Exactly realizable desired trajectories

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Abstract

Trajectory tracking of nonlinear dynamical systems with affine open-loop controls is investigated. The control task is to enforce the system state to follow a prescribed desired trajectory as closely as possible. We introduce exactly realizable desired trajectories as these trajectories which can be tracked exactly by an appropriate control. Exactly realizable trajectories are characterized mathematically by means of Moore-Penrose projectors constructed from the input matrix. The approach leads to differential-algebraic systems of equations and is considerably simpler than the related concept of system inversion. Furthermore, we identify a particularly simple class of nonlinear affine control systems. Systems in this class satisfy the so-called linearizing assumption and share many properties with linear control systems. For example, conditions for controllability can be formulated in terms of a rank condition for a controllability matrix analogously to the Kalman rank condition for linear time-invariant systems.

1. Introduction

A common approach to control is concerned with states as the object to be controlled [1, 2]. Suppose a controlled system, often called a plant in this context, has a certain point $\boldsymbol{x}_1$ in state space, sometimes called the operating point, at which the system works efficiently. The control task is then to bring the system to the operating point $\boldsymbol{x}_1$, and keep it there.

In contrast to that, here we develop an approach to control which centers on the state trajectory over time, $\boldsymbol{x}(t)$, as the object of interest. We distinguish between the controlled state trajectory $\boldsymbol{x}(t)$ and the desired trajectory $\boldsymbol{x}_d(t)$. The former is the trajectory which the time-dependent state $\boldsymbol{x}(t)$ traces out in state space under the action of a control signal, also called an input signal. The latter is a fictitious reference trajectory for the state over time. It is prescribed in analytical or numerical form by the experimenter. Depending on the choice of the desired trajectory $\boldsymbol{x}_d(t)$, the controlled state $\boldsymbol{x}(t)$ may or may not exactly follow $\boldsymbol{x}_d(t)$.

Of course, both approaches to control are closely related. A single operating point in state space at which the system is to be kept is nothing more than a degenerate state trajectory. Equivalently, any state trajectory can be approximated by a succession of working points.

Trajectory tracking aims at enforcing, via a control signal $\boldsymbol{u}(t)$, a system state $\boldsymbol{x}(t)$ to follow a prescribed desired trajectory $\boldsymbol{x}_d(t)$ as closely as possible within a time interval $t_0 \le t \le t_1$. The distance between $\boldsymbol{x}(t)$ and $\boldsymbol{x}_d(t)$ in function space can be measured by the functional

$$\mathcal{J} = \frac{1}{2}\int_{t_0}^{t_1} \mathrm{d}t \left(\boldsymbol{x}(t) - \boldsymbol{x}_d(t)\right)^2. \tag{1}$$

The smallest possible value $\mathcal{J} = 0$ is attained if and only if the state $\boldsymbol{x}(t)$ follows the desired trajectory exactly, i.e.,

$$\boldsymbol{x}(t) = \boldsymbol{x}_d(t) \tag{2}$$

for all times $t_0 \le t \le t_1$. We call a desired trajectory $\boldsymbol{x}_d(t)$ for which Eq. (2) holds an *exactly realizable desired trajectory*. Clearly, not every desired trajectory $\boldsymbol{x}_d(t)$ can be exactly realized. The question addressed in this article is how, for a given affine control system, exactly realizable desired trajectories can be characterized mathematically.

Tracking and regulation of desired outputs are common problems in applications and have a long history of research. The linear quadratic regulator [3] is a cornerstone of control theory. Further notable achievements are the solution of the linear time-invariant (LTI) regulator problem by Francis [4] and its generalization to nonlinear systems, the Byrnes-Isidori regulator [5]. These regulators track desired trajectories asymptotically and can deal with external disturbances and perturbations of initial conditions. In contrast to that, here we consider the exact tracking of desired trajectories in undisturbed systems by open-loop control.

A concept closely related to our work is that of an inversion of control systems. The idea there is to find a second controlled dynamical system which takes the desired output of the original system as the input and outputs the input of the original system. A control system is invertible when the corresponding input-output map is injective. The resulting control signal is open-loop and often referred to as feed-forward control. Stabilization of potential instabilities can be accomplished by an additional feedback control. Early work investigated the invertibility of LTI systems [6, 7, 8]. Hirschorn analyzed invertibility of nonlinear systems for single [9] and multivariable [10] input signals. Inversion is commonly investigated by generating and analyzing a hierarchy of auxiliary dynamical systems. Newer works focus on the stability of inversion-based output tracking by combining system inversion with feedback [11, 12].

The formalism necessary for the mathematical characterization of realizable trajectories is introduced in Section 2. Section 3 defines the notion of exactly realizable trajectories while Section 4 introduces output realizability. Section 5 proposes a basic assumption, called the linearizing assumption. This assumption defines a class of nonlinear control systems which, to a large extent, behave like linear systems. We demonstrate how controllability can be recovered in our approach for systems satisfying the linearizing assumption in Sections 6 and 7. Section 8 concludes with a discussion and outlook.

2. Formalism

Consider the affine control system for the state $\boldsymbol{x} \in \mathbb{R}^n$ with output $\boldsymbol{y} \in \mathbb{R}^m$ and control signal $\boldsymbol{u} \in \mathbb{R}^p$,

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{R}(\boldsymbol{x}(t)) + \boldsymbol{\mathcal{B}}(\boldsymbol{x}(t))\,\boldsymbol{u}(t), \quad \boldsymbol{y}(t) = \boldsymbol{h}(\boldsymbol{x}(t)). \tag{3}$$

The time derivative is denoted by $\dot{\boldsymbol{x}}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x}(t)$. The dynamical system (3) is supplemented with the initial condition $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$. The $n \times p$ input matrix $\boldsymbol{\mathcal{B}}(\boldsymbol{x})$ may be state dependent and is assumed to have full rank,

$$\operatorname{rank}(\boldsymbol{\mathcal{B}}(\boldsymbol{x})) = p, \tag{4}$$

for all $\boldsymbol{x} \in \mathbb{R}^n$. The main elements of the formalism introduced below are two complementary projection matrices defined in terms of the input matrix $\boldsymbol{\mathcal{B}}(\boldsymbol{x})$.

Definition 1. The Moore-Penrose pseudo inverse [13] of $\boldsymbol{\mathcal{B}}(\boldsymbol{x})$, denoted by $\boldsymbol{\mathcal{B}}^+(\boldsymbol{x})$, is defined as the $p \times n$ matrix

$$\boldsymbol{\mathcal{B}}^+(\boldsymbol{x}) = \left(\boldsymbol{\mathcal{B}}^T(\boldsymbol{x})\,\boldsymbol{\mathcal{B}}(\boldsymbol{x})\right)^{-1}\boldsymbol{\mathcal{B}}^T(\boldsymbol{x}). \tag{5}$$

The Moore-Penrose projectors $\boldsymbol{\mathcal{P}}(\boldsymbol{x})$ and $\boldsymbol{\mathcal{Q}}(\boldsymbol{x})$ are $n \times n$ matrices defined as

$$\boldsymbol{\mathcal{P}}(\boldsymbol{x}) = \boldsymbol{\mathcal{B}}(\boldsymbol{x})\,\boldsymbol{\mathcal{B}}^+(\boldsymbol{x}), \qquad \boldsymbol{\mathcal{Q}}(\boldsymbol{x}) = \mathbf{1} - \boldsymbol{\mathcal{P}}(\boldsymbol{x}). \tag{6}$$

Remark 1. Note that the $p \times p$ matrix $\boldsymbol{\mathcal{B}}^T(\boldsymbol{x})\,\boldsymbol{\mathcal{B}}(\boldsymbol{x})$ has full rank p because $\boldsymbol{\mathcal{B}}(\boldsymbol{x})$ is assumed to have full rank. Therefore, $\boldsymbol{\mathcal{B}}^T(\boldsymbol{x})\,\boldsymbol{\mathcal{B}}(\boldsymbol{x})$ is a quadratic symmetric non-singular matrix and its inverse exists. From the definitions (6) follow idempotence

$$\boldsymbol{\mathcal{P}}(\boldsymbol{x})\,\boldsymbol{\mathcal{P}}(\boldsymbol{x}) = \boldsymbol{\mathcal{P}}(\boldsymbol{x}), \qquad \boldsymbol{\mathcal{Q}}(\boldsymbol{x})\,\boldsymbol{\mathcal{Q}}(\boldsymbol{x}) = \boldsymbol{\mathcal{Q}}(\boldsymbol{x}), \tag{7}$$

and complementarity

$$\boldsymbol{\mathcal{Q}}(\boldsymbol{x})\,\boldsymbol{\mathcal{P}}(\boldsymbol{x}) = \boldsymbol{\mathcal{P}}(\boldsymbol{x})\,\boldsymbol{\mathcal{Q}}(\boldsymbol{x}) = \mathbf{0}. \tag{8}$$

The projectors are symmetric,

$$\boldsymbol{\mathcal{P}}^T(\boldsymbol{x}) = \boldsymbol{\mathcal{P}}(\boldsymbol{x}), \qquad \boldsymbol{\mathcal{Q}}^T(\boldsymbol{x}) = \boldsymbol{\mathcal{Q}}(\boldsymbol{x}), \tag{9}$$

and their ranks are

$$\operatorname{rank}(\boldsymbol{\mathcal{P}}(\boldsymbol{x})) = p, \qquad \operatorname{rank}(\boldsymbol{\mathcal{Q}}(\boldsymbol{x})) = n - p, \tag{10}$$

independent of $\boldsymbol{x}$. Furthermore, multiplying $\boldsymbol{\mathcal{P}}(\boldsymbol{x})$ and $\boldsymbol{\mathcal{Q}}(\boldsymbol{x})$ from the right with the input matrix $\boldsymbol{\mathcal{B}}(\boldsymbol{x})$ yields the important relations

$$\boldsymbol{\mathcal{P}}(\boldsymbol{x})\,\boldsymbol{\mathcal{B}}(\boldsymbol{x}) = \boldsymbol{\mathcal{B}}(\boldsymbol{x}), \qquad \boldsymbol{\mathcal{Q}}(\boldsymbol{x})\,\boldsymbol{\mathcal{B}}(\boldsymbol{x}) = \mathbf{0}. \tag{11}$$

Equation (11) shows that the p linearly independent columns of $\boldsymbol{\mathcal{B}}(\boldsymbol{x})$ are eigenvectors of $\boldsymbol{\mathcal{P}}(\boldsymbol{x})$ to eigenvalue one and eigenvectors of $\boldsymbol{\mathcal{Q}}(\boldsymbol{x})$ to eigenvalue zero. Alternatively, due to the idempotence of the projectors, the p eigenvectors of $\boldsymbol{\mathcal{P}}(\boldsymbol{x})$ to eigenvalue one are given by the $j \in \{1, \dots, p\}$ linearly independent columns $\boldsymbol{p}_j(\boldsymbol{x})$ of $\boldsymbol{\mathcal{P}}(\boldsymbol{x})$. The remaining $n - p$ eigenvectors are given by the $i \in \{1, \dots, n-p\}$ linearly independent columns $\boldsymbol{q}_i(\boldsymbol{x})$ of $\boldsymbol{\mathcal{Q}}(\boldsymbol{x})$.

Remark 2. Using $\mathbf{1} = \boldsymbol{\mathcal{P}}(\boldsymbol{x}) + \boldsymbol{\mathcal{Q}}(\boldsymbol{x})$, any state vector $\boldsymbol{x}$ can be split up as

$$\boldsymbol{x} = \boldsymbol{\mathcal{P}}(\boldsymbol{x})\,\boldsymbol{x} + \boldsymbol{\mathcal{Q}}(\boldsymbol{x})\,\boldsymbol{x} = \boldsymbol{v} + \boldsymbol{w}. \tag{12}$$

Because $\boldsymbol{\mathcal{P}}(\boldsymbol{x})$ has rank p, only $p \leq n$ of the n components of $\boldsymbol{v} = \boldsymbol{\mathcal{P}}(\boldsymbol{x})\,\boldsymbol{x}$ are independent. Similarly, only $n-p$ components of $\boldsymbol{w} = \boldsymbol{\mathcal{Q}}(\boldsymbol{x})\,\boldsymbol{x}$ are independent. p independent components $\hat{\boldsymbol{v}} \in \mathbb{R}^p$ of $\boldsymbol{v}$ and $n - p$ independent components $\hat{\boldsymbol{w}} \in \mathbb{R}^{n-p}$ of $\boldsymbol{w}$ can be obtained as $\hat{\boldsymbol{v}} = \hat{\boldsymbol{\mathcal{P}}}^T(\boldsymbol{x})\,\boldsymbol{x}$ and $\hat{\boldsymbol{w}} = \hat{\boldsymbol{\mathcal{Q}}}^T(\boldsymbol{x})\,\boldsymbol{x}$, respectively. Here, the $n \times p$ matrix $\hat{\boldsymbol{\mathcal{P}}}(\boldsymbol{x})$ and the $n \times (n-p)$ matrix $\hat{\boldsymbol{\mathcal{Q}}}(\boldsymbol{x})$ are constructed from the linearly independent columns $\boldsymbol{p}_i(\boldsymbol{x})$ of $\boldsymbol{\mathcal{P}}(\boldsymbol{x})$ and $\boldsymbol{q}_i(\boldsymbol{x})$ of $\boldsymbol{\mathcal{Q}}(\boldsymbol{x})$ as

$$\hat{\boldsymbol{\mathcal{P}}}(\boldsymbol{x}) = \left(\boldsymbol{p}_1(\boldsymbol{x})\,|\dots|\boldsymbol{p}_p(\boldsymbol{x})\right), \tag{13}$$

$$\hat{\boldsymbol{\mathcal{Q}}}(\boldsymbol{x}) = \left(\boldsymbol{q}_1(\boldsymbol{x})\,|\dots|\boldsymbol{q}_{n-p}(\boldsymbol{x})\right). \tag{14}$$

If the projectors $\boldsymbol{\mathcal{P}}(\boldsymbol{x}) = \boldsymbol{\mathcal{P}}$ and $\boldsymbol{\mathcal{Q}}(\boldsymbol{x}) = \boldsymbol{\mathcal{Q}}$ are independent of the state $\boldsymbol{x}$, the vectors $\boldsymbol{v} = \boldsymbol{\mathcal{P}}\boldsymbol{x}$ and $\boldsymbol{w} = \boldsymbol{\mathcal{Q}}\boldsymbol{x}$ are simply linear combinations of the original state components $\boldsymbol{x}$. If $\boldsymbol{\mathcal{P}}(\boldsymbol{x})$ and therefore $\boldsymbol{\mathcal{Q}}(\boldsymbol{x}) = \mathbf{1} - \boldsymbol{\mathcal{P}}(\boldsymbol{x})$ depends on $\boldsymbol{x}$, both vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ are nonlinear functions of the state $\boldsymbol{x}$. However, a state transformation can be found such that $\boldsymbol{v}$ and $\boldsymbol{w}$ attain a particularly simple form. Being a projector, $\boldsymbol{\mathcal{Q}}(\boldsymbol{x})$ can be diagonalized by a nonsingular $n \times n$ matrix $\boldsymbol{\mathcal{T}}(\boldsymbol{x})$, resulting in a diagonal $n \times n$ matrix $\boldsymbol{\mathcal{Q}}_D$,

$$\boldsymbol{\mathcal{Q}}_D = \boldsymbol{\mathcal{T}}^{-1}(\boldsymbol{x})\,\boldsymbol{\mathcal{Q}}(\boldsymbol{x})\,\boldsymbol{\mathcal{T}}(\boldsymbol{x}). \tag{15}$$

The first p entries on the diagonal of $\boldsymbol{\mathcal{Q}}_D$ are zero while the remaining $n-p$ entries on the diagonal of $\boldsymbol{\mathcal{Q}}_D$ are one. The same matrix $\boldsymbol{\mathcal{T}}(\boldsymbol{x})$ diagonalizes the projector $\boldsymbol{\mathcal{P}}(\boldsymbol{x})$ as well. Defining the transformed state vector $\tilde{\boldsymbol{x}}$ as

$$\tilde{\boldsymbol{x}} = \boldsymbol{\mathcal{T}}^{-1}(\boldsymbol{x})\,\boldsymbol{x}, \tag{16}$$

the separation of the state can be seen to attain the particularly simple form

$$\begin{aligned}\tilde{\boldsymbol{x}} &= \boldsymbol{\mathcal{T}}^{-1}(\boldsymbol{x})\,\boldsymbol{\mathcal{P}}(\boldsymbol{x})\,\boldsymbol{x} + \boldsymbol{\mathcal{T}}^{-1}(\boldsymbol{x})\,\boldsymbol{\mathcal{Q}}(\boldsymbol{x})\,\boldsymbol{x} \\ &= \boldsymbol{\mathcal{P}}_D\tilde{\boldsymbol{x}} + \boldsymbol{\mathcal{Q}}_D\tilde{\boldsymbol{x}} = \tilde{\boldsymbol{v}} + \tilde{\boldsymbol{w}}\end{aligned} \tag{17}$$

with

$$\tilde{\boldsymbol{v}} = \boldsymbol{\mathcal{P}}_D \tilde{\boldsymbol{x}} = \left(\tilde{x}_1, \ldots, \tilde{x}_p, 0, \ldots, 0\right)^T, \tag{18}$$

$$\tilde{\boldsymbol{w}} = \boldsymbol{\mathcal{Q}}_D \tilde{\boldsymbol{x}} = \left(0, \ldots, 0, \tilde{x}_{p+1}, \ldots, \tilde{x}_n\right)^T. \tag{19}$$

The matrix $\boldsymbol{\mathcal{T}}(\boldsymbol{x})$ can be constructed from the eigenvectors of $\boldsymbol{\mathcal{Q}}(\boldsymbol{x})$ in the usual manner. The state transformation $\boldsymbol{\mathcal{T}}^{-1}(\boldsymbol{x})$ leads to a new affine control system for $\tilde{\boldsymbol{x}}$. This new system may be viewed as a normal form of the affine control system (3) [14].

In the following theorem, we use the Moore-Penrose projectors to separate the controlled state equation in two equations. The first one provides a relation for the control signal, while the second equation is independent of the control.

Theorem 1. *Every affinely controlled state equation (3) can be split in two separate equations. The first equation*

$$\boldsymbol{\mathcal{Q}}(\boldsymbol{x})\left(\dot{\boldsymbol{x}} - \boldsymbol{R}(\boldsymbol{x})\right) = \boldsymbol{0}, \tag{20}$$

is independent of the control signal $\boldsymbol{u}$ and is called the constraint equation. The second equation yields an expression for the control $\boldsymbol{u}$ in terms of the controlled state $\boldsymbol{x}$,

$$\boldsymbol{u} = \boldsymbol{\mathcal{B}}^+(\boldsymbol{x})\left(\dot{\boldsymbol{x}} - \boldsymbol{R}(\boldsymbol{x})\right). \tag{21}$$

Proof of Theorem 1. The controlled state equation (3) can be written as

$$\frac{d}{dt}\left(\boldsymbol{\mathcal{P}}(\boldsymbol{x})\boldsymbol{x} + \boldsymbol{\mathcal{Q}}(\boldsymbol{x})\boldsymbol{x}\right) = \left(\boldsymbol{\mathcal{P}}(\boldsymbol{x}) + \boldsymbol{\mathcal{Q}}(\boldsymbol{x})\right)\boldsymbol{R}(\boldsymbol{x}) + \left(\boldsymbol{\mathcal{P}}(\boldsymbol{x}) + \boldsymbol{\mathcal{Q}}(\boldsymbol{x})\right)\boldsymbol{\mathcal{B}}(\boldsymbol{x})\boldsymbol{u}. \tag{22}$$

Multiplying with $\boldsymbol{\mathcal{Q}}(\boldsymbol{x})$ from the left and using Eq. (11) yields Eq. (20). Multiplying Eq. (3) by $\boldsymbol{\mathcal{B}}^T(\boldsymbol{x})$ from the left yields

$$\boldsymbol{\mathcal{B}}^T(\boldsymbol{x})\dot{\boldsymbol{x}} = \boldsymbol{\mathcal{B}}^T(\boldsymbol{x})\boldsymbol{R}(\boldsymbol{x}) + \boldsymbol{\mathcal{B}}^T(\boldsymbol{x})\boldsymbol{\mathcal{B}}(\boldsymbol{x})\boldsymbol{u}. \tag{23}$$

Multiplying with $\left(\boldsymbol{\mathcal{B}}^T(\boldsymbol{x})\boldsymbol{\mathcal{B}}(\boldsymbol{x})\right)^{-1}$ from the left yields Eq. (21) for the control. □

A formalism based on Moore-Penrose projectors can be introduced for the output as well. We assume an output of the form

$$\boldsymbol{y}(t) = \boldsymbol{\mathcal{C}}(\boldsymbol{x}(t))\boldsymbol{x}(t) + \boldsymbol{a}(t). \tag{24}$$

The vector $\boldsymbol{a}(t)$ is independent of the state and can be absorbed in the function $\boldsymbol{y}(t)$. We drop $\boldsymbol{a}(t)$ in the following. The $m \times n$ matrix $\boldsymbol{\mathcal{C}}(\boldsymbol{x})$ with $m \leq n$ is assumed to have full rank for all $\boldsymbol{x}$,

$$\operatorname{rank}\left(\boldsymbol{\mathcal{C}}(\boldsymbol{x})\right) = m. \tag{25}$$

Definition 2. The Moore-Penrose pseudo inverse of $\boldsymbol{\mathcal{C}}(\boldsymbol{x})$, denoted by $\boldsymbol{\mathcal{C}}^+(\boldsymbol{x})$, is the $n \times m$ matrix

$$\boldsymbol{\mathcal{C}}^+(\boldsymbol{x}) = \boldsymbol{\mathcal{C}}^T(\boldsymbol{x})\left(\boldsymbol{\mathcal{C}}(\boldsymbol{x})\boldsymbol{\mathcal{C}}^T(\boldsymbol{x})\right)^{-1}. \tag{26}$$

The Moore-Penrose projectors $\boldsymbol{\mathcal{M}}(\boldsymbol{x})$ and $\boldsymbol{\mathcal{N}}(\boldsymbol{x})$ are $n \times n$ matrices defined by

$$\boldsymbol{\mathcal{M}}(\boldsymbol{x}) = \boldsymbol{\mathcal{C}}^+(\boldsymbol{x})\boldsymbol{\mathcal{C}}(\boldsymbol{x}), \quad \boldsymbol{\mathcal{N}}(\boldsymbol{x}) = \boldsymbol{1} - \boldsymbol{\mathcal{M}}(\boldsymbol{x}). \tag{27}$$

Remark 3. The ranks of $\boldsymbol{\mathcal{M}}(\boldsymbol{x})$ and $\boldsymbol{\mathcal{N}}(\boldsymbol{x})$ are

$$\operatorname{rank}\left(\boldsymbol{\mathcal{M}}(\boldsymbol{x})\right) = m, \qquad \operatorname{rank}\left(\boldsymbol{\mathcal{N}}(\boldsymbol{x})\right) = n - m, \tag{28}$$

and they satisfy

$$\boldsymbol{\mathcal{M}}(\boldsymbol{x})\boldsymbol{\mathcal{C}}^T(\boldsymbol{x}) = \boldsymbol{\mathcal{C}}^T(\boldsymbol{x}), \quad \boldsymbol{\mathcal{N}}(\boldsymbol{x})\boldsymbol{\mathcal{C}}^T(\boldsymbol{x}) = \boldsymbol{0}. \tag{29}$$

With the help of $\boldsymbol{\mathcal{M}}(\boldsymbol{x})$ and $\boldsymbol{\mathcal{N}}(\boldsymbol{x})$ the state vector $\boldsymbol{x}(t)$ can be split up as

$$\boldsymbol{x} = \boldsymbol{\mathcal{M}}(\boldsymbol{x})\boldsymbol{x} + \boldsymbol{\mathcal{N}}(\boldsymbol{x})\boldsymbol{x} = \boldsymbol{\mathcal{C}}^+(\boldsymbol{x})\boldsymbol{y} + \boldsymbol{\mathcal{N}}(\boldsymbol{x})\boldsymbol{x}. \tag{30}$$

Thus the part $\boldsymbol{\mathcal{M}}(\boldsymbol{x})\boldsymbol{x}$ can be expressed in terms of the output $\boldsymbol{y}$ while the part $\boldsymbol{\mathcal{N}}(\boldsymbol{x})\boldsymbol{x}$ remains undetermined.

3. Exactly realizable desired trajectories

Not every desired trajectory $\boldsymbol{x}_d$ can be realized by control. Here, we formulate a condition which has to be satisfied by a desired trajectory to be exactly realizable.

Theorem 2. *The controlled state trajectory $\boldsymbol{x}(t)$ follows the desired trajectory $\boldsymbol{x}_d(t)$ exactly,*

$$\boldsymbol{x}(t) = \boldsymbol{x}_d(t), \tag{31}$$

if and only if

1. *$\boldsymbol{x}_d(t)$ satisfies the constraint equation*
$$\boldsymbol{\mathcal{Q}}(\boldsymbol{x}_d(t))\left(\dot{\boldsymbol{x}}_d(t) - \boldsymbol{R}(\boldsymbol{x}_d(t))\right) = \boldsymbol{0}, \tag{32}$$
2. *the initial value $\boldsymbol{x}_d(t_0)$ equals the initial value $\boldsymbol{x}_0$ of the controlled state equation*
$$\boldsymbol{x}_d(t_0) = \boldsymbol{x}_0, \tag{33}$$
3. *the control signal enforcing $\boldsymbol{x}_d(t)$ is given by*
$$\boldsymbol{u}(t) = \boldsymbol{\mathcal{B}}^+(\boldsymbol{x}_d(t))\left(\dot{\boldsymbol{x}}_d(t) - \boldsymbol{R}(\boldsymbol{x}_d(t))\right). \tag{34}$$

Proof of Theorem 2. From Theorem 1 and $\boldsymbol{x}(t) = \boldsymbol{x}_d(t)$ follows the necessity of conditions (32), (33) and (34). For sufficiency, expression (34) for the control is used in the controlled state equation (3) to obtain

$$\dot{\boldsymbol{x}} = \boldsymbol{R}(\boldsymbol{x}) + \boldsymbol{\mathcal{B}}(\boldsymbol{x})\boldsymbol{\mathcal{B}}^+(\boldsymbol{x}_d)\left(\dot{\boldsymbol{x}}_d - \boldsymbol{R}(\boldsymbol{x}_d)\right). \tag{35}$$

Note that $\boldsymbol{\mathcal{B}}$ depends on the actual system state $\boldsymbol{x}$ while $\boldsymbol{\mathcal{B}}^+$ depends on the desired state $\boldsymbol{x}_d$. We introduce the difference $\Delta\boldsymbol{x}(t)$ between true and desired state as

$$\Delta\boldsymbol{x}(t) = \boldsymbol{x}(t) - \boldsymbol{x}_d(t). \tag{36}$$

Using Eq. (35), the ODE for $\Delta\boldsymbol{x}$ reads

$$\begin{aligned} \Delta\dot{\boldsymbol{x}} &= \boldsymbol{R}(\Delta\boldsymbol{x} + \boldsymbol{x}_d) - \dot{\boldsymbol{x}}_d \\ &\quad + \boldsymbol{\mathcal{B}}(\Delta\boldsymbol{x} + \boldsymbol{x}_d)\boldsymbol{\mathcal{B}}^+(\boldsymbol{x}_d)(\dot{\boldsymbol{x}}_d - \boldsymbol{R}(\boldsymbol{x}_d)), \end{aligned} \tag{37}$$

$$\Delta\boldsymbol{x}(t_0) = \boldsymbol{x}(t_0) - \boldsymbol{x}_d(t_0). \tag{38}$$

If the desired trajectory $\boldsymbol{x}_d$ satisfies initially Eq. (33), the initial condition for Eq. (37) vanishes, $\Delta\boldsymbol{x}(t_0) = \boldsymbol{0}$. If condition (32) is satisfied, then $\Delta\boldsymbol{x}(t) = \boldsymbol{0}$ is a stationary point of Eq. (37),

$$\begin{aligned} \Delta\dot{\boldsymbol{x}} &= \boldsymbol{R}(\boldsymbol{x}_d) - \dot{\boldsymbol{x}}_d + \boldsymbol{\mathcal{P}}(\boldsymbol{x}_d)(\dot{\boldsymbol{x}}_d - \boldsymbol{R}(\boldsymbol{x}_d)) \\ &= \boldsymbol{\mathcal{Q}}(\boldsymbol{x}_d)(\dot{\boldsymbol{x}}_d - \boldsymbol{R}(\boldsymbol{x}_d)) = \boldsymbol{0}, \end{aligned} \tag{39}$$

and so $\boldsymbol{x}(t) = \boldsymbol{x}_d(t)$ remains a solution to Eq. (35) for all times. □

Remark 4. Because the control signal $\boldsymbol{u}(t)$ consists of $p \leq n$ independent components, at most p one-to-one relations between state components and control components can be found. Thus, maximally p components of $\boldsymbol{x}_d(t)$ can be prescribed by the experimenter, while the remaining $n-p$ components are free. The time evolution of these $n-p$ components is fixed by the constraint equation (32). This motivates the name constraint equation: for an arbitrary desired trajectory $\boldsymbol{x}_d(t)$ to be exactly realizable, it is constrained by Eq. (32). One possibility to obtain $n-p$ linearly independent equations from the constraint equation is to use the matrix $\hat{\boldsymbol{\mathcal{Q}}}$ defined in Eq. (14) as

$$\hat{\boldsymbol{\mathcal{Q}}}(\boldsymbol{x}_d)(\dot{\boldsymbol{x}}_d - \boldsymbol{R}(\boldsymbol{x}_d)) = \boldsymbol{0}. \tag{40}$$

Remark 5. The necessity to satisfy the initial conditions $\boldsymbol{x}_d(t_0) = \boldsymbol{x}(t_0) = \boldsymbol{x}_0$, Eq. (33), leaves us with two possibilities. Either the initial state $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$ of the system can be prepared such that it equals the initial value $\boldsymbol{x}_d(t_0)$ of a given desired trajectory $\boldsymbol{x}_d$. Or the desired trajectory $\boldsymbol{x}_d$ must be designed such that it starts from the observed initial state $\boldsymbol{x}_0$ of the system. The control signal as given by Eq. (34) does neither depend on the current nor on the previous state $\boldsymbol{x}$ of the system and is an open-loop control signal. Only the initial state $\boldsymbol{x}_0$ of the actual controlled system enters via the initial condition Eq. (33) for the constraint equation. In general, the controlled system may suffer from instability. For example, it is impossible to prepare a real systems exactly in the initial state $\boldsymbol{x}_0$. Furthermore, a mathematical model must be viewed as an approximation of a real system. Disturbances which are not taken into account in the model affect the time evolution of the state. An additional feedback control may be sufficient to stabilize unstable desired trajectories. However, a thorough discussion of these issues is outside the scope of this article.

Equation (34) employs the Moore-Penrose pseudo inverse $\boldsymbol{\mathcal{B}}^+(\boldsymbol{x})$ to obtain the control signal in terms of the desired trajectory. This choice for a generalized inverse matrix is not unique. Note that any $p \times n$ matrix $\boldsymbol{\mathcal{B}}^g$ satisfying $\boldsymbol{\mathcal{B}}\boldsymbol{\mathcal{B}}^g\boldsymbol{\mathcal{B}} = \boldsymbol{\mathcal{B}}$ is called a generalized inverse of the $n \times p$ matrix $\boldsymbol{\mathcal{B}}$. Indeed, any $p \times n$ matrix $\boldsymbol{\mathcal{K}}(\boldsymbol{x})$ with the property

$$\operatorname{rank}(\boldsymbol{\mathcal{K}}(\boldsymbol{x})\boldsymbol{\mathcal{B}}(\boldsymbol{x})) = p \tag{41}$$

for all $\boldsymbol{x}$ can be used to construct a generalized inverse $p \times n$ matrix $\boldsymbol{\mathcal{B}}^g(\boldsymbol{x})$ of $\boldsymbol{\mathcal{B}}(\boldsymbol{x})$ as

$$\boldsymbol{\mathcal{B}}^g(\boldsymbol{x}) = (\boldsymbol{\mathcal{K}}(\boldsymbol{x})\boldsymbol{\mathcal{B}}(\boldsymbol{x}))^{-1}\boldsymbol{\mathcal{K}}(\boldsymbol{x}). \tag{42}$$

With the choice $\boldsymbol{\mathcal{K}}(\boldsymbol{x}) = \boldsymbol{\mathcal{B}}^T(\boldsymbol{x})$, $\boldsymbol{\mathcal{B}}^g(\boldsymbol{x})$ becomes the Moore-Penrose pseudo inverse $\boldsymbol{\mathcal{B}}^+(\boldsymbol{x})$. We demonstrate the uniqueness of the control solution Eq. (34) and its independence of the choice of $\boldsymbol{\mathcal{K}}(\boldsymbol{x})$ in the following theorem. In principle, any generalized inverse constructed as in Eq. (42) may be used to formulate the constraint equation and the control solution.

Theorem 3. *Let $\boldsymbol{\mathcal{K}}_i(\boldsymbol{x})$ with $i \in \{1, 2\}$ be two $p \times n$ matrices with the property*

$$\operatorname{rank}(\boldsymbol{\mathcal{K}}_i(\boldsymbol{x})\boldsymbol{\mathcal{B}}(\boldsymbol{x})) = p. \tag{43}$$

Define two generalized inverses as

$$\boldsymbol{\mathcal{B}}_i^g(\boldsymbol{x}) = (\boldsymbol{\mathcal{K}}_i(\boldsymbol{x})\boldsymbol{\mathcal{B}}(\boldsymbol{x}))^{-1}\boldsymbol{\mathcal{K}}_i(\boldsymbol{x}), \tag{44}$$

and corresponding projectors

$$\boldsymbol{\mathcal{P}}_i^g(\boldsymbol{x}) = \boldsymbol{\mathcal{B}}(\boldsymbol{x})\boldsymbol{\mathcal{B}}_i^g(\boldsymbol{x}), \quad \boldsymbol{\mathcal{Q}}_i^g(\boldsymbol{x}) = \boldsymbol{1} - \boldsymbol{\mathcal{P}}_i^g(\boldsymbol{x}). \tag{45}$$

The control signals expressed in terms of the desired trajectory are

$$\boldsymbol{u}_i(t) = \boldsymbol{\mathcal{B}}_i^g(\boldsymbol{x}_d(t))(\dot{\boldsymbol{x}}_d(t) - \boldsymbol{R}(\boldsymbol{x}_d(t))), \tag{46}$$

with desired trajectory $\boldsymbol{x}_d$ constrained by

$$\boldsymbol{0} = \boldsymbol{\mathcal{Q}}_i^g(\boldsymbol{x}_d(t))(\dot{\boldsymbol{x}}_d(t) - \boldsymbol{R}(\boldsymbol{x}_d(t))). \tag{47}$$

Then both control signals are identical,

$$\boldsymbol{u}_1(t) = \boldsymbol{u}_2(t). \tag{48}$$

Proof of Theorem 3. Multiplying the difference $\boldsymbol{u}_1 - \boldsymbol{u}_2$ by $\boldsymbol{\mathcal{B}}(\boldsymbol{x}_d)$ and exploiting the definitions of the projectors as well as the constraint equations yields

$$\begin{aligned} &\boldsymbol{\mathcal{B}}(\boldsymbol{x}_d)(\boldsymbol{u}_1 - \boldsymbol{u}_2) \\ &= (\boldsymbol{\mathcal{B}}(\boldsymbol{x}_d)\boldsymbol{\mathcal{B}}_1^g(\boldsymbol{x}_d) - \boldsymbol{\mathcal{B}}(\boldsymbol{x}_d)\boldsymbol{\mathcal{B}}_2^g(\boldsymbol{x}_d))(\dot{\boldsymbol{x}}_d - \boldsymbol{R}(\boldsymbol{x}_d)) \\ &= (\boldsymbol{\mathcal{P}}_1^g(\boldsymbol{x}_d) - \boldsymbol{\mathcal{P}}_2^g(\boldsymbol{x}_d))(\dot{\boldsymbol{x}}_d - \boldsymbol{R}(\boldsymbol{x}_d)) \\ &= (\boldsymbol{\mathcal{Q}}_2^g(\boldsymbol{x}_d) - \boldsymbol{\mathcal{Q}}_1^g(\boldsymbol{x}_d))(\dot{\boldsymbol{x}}_d - \boldsymbol{R}(\boldsymbol{x}_d)) = \boldsymbol{0}. \end{aligned} \tag{49}$$

Thus $\boldsymbol{u}_1 - \boldsymbol{u}_2$ lies in the null space of the input matrix $\boldsymbol{\mathcal{B}}(\boldsymbol{x}_d)$. Because $\boldsymbol{\mathcal{B}}(\boldsymbol{x}_d)$ has full rank p by assumption, its null space contains only $\boldsymbol{0}$ and so $\boldsymbol{u}_1(t) = \boldsymbol{u}_2(t)$. □

4. Output realizability

The constraint equation does not dictate which state components are prescribed and which are fixed by the constraint equation. In general, we have the freedom to choose an output $\boldsymbol{y} = \boldsymbol{h}(\boldsymbol{x})$ with m components such that under the action of control, a prescribed desired output $\boldsymbol{y}_d(t)$ is exactly realized, $\boldsymbol{y}_d(t) = \boldsymbol{y}(t)$.

Theorem 4. *Let $\boldsymbol{x}_d(t)$ be an exactly realizable trajectory, i.e., it satisfies the constraint equation (32) and the initial condition Eq. (33). If $\boldsymbol{x}_d(t)$ additionally satisfies*

$$\boldsymbol{y}_d(t) = \boldsymbol{h}(\boldsymbol{x}_d(t)), \tag{50}$$

then the output $\boldsymbol{y}_d(t)$ is realized exactly, i.e.,

$$\boldsymbol{y}(t) = \boldsymbol{y}_d(t). \tag{51}$$

Proof of Theorem 4. Let $\Delta\boldsymbol{y}(t)$ be defined as

$$\begin{aligned}\Delta\boldsymbol{y}(t) &= \boldsymbol{y}_d(t) - \boldsymbol{y}(t) = \boldsymbol{h}(\boldsymbol{x}_d(t)) - \boldsymbol{h}(\boldsymbol{x}(t)) \\ &= \boldsymbol{h}(\boldsymbol{x}_d(t)) - \boldsymbol{h}(\Delta\boldsymbol{x}(t) + \boldsymbol{x}_d(t)),\end{aligned} \tag{52}$$

with $\Delta\boldsymbol{x}(t) = \boldsymbol{x}(t) - \boldsymbol{x}_d(t)$. For an exactly realizable trajectory we have $\Delta\boldsymbol{x}(t) = \boldsymbol{0}$ and so $\boldsymbol{y}(t) = \boldsymbol{y}_d(t)$. □

The solution to a control problem consists of a solution $\boldsymbol{x}$ to the controlled state equation (3) and a solution $\boldsymbol{u}$ for the control signal. Within the framework of exactly realizable trajectories, these are given by

$$\boldsymbol{u} = \boldsymbol{\mathcal{B}}^{+}(\boldsymbol{x}_d)(\dot{\boldsymbol{x}}_d - \boldsymbol{R}(\boldsymbol{x}_d)), \qquad \boldsymbol{x} = \boldsymbol{x}_d. \tag{53}$$

For $\boldsymbol{x} = \boldsymbol{x}_d$ to hold, $\boldsymbol{x}_d$ must satisfy the constraint equation (32). For output realizability, $\boldsymbol{x}_d$ additionally has to satisfy the output relation (50). Thus the only equations which remain to be solved is the system of $n - p + m$ inhomogeneous differential-algebraic equations (DAE) given by

$$\boldsymbol{y}_d = \boldsymbol{h}(\boldsymbol{x}_d), \qquad \boldsymbol{\mathcal{Q}}(\boldsymbol{x}_d)(\dot{\boldsymbol{x}}_d - \boldsymbol{R}(\boldsymbol{x}_d)) = \boldsymbol{0}. \tag{54}$$

The desired output $\boldsymbol{y}_d(t)$ represents an inhomogeneity and renders Eqs. (54) a non-autonomous DAE. Equations (54) have to be solved for $n-p+m$ components of $\boldsymbol{x}_d$ together with the n initial conditions $\boldsymbol{x}_d(t_0) = \boldsymbol{x}_0$. In principle, solutions can exist as long as $m \leq p$. If $m < p$, $p - m$ components of $\boldsymbol{x}_d$ may be freely chosen as long as they do not violate the initial conditions. We expect a solution $\boldsymbol{x}_d(t)$ to Eqs. (54) to depend on the entire history of the output $\boldsymbol{y}_d$ from the initial time t_0 up to the current time t. In a final step, the solution for $\boldsymbol{x}_d$ obtained from Eqs. (54) is used in the control signal $\boldsymbol{u}$, Eq. (34), to eliminate $n-p+m$ components of $\boldsymbol{x}_d$. In general, $\boldsymbol{u}(t)$ depends on the entire history of $\boldsymbol{y}_d$ up to the current time t.

Being a system of DAEs, Eqs. (54) cannot accommodate all n initial conditions. Consequences of the initial conditions can be distinguished as follows. First, evaluating the output relation at $t = t_0$ imposes m conditions on the desired output as

$$\boldsymbol{y}_d(t_0) = \boldsymbol{h}(\boldsymbol{x}_0). \tag{55}$$

Second, r initial conditions are accommodated by the constants of integration arising in the constraint equation. Third, in case that $n-m-r = l > 0$, l additional relations between $\boldsymbol{y}_d(t_0)$ and $\boldsymbol{x}_0$ must be satisfied. The latter conditions also involve the time derivative of $\boldsymbol{y}_d$ at the initial time $t = t_0$. They ensure that the constraint equation is satisfied also at $t = t_0$,

$$\boldsymbol{\mathcal{Q}}(\boldsymbol{x}_d(t_0))(\dot{\boldsymbol{x}}_d(t_0) - \boldsymbol{R}(\boldsymbol{x}_d(t_0))) = \boldsymbol{0}. \tag{56}$$

We illustrate output realizability with the help of two examples.

Example 1. Consider the system

$$\dot{x}_1(t) = x_2(t), \tag{57}$$

$$\dot{x}_2(t) = R(x_1(t), x_2(t)) + B(x_1(t), x_2(t))\, u(t), \tag{58}$$

with state vector $\boldsymbol{x} = (x_1, x_2)^T$, nonlinearity $\boldsymbol{R}(\boldsymbol{x}) = (x_2, R(x_1, x_2))^T$, and input matrix $\boldsymbol{\mathcal{B}}(\boldsymbol{x}) = (0, B(x_1, x_2))^T$. The initial condition is $\boldsymbol{x}(t_0) = \boldsymbol{x}_0 = (x_{1,0}, x_{2,0})^T$. The assumption of full rank of $\boldsymbol{\mathcal{B}}(\boldsymbol{x})$ for all $\boldsymbol{x}$ implies $B(x_1, x_2) \neq 0$. The system represents Newton's equation of motion for a point particle with unit mass. The particle moves with position x_1 and velocity x_2 in one spatial dimensional under the influence of an external force R and a control force Bu. The Moore-Penrose pseudo inverse of the input matrix $\boldsymbol{\mathcal{B}}$ and the corresponding projectors are

$$\boldsymbol{\mathcal{B}}^{+}(\boldsymbol{x}) = B(x_1, x_2)^{-2}(0, B(x_1, x_2)), \tag{59}$$

$$\boldsymbol{\mathcal{P}}(\boldsymbol{x}) = \boldsymbol{\mathcal{P}} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \tag{60}$$

$$\boldsymbol{\mathcal{Q}}(\boldsymbol{x}) = \boldsymbol{\mathcal{Q}} = \boldsymbol{1} - \boldsymbol{\mathcal{P}} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{61}$$

The constraint equation and the control signal are

$$\dot{x}_{1,d}(t) = x_{2,d}(t), \quad u(t) = \frac{\dot{x}_{2,d}(t) - R(x_{1,d}(t), x_{2,d}(t))}{B(x_{1,d}(t), x_{2,d}(t))}, \tag{62}$$

respectively.

Let us first assume that the output y is given by the velocity x_2, $y(t) = x_2(t)$. The control task is then to enforce a velocity over time prescribed by the experimenter in form of the desired output y_d. The system of DAEs for output realizability (54) becomes

$$x_{2,d}(t) = y_d(t), \qquad \dot{x}_{1,d}(t) = x_{2,d}(t). \tag{63}$$

The constraint equation is a differential equation for the position $x_{1,d}$ and yields

$$x_{1,d}(t) = x_{1,d}(t_0) + \int_{t_0}^{t} \mathrm{d}\tau\, y_d(\tau). \tag{64}$$

To satisfy the initial condition $\boldsymbol{x}_d(t_0) = \boldsymbol{x}_0$, we must have

$$x_{1,d}(t_0) = x_{1,0}, \qquad x_{2,d}(t_0) = x_{2,0} = y_d(t_0). \tag{65}$$

Thus, for a given desired output y_d, the system has to be prepared in the state $\boldsymbol{x}_0 = (x_{1,0}, y_d(t_0))^T$, with $x_{1,0}$ a free parameter. Or, for a given system with initial state $\boldsymbol{x}_0$, the desired output must be chosen such that initially $y_d(t_0) = x_{2,0}$. The control signal becomes an expression depending only on the desired output y_d and the initial value $x_{1,0}$,

$$u(t) = \frac{\dot{y}_d(t) - R\left(x_{1,0} + \int_{t_0}^{t} \mathrm{d}\tau y_d(\tau), y_d(t)\right)}{B\left(x_{1,0} + \int_{t_0}^{t} \mathrm{d}\tau y_d(\tau), y_d(t)\right)}. \tag{66}$$

The context of a mechanical control system allows the following interpretation of our approach. The constraint equation can be viewed as the definition of the velocity of a point particle. Neither an external force R nor a control force Bu can change that definition. With only a single control force, position x_1 and velocity x_2 over time cannot be controlled independently from each other.

Instead of choosing the velocity x_2 as the output, we may choose the position $y(t) = x_1(t)$ as well. The DAE (54) becomes

$$x_{1,d}(t) = y_d(t), \qquad \dot{x}_{1,d}(t) = x_{2,d}(t). \tag{67}$$

Here, the constraint equation is an algebraic equation for the desired velocity $x_{2,d}$, $x_{2,d}(t) = \dot{y}_d(t)$, which is used eliminate $x_{2,d}(t)$ from the control signal. We obtain an expression which depends on the desired output only,

$$u(t) = \frac{\ddot{y}_d(t) - R(y_d(t), \dot{y}_d(t))}{B(y_d(t), \dot{y}_d(t))}. \tag{68}$$

To satisfy the initial condition $\boldsymbol{x}_d(t_0) = \boldsymbol{x}_0$, the desired output $y_d(t)$ must satisfy the two initial conditions

$$x_{1,d}(t_0) = x_{1,0} = y_d(t_0), \quad x_{2,d}(t_0) = x_{2,0} = \dot{y}_d(t_0). \tag{69}$$

In this case, the initial desired output has to satisfy two conditions because the constraint equation is a purely algebraic equation which does not allow for an initial condition. Equation (68) is known as a variant of the so-called computed torque formula and has found widespread application in robotics [15, 16, 17].

Example 2. We consider the example from [11] with $n = 4$ and $p = 1$. The output is $y = x_1 - 3x_3$, and the nonlinearity $\boldsymbol{R}$ and input matrix $\boldsymbol{\mathcal{B}}$ are given by

$$\boldsymbol{R}(\boldsymbol{x}) = \begin{pmatrix} x_2 - x_1 \\ x_1^3 - 3x_2 \\ x_1 - 2x_3 \\ x_3^2 - x_4 \end{pmatrix}, \quad \boldsymbol{\mathcal{B}}(\boldsymbol{x}) = \begin{pmatrix} 0 \\ 2 + \sin^2(x_4) \\ 0 \\ 0 \end{pmatrix}. \tag{70}$$

The control signal in terms of the desired trajectory is

$$u(t) = \frac{\dot{x}_{2,d}(t) - x_{1,d}^3(t) + 3x_{2,d}(t)}{2 + \sin^2(x_{4,d}(t))}. \tag{71}$$

Equations (54) becomes

$$y_d(t) = x_{1,d}(t) - 3x_{3,d}(t), \tag{72}$$
$$\dot{x}_{1,d}(t) = -x_{1,d}(t) + x_{2,d}(t), \tag{73}$$
$$\dot{x}_{3,d}(t) = x_{1,d}(t) - 2x_{3,d}(t), \tag{74}$$
$$\dot{x}_{4,d}(t) = x_{3,d}^2(t) - x_{4,d}(t), \tag{75}$$

which must be solved for $\boldsymbol{x}_d(t)$. We obtain

$$x_{1,d}(t) = 3\int_{t_0}^{t} \mathrm{d}\tau \mathrm{e}^{t-\tau} y_d(\tau) + y_d(t) + \mathrm{e}^{t-t_0}(x_{1,d}(t_0) - y_d(t_0)), \tag{76}$$
$$x_{2,d}(t) = \dot{x}_{1,d}(t) + x_{1,d}(t), \tag{77}$$
$$x_{3,d}(t) = \frac{1}{3}(x_{1,d}(t) - y_d(t)), \tag{78}$$
$$x_{4,d}(t) = \mathrm{e}^{t_0-t} x_4(t_0) + \frac{1}{9}\int_{t_0}^{t} \mathrm{d}\tau \mathrm{e}^{\tau+t-2t_0}(f(\tau))^2, \tag{79}$$

with

$$f(t) = x_1(t_0) - y_d(t_0) + 3\int_{t_0}^{t} \mathrm{d}\tau \mathrm{e}^{t_0-\tau} y_d(\tau). \tag{80}$$

Enforcing the initial condition $\boldsymbol{x}_d(t_0) = \boldsymbol{x}_0$ yields two additional relations for the initial desired output,

$$x_{2,d}(t_0) = x_{2,0} = 2(x_{1,0} + y_d(t_0)) + \dot{y}_d(t_0), \tag{81}$$
$$x_{3,d}(t_0) = x_{3,0} = \frac{1}{3}(x_{1,0} - y_d(t_0)), \tag{82}$$

while two values of the initial conditions are free parameters,

$$x_{1,d}(t_0) = x_{1,0}, \qquad x_{4,d}(t_0) = x_{4,0}. \tag{83}$$

5. Linearizing assumption

Instead of having to solve the controlled dynamical system (3) with the control signal acting as an inhomogeneity, the approach of exactly realizable trajectories leads to Eqs. (54) with the desired output $\boldsymbol{y}_d$ acting as an inhomogeneity. As the examples of the last section demonstrate, Eqs. (54) are often considerably simpler than the original controlled dynamical system. In both cases, no simple analytical solution in closed form can be found for the controlled dynamical system. In general, if the constraint equation as well as the output relation is linear, the entire controlled system can be regarded, in some sense and to some extent, as being linear, even though the original system (3) is nonlinear. Three conditions must be met for Eqs. (54) to be linear.

Assumption 1. *Linearizing assumption.*

1. The projectors $\boldsymbol{\mathcal{P}}(\boldsymbol{x})$ and $\boldsymbol{\mathcal{Q}}(\boldsymbol{x})$ must be independent of the state $\boldsymbol{x}$,

$$\boldsymbol{\mathcal{P}}(\boldsymbol{x}) = \boldsymbol{\mathcal{P}} = \mathbf{1} - \boldsymbol{\mathcal{Q}} = \text{const}, \tag{84}$$

2. the nonlinearity $\boldsymbol{R}(\boldsymbol{x})$ must be linear with respect to the input matrix $\boldsymbol{\mathcal{B}}(\boldsymbol{x})$ in the sense that

$$\boldsymbol{\mathcal{Q}}\boldsymbol{R}(\boldsymbol{x}) = \boldsymbol{\mathcal{Q}}\boldsymbol{\mathcal{A}}\boldsymbol{x} + \boldsymbol{\mathcal{Q}}\boldsymbol{b} \tag{85}$$

   with $n \times n$ matrix $\boldsymbol{\mathcal{A}}$ and n-component vector $\boldsymbol{b}$ independent of the state $\boldsymbol{x}$,
3. the output relation must be linear,

$$\boldsymbol{y}(t) = \boldsymbol{\mathcal{C}}\boldsymbol{x}(t). \tag{86}$$

Employing these three assumptions together with the state separation Eq. (30) yields

$$\boldsymbol{\mathcal{Q}}\boldsymbol{\mathcal{N}}\dot{\boldsymbol{x}}_d = \boldsymbol{\mathcal{Q}}\boldsymbol{\mathcal{A}}\boldsymbol{\mathcal{N}}\boldsymbol{x}_d + \boldsymbol{\mathcal{Q}}\boldsymbol{b} + \boldsymbol{\mathcal{Q}}\boldsymbol{\mathcal{A}}\boldsymbol{\mathcal{C}}^+\boldsymbol{y}_d - \boldsymbol{\mathcal{Q}}\boldsymbol{\mathcal{C}}^+\dot{\boldsymbol{y}}_d. \tag{87}$$

This is a system of $n-p-r$ algebraic and r differential equations for the $n-m$ independent components of $\boldsymbol{\mathcal{N}}\boldsymbol{x}_d(t)$. The number r is given by the rank of the matrix product $\boldsymbol{\mathcal{Q}}\boldsymbol{\mathcal{N}}$,

$$\begin{aligned} r &= \text{rank}(\boldsymbol{\mathcal{Q}}\boldsymbol{\mathcal{N}}) \leq \min(\text{rank}(\boldsymbol{\mathcal{Q}}), \text{rank}(\boldsymbol{\mathcal{N}})) \\ &= \min(n-p, n-m). \end{aligned} \tag{88}$$

Being an inhomogeneous linear differential-algebraic equation, closed form solutions for Eq. (87) can be obtained, see [18, 19, 20]. However, a complete discussion is outside the scope of this article and reserved for later investigations. Equation (87) assumes a particularly simple form if $m = p$ and $r = n-p$. This is the case for the output $\boldsymbol{\mathcal{C}} = \hat{\boldsymbol{\mathcal{P}}}^T$, or, as long as the input matrix is constant, $\boldsymbol{\mathcal{C}} = \boldsymbol{\mathcal{B}}^T$. These outputs imply $\boldsymbol{\mathcal{M}} = \boldsymbol{\mathcal{P}}$ and $\boldsymbol{\mathcal{N}} = \boldsymbol{\mathcal{Q}}$, and Eq. (87) becomes a system of $n-p$ differential equations for the $n-p$ independent components of $\boldsymbol{\mathcal{Q}}\boldsymbol{x}_d$,

$$\boldsymbol{\mathcal{Q}}\dot{\boldsymbol{x}}_d = \boldsymbol{\mathcal{Q}}\boldsymbol{\mathcal{A}}\boldsymbol{\mathcal{Q}}\boldsymbol{x}_d + \boldsymbol{\mathcal{Q}}\boldsymbol{\mathcal{A}}\boldsymbol{\mathcal{P}}\boldsymbol{x}_d + \boldsymbol{\mathcal{Q}}\boldsymbol{b}. \tag{89}$$

Remark 6. Note that assumption (84) does neither imply that the input matrix $\boldsymbol{\mathcal{B}}(\boldsymbol{x})$ nor its Moore-Penrose pseudo inverse $\boldsymbol{\mathcal{B}}^+(\boldsymbol{x})$ is independent of $\boldsymbol{x}$. Furthermore, while the constraint equation is linear, the control signal $\boldsymbol{u} = \boldsymbol{\mathcal{B}}^+(\boldsymbol{x}_d)(\dot{\boldsymbol{x}}_d - \boldsymbol{R}(\boldsymbol{x}_d))$ may still depend nonlinearly on the desired state $\boldsymbol{x}_d$.

Remark 7. Condition (85) is very restrictive. It enforces $n-p$ components to depend only linearly on the state. However, some important models of nonlinear dynamics satisfy the linearizing assumption. Among these are all one-dimensional mechanical control systems discussed in Example 1. Another prominent example is the FitzHugh-Nagumo model [21], a prototype model for excitable systems as e.g. the neuron, with a control acting on the activator variable [14].

Remark 8. Systems satisfying the linearizing assumption are feedback linearizable [2] without a state transform. This can be seen as follows. Let the feedback-controlled system be

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{R}(\boldsymbol{x}(t)) + \boldsymbol{\mathcal{B}}(\boldsymbol{x}(t))\boldsymbol{u}(\boldsymbol{x}(t)). \tag{90}$$

The control signal $\boldsymbol{u} \in \mathbb{R}^p$ is transformed to the new control signal $\boldsymbol{v} \in \mathbb{R}^p$ with the help of the $n \times p$ matrix $\boldsymbol{\mathcal{H}}$ as

$$\boldsymbol{u}(\boldsymbol{x}) = -\boldsymbol{\mathcal{B}}^+(\boldsymbol{x})(\boldsymbol{R}(\boldsymbol{x}) - \boldsymbol{\mathcal{H}}\boldsymbol{v}(\boldsymbol{x})) \tag{91}$$

to obtain

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{\mathcal{Q}}\boldsymbol{\mathcal{A}}\boldsymbol{x}(t) + \boldsymbol{\mathcal{Q}}\boldsymbol{b} + \boldsymbol{\mathcal{P}}\boldsymbol{\mathcal{H}}\boldsymbol{v}(\boldsymbol{x}(t)), \tag{92}$$

where the linearizing assumption has been applied. Let the projector $\boldsymbol{\mathcal{P}}$ with rank p be rank-decomposed as $\boldsymbol{\mathcal{P}} = \boldsymbol{\mathcal{F}}\boldsymbol{\mathcal{G}}$ with constant $n \times p$ matrix $\boldsymbol{\mathcal{F}}$ and constant $p \times n$ matrix $\boldsymbol{\mathcal{G}}$. We introduce the new control signal $\tilde{\boldsymbol{v}}(\boldsymbol{x}) = \boldsymbol{\mathcal{G}}\boldsymbol{\mathcal{H}}\boldsymbol{v}(\boldsymbol{x})$ to get a linear feedback-controlled system,

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{\mathcal{Q}}\boldsymbol{\mathcal{A}}\boldsymbol{x}(t) + \boldsymbol{\mathcal{Q}}\boldsymbol{b} + \boldsymbol{\mathcal{F}}\tilde{\boldsymbol{v}}(\boldsymbol{x}(t)). \tag{93}$$

6. Controllability

A system is called controllable, or full state controllable, if it is possible to achieve a transfer from an initial state $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$ at time $t = t_0$ to a final state $\boldsymbol{x}(t_1) = \boldsymbol{x}_1$ at the terminal time $t = t_1$ [22]. Along which trajectory the transfer is achieved is irrelevant. While for LTI systems conditions for controllability are easily expressed in terms of a Kalman rank condition, these conditions are more difficult for nonlinear control systems [2, 23]. Here, we derive a similar rank condition within the framework of exactly realizable trajectories. This rank condition also applies to systems satisfying the linearizing assumption 1. We consider the controlled state equation (3) together with the linearizing assumption Eq. (85). This implies a linear constraint equation (32),

$$\boldsymbol{\mathcal{Q}}\dot{\boldsymbol{x}}_d = \boldsymbol{\mathcal{Q}}\boldsymbol{\mathcal{A}}\boldsymbol{\mathcal{Q}}\boldsymbol{x}_d + \boldsymbol{\mathcal{Q}}\boldsymbol{\mathcal{A}}\boldsymbol{\mathcal{P}}\boldsymbol{x}_d + \boldsymbol{\mathcal{Q}}\boldsymbol{b}. \tag{94}$$

Because the trajectory in between $\boldsymbol{x}_0$ and $\boldsymbol{x}_1$ is irrelevant, we may assume that the state components $\boldsymbol{\mathcal{P}}\boldsymbol{x}_d(t)$ are prescribed while the components $\boldsymbol{\mathcal{Q}}\boldsymbol{x}_d(t)$ are governed by Eq. (94). Equation (94) is an inhomogeneous linear dynamical system for $\boldsymbol{\mathcal{Q}}\boldsymbol{x}_d(t)$. Its solution with initial condition $\boldsymbol{\mathcal{Q}}\boldsymbol{x}_d(t_0) = \boldsymbol{\mathcal{Q}}\boldsymbol{x}_0$ yields

$$\begin{aligned} \boldsymbol{\mathcal{Q}}\boldsymbol{x}_d(t) = &\int_{t_0}^{t} \mathrm{d}\tau \exp(\boldsymbol{\mathcal{Q}}\boldsymbol{\mathcal{A}}\boldsymbol{\mathcal{Q}}(t-\tau))\,\boldsymbol{\mathcal{Q}}(\boldsymbol{\mathcal{A}}\boldsymbol{\mathcal{P}}\boldsymbol{x}_d(\tau) + \boldsymbol{b}) \\ &+ \exp(\boldsymbol{\mathcal{Q}}\boldsymbol{\mathcal{A}}\boldsymbol{\mathcal{Q}}(t-t_0))\,\boldsymbol{\mathcal{Q}}\boldsymbol{x}_0. \end{aligned} \tag{95}$$

Theorem 5. *Controllability.*
A nonlinear system Eq. (3) which satisfies the linearizing

assumption 1 is controllable if the $n \times n^2$ controllability matrix

$$\boldsymbol{\mathcal{K}} = \left(\boldsymbol{\mathcal{QAP}}|\boldsymbol{\mathcal{QAQAP}}|\cdots|\left(\boldsymbol{\mathcal{QAQ}}\right)^{n-1}\boldsymbol{\mathcal{QAP}}\right). \quad (96)$$

satisfies the rank condition

$$\operatorname{rank}\left(\boldsymbol{\mathcal{K}}\right) = n - p. \quad (97)$$

Proof of Theorem 5. Achieving a transfer from an initial to a finite state means the desired trajectory $\boldsymbol{x}_d(t)$ has to satisfy $\boldsymbol{x}_d(t_0) = \boldsymbol{x}_0$ and $\boldsymbol{x}_d(t_1) = \boldsymbol{x}_1$. Consequently, $\boldsymbol{\mathcal{P}}\boldsymbol{x}_d$ and $\boldsymbol{\mathcal{Q}}\boldsymbol{x}_d$ have to satisfy the initial and terminal conditions

$$\boldsymbol{\mathcal{P}}\boldsymbol{x}_d(t_0) = \boldsymbol{\mathcal{P}}\boldsymbol{x}_0, \qquad \boldsymbol{\mathcal{P}}\boldsymbol{x}_d(t_1) = \boldsymbol{\mathcal{P}}\boldsymbol{x}_1, \quad (98)$$

$$\boldsymbol{\mathcal{Q}}\boldsymbol{x}_d(t_0) = \boldsymbol{\mathcal{Q}}\boldsymbol{x}_0, \qquad \boldsymbol{\mathcal{Q}}\boldsymbol{x}_d(t_1) = \boldsymbol{\mathcal{Q}}\boldsymbol{x}_1. \quad (99)$$

The part $\boldsymbol{\mathcal{P}}\boldsymbol{x}_d$ is prescribed by the experimenter such that it satisfies the initial and terminal conditions Eq. (98). Consequently, all initial and terminal conditions except $\boldsymbol{\mathcal{Q}}\boldsymbol{x}_d(t_1) = \boldsymbol{\mathcal{Q}}\boldsymbol{x}_1$ are satisfied. Enforcing this remaining condition onto the solution Eq. (95) of the constraint equation yields

$$\boldsymbol{\mathcal{Q}}\boldsymbol{x}_1 = \boldsymbol{\mathcal{Q}}\boldsymbol{x}_d(t_1) = \exp\left(\boldsymbol{\mathcal{QAQ}}(t_1 - t_0)\right)\boldsymbol{\mathcal{Q}}\boldsymbol{x}_0 + \int_{t_0}^{t_1} \mathrm{d}\tau \exp\left(\boldsymbol{\mathcal{QAQ}}(t_1 - \tau)\right)\boldsymbol{\mathcal{Q}}\left(\boldsymbol{\mathcal{AP}}\boldsymbol{x}_d(\tau) + \boldsymbol{b}\right). \quad (100)$$

This can actually be viewed as a condition for the part $\boldsymbol{\mathcal{P}}\boldsymbol{x}_d$. The transfer from $\boldsymbol{x}_0$ to $\boldsymbol{x}_1$ is achieved as long as $\boldsymbol{\mathcal{P}}\boldsymbol{x}_d$ satisfies Eq. (100).

Similarly to the proof of the Kalman rank condition in [1], conditions on the state matrix $\boldsymbol{\mathcal{A}}$ and the projectors $\boldsymbol{\mathcal{P}}$ and $\boldsymbol{\mathcal{Q}}$ can be given such that Eq. (100) is satisfied. Due to the Cayley-Hamilton theorem [24], any power of matrices with $i \geq n$ can be expanded in terms of lower order matrix powers as

$$\left(\boldsymbol{\mathcal{QAQ}}\right)^i = \sum_{k=0}^{n-1} d_{ik}\left(\boldsymbol{\mathcal{QAQ}}\right)^k. \quad (101)$$

The term involving $\boldsymbol{\mathcal{P}}\boldsymbol{x}_d(\tau)$ in Eq. (100) can be simplified,

$$\int_{t_0}^{t_1} \mathrm{d}\tau \exp\left(\boldsymbol{\mathcal{QAQ}}(t_0 - \tau)\right)\boldsymbol{\mathcal{QAP}}\boldsymbol{x}_d(\tau) = \sum_{k=0}^{n-1}\left(\boldsymbol{\mathcal{QAQ}}\right)^k\boldsymbol{\mathcal{QAP}} \times \int_{t_0}^{t_1}\mathrm{d}\tau\left(\frac{(t_0-\tau)^k}{k!} + \sum_{i=n}^{\infty} d_{ik}\frac{(t_0-\tau)^i}{i!}\right)\boldsymbol{\mathcal{P}}\boldsymbol{x}_d(\tau), \quad (102)$$

such that Eq. (100) becomes a truncated sum

$$\exp\left(-\boldsymbol{\mathcal{QAQ}}(t_1 - t_0)\right)\boldsymbol{\mathcal{Q}}\boldsymbol{x}_1 - \boldsymbol{\mathcal{Q}}\boldsymbol{x}_0 - \int_{t_0}^{t_1}\mathrm{d}\tau\exp\left(\boldsymbol{\mathcal{QAQ}}(t_0-\tau)\right)\boldsymbol{\mathcal{Q}}\boldsymbol{b} = \sum_{k=0}^{n-1}\left(\boldsymbol{\mathcal{QAQ}}\right)^k\boldsymbol{\mathcal{QAP}}\boldsymbol{\alpha}_k(t_1, t_0). \quad (103)$$

We defined the $n \times 1$ vectors $\boldsymbol{\alpha}_k$ for $k \in \{0, \ldots, n-1\}$ as

$$\boldsymbol{\alpha}_k(t_1, t_0) = \int_{t_0}^{t_1}\mathrm{d}\tau\left(\frac{(t_0-\tau)^k}{k!} + \sum_{i=n}^{\infty} d_{ik}\frac{(t_0-\tau)^i}{i!}\right)\boldsymbol{\mathcal{P}}\boldsymbol{x}_d(\tau). \quad (104)$$

The right hand side of Eq. (103) can be written with the help of the $n^2 \times 1$ vector

$$\boldsymbol{\alpha}(t_1, t_0) = \left(\boldsymbol{\alpha}_0(t_1, t_0), \ldots, \boldsymbol{\alpha}_{n-1}(t_1, t_0)\right)^T \quad (105)$$

as

$$\exp\left(-\boldsymbol{\mathcal{QAQ}}(t_1 - t_0)\right)\boldsymbol{\mathcal{Q}}\boldsymbol{x}_1 - \boldsymbol{\mathcal{Q}}\boldsymbol{x}_0 - \int_{t_0}^{t_1}\mathrm{d}\tau\exp\left(\boldsymbol{\mathcal{QAQ}}(t_0-\tau)\right)\boldsymbol{\mathcal{Q}}\boldsymbol{b} = \boldsymbol{\mathcal{K}}\boldsymbol{\alpha}(t_1, t_0). \quad (106)$$

We defined the $n \times n^2$ controllability matrix $\boldsymbol{\mathcal{K}}$ as given by Eq. (96). The left hand side of Eq. (106) can be any point in $\boldsymbol{\mathcal{Q}}\mathbb{R}^n = \mathbb{R}^{n-p}$. The mapping from $\boldsymbol{\mathcal{Q}}\boldsymbol{x}_1$ to $\boldsymbol{\alpha}$ is surjective, i.e., every element on the left hand side has a corresponding element on the right hand side, if $\boldsymbol{\mathcal{K}}$ has full rank $n - p$. □

Remark 9. Using the complementary projectors $\boldsymbol{\mathcal{P}}$ and $\boldsymbol{\mathcal{Q}}$, the state matrix $\boldsymbol{\mathcal{A}}$ can be split up in four parts as

$$\boldsymbol{\mathcal{A}} = \boldsymbol{\mathcal{PAP}} + \boldsymbol{\mathcal{PAQ}} + \boldsymbol{\mathcal{QAP}} + \boldsymbol{\mathcal{QAQ}}. \quad (107)$$

Note that the controllability matrix $\tilde{\boldsymbol{\mathcal{K}}}$, Eq. (96), does only depend on the parts $\boldsymbol{\mathcal{QAP}}$ and $\boldsymbol{\mathcal{QAQ}}$, but not on $\boldsymbol{\mathcal{PAP}}$ and $\boldsymbol{\mathcal{PAQ}}$. Consequently, only knowledge of the parts $\boldsymbol{\mathcal{QAP}}$ and $\boldsymbol{\mathcal{QAQ}}$ is required to decide if a system is controllable. This might be advantageous for applications with incomplete knowledge about the underlying dynamics.

7. Output controllability

Theorem 6. *Output controllability.*
The system (3) satisfying the linearizing assumption 1 is output controllable if the output controllability matrix

$$\boldsymbol{\mathcal{K}}_{\boldsymbol{\mathcal{C}}} = \left(\boldsymbol{\mathcal{CP}}|\boldsymbol{\mathcal{CQAP}}|\cdots|\boldsymbol{\mathcal{C}}\left(\boldsymbol{\mathcal{QAQ}}\right)^{n-1}\boldsymbol{\mathcal{QAP}}\right) \quad (108)$$

satisfies the rank condition

$$\operatorname{rank}\left(\boldsymbol{\mathcal{K}}_{\boldsymbol{\mathcal{C}}}\right) = m. \quad (109)$$

Proof of Theorem 6. Using Eq. (95), the solution for the linear output reads

$$\boldsymbol{y}_d(t) = \boldsymbol{\mathcal{CP}}\boldsymbol{x}_d(t) + \boldsymbol{\mathcal{C}}\exp\left(\boldsymbol{\mathcal{QAQ}}(t-t_0)\right)\boldsymbol{\mathcal{Q}}\boldsymbol{x}_0 + \boldsymbol{\mathcal{C}}\int_{t_0}^{t}\mathrm{d}\tau\exp\left(\boldsymbol{\mathcal{QAQ}}(t-\tau)\right)\boldsymbol{\mathcal{QAP}}\boldsymbol{x}_d(\tau). \quad (110)$$

Exploiting the Cayley-Hamilton theorem, Eq. (110) becomes

$$\begin{aligned} &\boldsymbol{y}_d(t) - \boldsymbol{\mathcal{C}} \exp\left(\boldsymbol{\mathcal{QAQ}}\left(t-t_0\right)\right) \boldsymbol{\mathcal{Q}} \boldsymbol{x}_0 \\ &= \boldsymbol{\mathcal{CP}} \boldsymbol{x}_d(t) + \boldsymbol{\mathcal{C}} \sum_{k=0}^{n-1} \left(\boldsymbol{\mathcal{QAQ}}\right)^k \boldsymbol{\mathcal{QAP}} \boldsymbol{\beta}_k\left(t, t_0\right). \end{aligned} \tag{111}$$

The remainder of the proof proceeds analogously to the proof of Theorem 5 for full state controllability. □

Remark 10. For $\boldsymbol{\mathcal{C}} = \mathbf{1}$ and $m = n$, output controllability should reduce to full state controllability. Indeed, if the controllability matrix $\boldsymbol{\mathcal{K}}$, Eq. (96), satisfies the rank condition $\operatorname{rank}(\boldsymbol{\mathcal{K}}) = n - p$, then the matrix $\tilde{\boldsymbol{\mathcal{K}}} = (\boldsymbol{\mathcal{P}}|\boldsymbol{\mathcal{K}})$ satisfies $\operatorname{rank}\left(\tilde{\boldsymbol{\mathcal{K}}}\right) = n$, and $\boldsymbol{\mathcal{K}}_{\boldsymbol{\mathcal{C}}}$ reduces to $\tilde{\boldsymbol{\mathcal{K}}}$ for $\boldsymbol{\mathcal{C}} = \mathbf{1}$ and $m = n$.

Remark 11. Similar as for controllability, we can use the Moore-Penrose projectors $\boldsymbol{\mathcal{M}}$ and $\boldsymbol{\mathcal{N}}$ constructed from the output matrix $\boldsymbol{\mathcal{C}}$ in Definition 2 to express observability in terms of a rank condition for an observability matrix [14].

8. Conclusions and outlook

Exactly realizable desired trajectories are the subset of desired trajectories $\boldsymbol{x}_d(t)$ for which a control exists such that the state over time $\boldsymbol{x}(t)$ follows the desired trajectory exactly, $\boldsymbol{x}(t) = \boldsymbol{x}_d(t)$. By means of the Moore-Penrose projectors defined in Eqs. (6), we propose a separation of the state equation (3) in two parts. The first part, called the constraint equation (32), is independent of the control signal $\boldsymbol{u}$. The second part Eq. (34) establishes a one-to-one relationship between the p-dimensional control signal $\boldsymbol{u}(t)$ and p out of n components of the desired trajectory $\boldsymbol{x}_d(t)$. The constraint equation fixes those $n-p$ components of the desired trajectory $\boldsymbol{x}_d(t)$ for which no one-to-one relationship with the control signal exists. A desired trajectory is exactly realizable if and only if it satisfies the constraint equation.
We can distinguish 3 classes of desired trajectories $\boldsymbol{x}_d$:

(A) desired trajectories $\boldsymbol{x}_d$ which are solutions to the uncontrolled system,
(B) desired trajectories $\boldsymbol{x}_d$ which are exactly realizable,
(C) arbitrary desired trajectories $\boldsymbol{x}_d$.

Desired trajectories of class (A) satisfy the uncontrolled state equation

$$\dot{\boldsymbol{x}}_d(t) = \boldsymbol{R}\left(\boldsymbol{x}_d(t)\right). \tag{112}$$

This constitutes the most specific class of desired trajectories. Because of Eq. (112), the constraint equation (32) is trivially satisfied and the control signal given by Eq. (34) vanishes, $\boldsymbol{u}(t) = \mathbf{0}$. Class (A) encompasses several important control tasks, as e.g. the stabilization of unstable stationary states and periodic orbits [25, 26]. Only for desired trajectories of class (A) it is possible to find non-invasive controls. Non-invasive control signals vanish upon achieving the control target. The open-loop control approach developed here cannot be employed to trajectories of class (A). Instead, the stabilization of unstable solutions to uncontrolled systems requires feedback control.
Desired trajectories of class (B) satisfy the constraint equation (32) and yield a non-vanishing control signal $\boldsymbol{u}(t) \neq \mathbf{0}$. The approach developed here applies to this class. Several other techniques developed in mathematical control theory as e.g. feedback linearization and differential flatness, also work with this class of desired trajectories [2, 27]. Class (B) contains the desired trajectories from class (A) as a special case. For desired trajectories of class (A) and class (B), the solution of the controlled state is simply given by $\boldsymbol{x}(t) = \boldsymbol{x}_d(t)$.
Finally, class (C) is the most general class of desired trajectories and contains class (A) and (B) as special cases. In general, these desired trajectories do not satisfy the constraint equation,

$$\mathbf{0} \neq \boldsymbol{\mathcal{Q}}\left(\boldsymbol{x}_d(t)\right)\left(\dot{\boldsymbol{x}}_d(t) - \boldsymbol{R}\left(\boldsymbol{x}_d(t)\right)\right), \tag{113}$$

such that the approach developed here cannot be applied to all desired trajectories of class (C). A general expression for the control signal in terms of the desired trajectory $\boldsymbol{x}_d(t)$ is not available. Furthermore, the solution for the controlled state trajectory $\boldsymbol{x}(t)$ is usually not simply given by $\boldsymbol{x}_d(t)$, $\boldsymbol{x}(t) \neq \boldsymbol{x}_d(t)$. A solution to control problems defined by class (C) does not only consist in finding an expression for the control signal, but also involves finding a solution for the controlled state $\boldsymbol{x}(t)$. One possible method to solve such control problems is optimal trajectory tracking. This technique is concerned with minimizing the distance between $\boldsymbol{x}(t)$ and $\boldsymbol{x}_d(t)$ in function space as measured by the functional

$$\mathcal{J} = \frac{1}{2} \int_{t_0}^{t_1} \mathrm{d}t \left(\boldsymbol{x}(t) - \boldsymbol{x}_d(t)\right)^2 + \frac{\epsilon^2}{2} \int_{t_0}^{t_1} \mathrm{d}t \left(\boldsymbol{u}(t)\right)^2. \tag{114}$$

The functional $\mathcal{J}$ is to be minimized subject to the constraint that $\boldsymbol{x}(t)$ is given as the solution to the controlled dynamical system (3). The regularization term with small coefficient $0 < \epsilon \ll 1$ ensures the existence of solutions for $\boldsymbol{x}(t)$ and $\boldsymbol{u}(t)$ within appropriate function spaces [3]. For $\epsilon \to 0$, the state as well as the control may diverge [14] and the optimization procedure becomes a so-called singular optimal control problem [3]. However, because exactly realizable desired trajectories satisfy $\boldsymbol{x}(t) = \boldsymbol{x}_d(t)$, they can be viewed as bounded solutions to unregularized ($\epsilon = 0$) optimal trajectory tracking problems.

The linearizing assumption defines a class of nonlinear control systems which essentially behave like linear control system. Systems satisfying the linearizing assumption allow exact analytical solutions in closed form even if no analytical solutions for the uncontrolled system exist. The linearizing assumption uncovers a hidden linear structure underlying nonlinear open-loop control systems. Similarly, feedback linearization defines a huge class of nonlinear control systems possessing an underlying linear structure [2]. The class of feedback linearizable systems contains the systems satisfying the linearizing assumption as a trivial case. However, the linearizing assumption defined here goes much further than feedback linearization. We were able to apply the relatively simple notion of controllability in terms of a rank condition to systems satisfying the linearizing assumption. This is a direct extension of the properties of linear control systems to a class of nonlinear control systems. Furthermore, we may combine the linearizing assumption with the viewpoint that exactly realizable trajectories solve an unregularized optimal control problem. This reveals the possibility of linear structures underlying nonlinear optimal trajectory tracking in the limit of vanishing regularization parameter $\epsilon \to 0$ [14].

Finally, we mention a possible extension of the ideas expounded here to spatio-temporal systems. While generalizing the notion of controllability to spatiotemporal systems encounters difficulties due to an infinite-dimensional state space, generalizing the notion of an exactly realizable trajectory is straightforward [14]. We applied these ideas in a slightly different form to control the position, orientation, and shape of wave patterns in reaction-diffusion systems in [28, 29, 30, 31].

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