dot{x} = (I_N \otimes A)x$. This results in $\inf_{t \in \Re} \|x(t)\| = 0$, verifying that $(h^V_{\Lambda'}, f_c)$ is WZSD.

Lemma 6: Suppose (C1) holds. With $C = \sqrt{-PA - A^{\mathrm{T}} P}$, $((C^T, PB)^T, A)$ is observable.

By applying Theorem 6 and [18, Lemma 1], the system (46) is uniformly globally exponentially stable w.r.t. $\tilde{\Phi}$. Since $\Phi \subseteq \tilde{\Phi}$, it is also uniformly globally exponentially stable w.r.t. Φ. Theorem 8 summarizes the previous discussions.

Theorem 8: Consider the switched system (46). Let $S \subseteq \Re^{Nk}$ be a nonempty set, Φ be a set of solution pairs $(x, \lambda) : \Re_+ \to S^\perp \times \Lambda$. Suppose that (C1) and (C2) hold. Then, the origin is uniformly globally exponentially stable w.r.t. Φ. ■

Remark 20: In [36], $\Phi = \{(x, \lambda)\}$ was considered with two stronger conditions:

1) (A, B) is controllable and
2) $(B^T P, A)$ is observable.

Assumption (C1) only requires a weaker condition that (A, B) is stabilizable. Moreover, in order to ensure the UGAS condition, a dwell time condition is also needed in [36]. This, together with Assumption 3 of [36], implies (C2), see Remarks 14 and 15. Thus, Theorem 8 can show uniform global exponential stability (instead of attractivity) under weaker assumptions [(C1) and (C2)]. This demonstrates the power of the proposed framework in the stability analysis of switched systems. ■

Remark 21: When $S = \{0\}$, (48) just indicates that $\sum_{\zeta \in \Lambda'} L_\zeta$ is nonsingular. Or equivalently, every node is reachable from node 0 in the union graph [28]. Since $S^\perp = \Re^{Nk}$, the constraint $x(t) \in S^\perp$ naturally holds. The obtained result is thus applicable to a leader-following consensus problem, see [36, (ii) of Th. 1].

Remark 22: Consider another case as $S = \{\,(1_N \otimes I_k)\, \upsilon \mid \upsilon \in \Re^k\}$. Under such a situation, (48) is equivalent to the following statement:

$$\left(\sum_{\zeta \in \Lambda'} L_\zeta\right) w = 0, w \in R^N \;\Rightarrow\; w = \theta 1_N \text{ for some } \theta \in \Re.$$

This condition can be interpreted as the union graph $\zeta_{\Lambda'}$ is connected where the edge set $E_{\Lambda'}$ of the graph $\zeta_{\Lambda'}$ is defined as $E_{\Lambda'} = \bigcup_{\zeta\in\Lambda'} E(\zeta)$ [28]. This shows that Theorem 8 is also applicable to a leaderless consensus problem, see [36, (i) of Th. 1] and [41]. ■

VI. Conclusion

WZSD plays an important role in checking UGAS. This paper introduced the concept of CZOS in order to facilitate checking WZSD of switched NLTV systems. This in turn led to a generalized Krasovskii–LaSalle theorem to infer UGAS. When reduced to switched NLTI systems, the needed detectability condition was further simplified, leading to several new stability results. For switched LTI systems, a systematic way to generate a CZOS was proposed. The effectiveness of the ideas was shown in the example of two consensus problems. Future work will focus on developing analysis guidelines in the context of particular classes of systems, to further the ease with which these tools can be utilized. It is intended that these guidelines will allow the inference of design guidelines for the construction of switched systems.

Appendix A
Proof of Theorem 1

In the following, the Contradiction method is used to prove Theorem 1. Suppose (h, f) is not WZSD w. r. t. Φ. Then, there exist a constant $0 < \varepsilon_0 < 1$, a time sequence $\{t_n\} \subseteq \Re_+$ and a sequence $\{(x_n, \lambda_n)\} \subseteq \Phi$ such that for each $n \in \aleph$, $t_n \geq t_0(x_n) + 2n$, $\varepsilon_0 \leq \|x_n(t+t_n)\| \leq 1/\varepsilon_0, \forall -n \leq t \leq n$, and

$$\lim_{n\to\infty} h(t+t_n, x_n(t+t_n), \lambda_n(t+t_n)) = 0 \qquad \text{(A1)}$$

for almost all t in $\Re$. Particularly, the family $\{x_n(\cdot + t_n) : [-n, n] \to \Re^p\}$ is uniformly bounded and equicontinuous as Λ is finite and $f_\zeta, \forall\zeta \in \Lambda$, are almost uniformly bounded. By Arzela–Ascoli lemma, there exists a subsequence $\{x_{n_k}(\cdot + t_{n_k})\}$ of $\{x_n(\cdot + t_n)\}$ converging uniformly to a continuous function $\bar{x} : \Re \to X$ on every compact subset of $\Re$ [29]. Since f_c is asymptotically almost periodic, we may assume that $\{f_c(\cdot + t_{n_k}, \cdot)\}$ also converges uniformly to a limiting function $\bar{f}_c(\cdot, \cdot)$ on every compact subset of $\Re \times X$ w. r. t. $\{t_{n_k}\}$, by taking a suitable subsequence. Employing (A1), Assumption 3 and the fact that Λ is finite, the following equation holds:

$$\lim_{k\to\infty} [f\ (t+t_{n_k}, x_{n_k}(t+t_{n_k}), \lambda_{n_k}(t+t_{n_k})) - f_c(t+t_{n_k}, x_{n_k}(t+t_{n_k}))] = 0 \qquad \text{(A2)}$$

for almost all t in $\Re$. Now the following equations can be derived:

$$\begin{aligned}\bar{x}(t) - \bar{x}(s) &= \lim_{k\to\infty}\ [x_{n_k}(t+t_{n_k}) - x_{n_k}(s+t_{n_k})]\\ &= \int_s^t \lim_{k\to\infty} [f(\tau+t_{n_k}, x_{n_k}(\tau+t_{n_k}), \lambda_{n_k}(\tau+t_{n_k}))\\ &\quad - f_c(\tau+t_{n_k}, x_{n_k}(\tau+t_{n_k}))]\, d\tau\\ &\quad + \int_s^t \lim_{k\to\infty}\ [f_c(\tau+t_{n_k}, x_{n_k}(\tau+t_{n_k}))\\ &\quad - \bar{f}_c(\tau, x_{n_k}(\tau+t_{n_k}))]\ d\tau\\ &\quad + \int_s^t \lim_{k\to\infty}\ [\bar{f}_c(\tau, x_{n_k}(\tau+t_{n_k}))\\ &\quad - \bar{f}_c(\tau, \bar{x}(\tau))]\, d\tau + \int_s^t \bar{f}_c(\tau, \bar{x}(\tau))\ d\tau\\ &= \int_s^t \bar{f}_c(\tau, \bar{x}(\tau))\ d\tau, \forall t \geq s,\end{aligned}$$

where the second equality used the Lebesgue dominance theorem [30, Th. 11.32] and the almost uniformly bounded property of f_ζ and f_c [17], while the last equality used (A2), the definition of limiting functions and continuity of $\bar{f}_c$. Therefore $\bar{x}$ is a solution of $\dot{\bar{x}} = \bar{f}_c(t, \bar{x})$. Since $\varepsilon_0 \leq \|x_n(t+t_n)\| \leq 1/\varepsilon_0, \forall -n \leq t \leq n$, we have $\varepsilon_0 \leq \|\bar{x}(t)\| \leq 1/\varepsilon_0$, for all t in $\Re$. Particularly, $\bar{x}$ is bounded. Since $h_\zeta(t, x), \forall\zeta \in \Lambda$, are continuous in x, uniformly in t, it has

$$\begin{aligned}&\lim_{k\to\infty} h(t+t_{n_k}, \bar{x}(t), \lambda_{n_k}(t+t_{n_k}))\\ &= \lim_{k\to\infty} h(t+t_{n_k}, x_{n_k}(t+t_{n_k}), \lambda_{n_k}(t+t_{n_k})) = 0\end{aligned}$$

for almost all t in $\Re$ based on (A1). According to Assumption 4, the following contradiction appears:

$$\varepsilon_0 \leq \inf_{t\in\Re} \|\bar{x}(t)\| = 0.$$

This shows that (h, f) is WZSD w. r. t. Φ. The proof of the theorem is then completed. ■

Appendix B
Proof of Theorem 6

In view of Theorem 3, it remains to show Assumption (I3). Let $x : \Re \to X$ be any bounded solution of $\dot{x} = f_c(x)$ and $\{\lambda_n\} \subseteq \Phi^{sw}$ and $t_n \to \infty$ be two sequences such that (22) in (I3) holds. By (I6), for all sufficiently large $n \in \aleph$, there exists $\Lambda_n \subseteq \Lambda$ such that $\lambda_n \in \Gamma(\Lambda_n, T_0, \tau_0, t_n)$ and $(h^V_{\Lambda_n}, f_c)$ is WZSD. Since Λ is finite, its power set is finite. Thus, there exist $\Lambda' \subseteq \Lambda$ and infinitely many $n \in \aleph$ such that $t_n \geq T_0$ and $\Lambda_n = \Lambda'$. Under (22) in (I3), for each $\zeta \in \Lambda'$, it is claimed that there is an infinite set $\Omega_\zeta \subseteq [-T_0, T_0]$ such that

$$\nabla V(x(t)) f_\zeta(x(t)) = 0, \forall t \in \Omega_\zeta. \qquad \text{(A3)}$$

If the claim is false, by continuity of x, there exists $\varepsilon_0 > 0$ such that $m(E_\zeta) \leq \tau_0/2$ where

$$E_\zeta = \{\, t \in [-T_0, T_0] \,|\, |\nabla V(x(t)) f_\zeta(x(t))| < \varepsilon_0\}\,.$$

In view of (22) and (32) in Definition 7, with $\Omega^n_\zeta = \{\, t \in [-T_0, T_0] \,|\, \lambda_n(t+t_n) = \zeta\}$, the following inequality holds:

$$0=\int_{-T_0}^{T_0}\lim_{n\to\infty}\left|\nabla V(x(t))f_{\lambda_n(t+t_n)}(x(t))\right|dt$$

$$=\limsup_{n\to\infty}\int_{-T_0}^{T_0}\left|\nabla V(x(t))f_{\lambda_n(t+t_n)}(x(t))\right|dt$$

$$\geq\limsup_{n\to\infty}\int_{E_\zeta^c\cap\Omega_\zeta^n}|\nabla V(x(t))f_\zeta(x(t))|\,dt$$

$$\geq\varepsilon_0\limsup_{n\to\infty}m(E_\zeta^c\cap\Omega_\zeta^n)\geq\varepsilon_0(\tau_0-m(E_\zeta))$$

$$\geq\varepsilon_0\tau_0/2>0,$$

by using the Lebesgue dominance theorem and the fact that $m(\Omega_\zeta^n)\geq\tau_0$ for infinitely many $n\in\aleph$. A contradiction is reached. By the claim [see (A3)], (I4) and the principle of permanence (Lemma 3)

$$\nabla V(x(t))f_\zeta(x(t))=0\ \forall t\in\Re,\forall\zeta\in\Lambda'.$$

Hence $h_{\Lambda'}^V(x(t))\equiv 0$. This implies $\inf_{t\in\Re}\|x(t)\|=0$ by WZSD of $(h_{\Lambda'}^V,f_c)$. So (I3) holds. This completes the proof of the theorem. ∎

Appendix C
Proof of Lemma 4

For each $\zeta\in\Lambda$, let $h_\zeta=(x^T\otimes I_N)C_\zeta x$ be a virtual output and $V=x^TQx$ be a "common" Lyapunov like function. By direct computation, $\dot V_\zeta=2x^TQA_\zeta x$. It can be checked that h_ζ, V and $\dot V_\zeta$ are all almost uniformly bounded. Moreover, (39) implies that for each $\zeta\in\Lambda$, the pair $(h_\zeta,\dot V_\zeta)=((x^T\otimes I_N)C_\zeta x,2x^TQA_\zeta x)$ is a nonpositive pair [19], [20]. All the assumptions of [20, Th. 6] are satisfied. So the OPE condition of $((x^T\otimes I_N)C_\zeta x,Ax)$ is concluded by the OPE condition of $((x^T\otimes I_{(N+1)})\tilde C_\zeta x,Ax)$ where the fact

$$(h_\zeta,\dot V_\zeta)=((u^T\otimes I_N)C_\zeta u,\ 2u^TQA_\zeta u)$$
$$=0\Leftrightarrow(u^T\otimes I_{(N+1)})\tilde C_\zeta u=0$$

and the tool of changing output [20, Th. 5] were applied. This completes the proof of the lemma. ∎

Appendix D
Proof of Theorem 7

For the case of arbitrary switching, Φ is the set of all solution pairs. Since all modes are linear, Φ is scaling invariant [18]. In view of [18, Lemma 1], we only need to show that the origin is UGAS.

First, Theorem 1 is used to show WZSD of $(h_\zeta^N(x),A_\zeta x)$ with $X=\Re^p$. Notice that the system function $A_\zeta x$ is almost uniformly bounded and the output function $h_\zeta^N(x)$ is continuous in *x*, uniformly in *t*. Moreover, $h_\zeta^N(x)$ satisfies Assumption 1. Since $f_c=A_cx$ is continuous and time-invariant, it is an asymptotically almost periodic function. Based on Lemma 2, Assumption 3 can be deduced by (41) in (S2). Using a similar argument as in Theorem 5, it is possible to show that (S2) and (S3) result in Assumption 4.

Indeed, every limiting function $\bar f_c$ is equal to $f_c=A_cx$. Due to $h_\zeta^N(x)$ being time-invariant, (11) becomes

$$\lim_{n\to\infty}h_{\lambda_n(t+t_n)}^N(x(t))=0 \tag{A4}$$

for almost all *t* in $\Re$ where $x:\Re\to\Re^p$ is a bounded solution of $\dot x=A_cx$, $\{\lambda_n:\Re_+\to\Lambda\}\subseteq\Phi^{sw}$ and $t_n\to\infty$. As in the proof of Theorem 4, consider the following common output:

$$h_c=\prod\nolimits_{\zeta\in\Lambda}h_\zeta^N(u)\ \forall u\in\Re^p.$$

Then, (A4) implies $h_c(x(t))\equiv 0$ [29], [30]. Following the same argument around (31), there exist $\zeta_0\in\Lambda$ and a sequence $t_n\to 0$, with $t_n\neq 0$ and such that

$$h_{\zeta_0}^N(x(t_n))=0\ \forall n\in\aleph.$$

By the definition of $h_{\zeta_0}^N$ in (42), $h_{\zeta_0}^N(x(t))=h_{\zeta_0}^N(e^{A_ct}x(0))$ is an analytic function. Hence it is a zero function according to Lemma 3. In view of (41) in (S2), $x|_{\Re_+}$ is a bounded solution of $\dot x=A_cx=A_{\zeta_0}x$ with $h_{\zeta_0}^N(x(t))=0,\forall t\in\Re_+$. From (S3), we deduce that $\lim_{t\to\infty}x(t)=0$. This turns to imply $\inf_{t\in\Re}\|x(t)\|=0$. Thus, Assumption 4 holds. Therefore $(h_\zeta^N(x),A_\zeta x)$ is WZSD based on Theorem 1.

By (S1), the origin is UGS and Assumption 2 holds with $h_\zeta=\sqrt{-x^T(PA_\zeta+A_\zeta^TP)x}$ [18]. Moreover, $h_\zeta(x)$ satisfies Assumption 1. By Proposition 1, it remains to check the OPE condition of $(h_\zeta(x),A_\zeta x)$. According to Lemma 1, Lemma 4, (S2), and [20, Th. 5], the following implications hold:

$(h_\zeta^N(x),A_\zeta x)$: WZSD $\Rightarrow(h_\zeta^N(x),A_\zeta x)$: OPE $\Rightarrow(h_\zeta^{N-1}(x),$
$A_\zeta x)$: OPE $\Rightarrow\cdots\Rightarrow$
$(h_\zeta^1(x),A_\zeta x)$: OPE $\Rightarrow(h_\zeta(x),A_\zeta x)$: OPE,

where the first implication used Lemma 1, the last implication used $P_1=P$ and the tool of changing output [20, Th. 5], and the other implications used (S2) and Lemma 4. By Proposition 1, the origin is UGAS. This completes the proof of the theorem. ∎

Appendix E
Proof of Lemma 5

The "only if" part is trivial. Let us show the "if" part. Consider the vector spaces W_1+W_2 and $[W_1+W_2]^\perp$. By the Gram-Schmidt process, it can be shown that $\Re^p=U+U^\perp$ for each subspace *U* of $\Re^p$. Particularly,

$$\Re^p=(W_1+W_2)+[W_1+W_2]^\perp.$$

Define a linear transformation $T_c:\Re^p\to\Re^p$ as

$$T_c(u+v+w)=A_1u+A_2v\ \forall u\in W_1\ \forall v$$
$$\in W_2\ \forall w\in[W_1+W_2]^\perp.$$

It is easy to check that T_c is a linear transformation. If $A_1\hat u=A_2\hat u,\forall\hat u\in W_1\cap W_2$, let us show that T_c is well defined. Suppose $u+v+w=u'+v'+w'$ for some

$$u,u'\in W_1,v,v'\in W_2,w,w'\in[W_1+W_2]^\perp.$$

Then

$$(u - u') + (v - v') = w' - w \in (W_1 + W_2) \cap [W_1 + W_2]^\perp = \{0\}$$

Thus, $w = w'$ and $u + v = u' + v'$. The latter implies $u - u' = v' - v \in W_1 \cap W_2$. Hence, $A_1(u - u') = A_2(v' - v)$. Particularly, $A_1 u + A_2 v = A_1 u' + A_2 v'$. This shows that T_c is well-defined and there is a unique $p \times p$ matrix A_c such that $T_c(u) = A_c u, \forall u \in \Re^p$. Moreover,

$$A_c u = T_c(u) = A_1 u, \forall u \in W_1,$$
$$A_c v = T_c(v) = A_2 v, \forall v \in W_2.$$

This completes the proof of the lemma. ■

Appendix F
Proof of Lemma 6

To check observability, the well-known PBH test is used [11]. If the system matrix $[zI_k - A^T \quad (C^T, PB)]^T$ has an invariant zero, say $z = \sigma$, we will find a contradiction. Indeed, in this case, there is a nonzero complex valued vector v such that $Cv = 0$, $B^T P v = 0$ and $Av = \sigma v$. By the definition of C, $(PA + A^T P)v = 0$. Then

$$0 = v^*(PA + A^T P)v = (\sigma + \bar{\sigma})v^* P v$$

where $v^* = \bar{v}^T$ and $\bar{v}$ is the conjugate vector of v. Since P is positive definite, $\mathrm{Re}(\sigma) = (\sigma + \bar{\sigma})/2 = 0$. Notice that $w = Pv \neq 0, w^T B = (B^T P v)^T = 0$ and

$$w^T A = (A^T P v)^T = -(P(Av))^T = -\sigma\, w^T.$$

Hence $z = -\sigma$, with the real part being zero, is an invariant zero of the system matrix $[zI_k - A \;\; B]$. Since (A, B) is stabilizable, a contradiction is reached according to the PBH test [11]. Therefore $[zI_k - A^T \quad (C^T, PB)]^T$ has no invariant zeros. Again by the PBH test, $((C^T, PB)^T, A)$ is observable. This completes the proof of the lemma. ■

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