On Switching Stabilizability for Continuous-Time Switched Linear Systems

Yueyun Lu and Wei Zhang

***Abstract*—This technical note studies switching stabilization problems for continuous-time switched linear systems. We consider four types of switching stabilizability defined under different assumptions on the switching control input. The most general switching stabilizability is defined as the existence of a measurable switching signal under which the resulting time-varying system is asymptotically stable. Discrete switching stabilizability is defined similarly but requires the switching signal to be piecewise constant on intervals of uniform length. In addition, we define feedback stabilizability in Filippov sense (respectively, sample-and-hold sense) as the existence of a feedback law under which closed-loop Filippov solution (respectively, sample-and-hold solution) is asymptotically stable. It is proved that the four switching stabilizability notions are equivalent and their *sufficient and necessary* condition is the existence of a piecewise quadratic control-Lyapunov function that can be expressed as the pointwise minimum of a finite number of quadratic functions.**

***Index Terms*—Control-Lyapunov function, sliding motion, switching stabilization.**

I. Introduction

This technical note studies switching stabilization problems for continuous-time switched linear systems (SLSs). Existing works in this area mostly focus on deriving sufficient conditions for switching stabilizability. These conditions often guarantee the existence of certain forms of control-Lyapunov functions (CLFs). Examples include quadratic CLFs [1], piecewise quadratic CLFs [2], composite CLFs that are obtained by taking the pointwise min, or pointwise max, or convex hull of a finite number of quadratic functions [3]. Despite the extensive results on sufficient conditions, establishing effective necessary conditions for switching stabilizability remains an open problem of fundamental importance.

To establish necessary conditions, it is important to note that switching stabilizability can be defined in many ways depending on the assumptions on the switching control input σ. One can require σ to be piecewise constant [4], or to have an average or minimum dwell time bigger than some threshold value [5], or to be generated by a state-feedback switching law [3]. Among the cases using feedback switching laws, switching stabilizability depends further on the solution notion used to define closed-loop trajectories, such as classical solution, Caratheodory solution, Filippov solution, or sample-and-hold solution [6]. Therefore, the study of switching stabilizability depends crucially on the assumptions on the admissible switching input and the adopted solution notion. Unfortunately, the complication arising from different definitions of switching stabilizability has not been adequately studied in the literature.

We consider four types of switching stabilizability. The most general *switching stabilizability* is defined as the existence of a measurable switching signal under which the resulting time-varying system is asymptotically stable. *Discrete switching stabilizability* is then defined by admitting only piecewise constant signals with switching intervals of uniform length. On the other hand, we also consider switching stabilizability under state-feedback switching laws. We call a SLS *feedback stabilizable in Filippov sense* (resp. *sample-and-hold sense*) if there exists a feedback law under which closed-loop Filippov solution (resp. sample-and-hold solution) is asymptotically stable.

We will introduce and study all the four switching stabilizability notions. The main contribution is the equivalence of the following statements for a continuous-time SLS:

i) The system is switching stabilizable;
ii) The system is feedback stabilizable in Filippov sense;
iii) The system is feedback stabilizable in sample-and-hold sense with bounded sampling rate;
iv) The system is discrete switching stabilizable;
v) There exists a piecewise quadratic CLF that can be expressed as the pointwise minimum of a finite number of quadratic functions.

The above result represents a significant contribution to the field of switched systems. Most existing works focus on developing sufficient conditions for feedback stabilizability in Filippov sense [1]–[3], [7], some of which even need to exclude sliding motions [2]. In fact, sufficient and necessary conditions are not available even for the well studied feedback stabilization problems in Filippov sense, not to mention other stabilizability notions. In contrast, we prove a unified sufficient and necessary condition for all the four switching stabilizability definitions. The result provides a fundamental insight that the class of piecewise quadratic CLFs is sufficiently rich to study switching stabilization problems under various assumptions on the switching control input. It justifies many existing works that have adopted quadratic or piecewise quadratic CLFs for simplicity or heuristic reasons [1]–[3], [7].

II. Switching Stabilizability Definitions

We consider the following continuous-time switched linear system (SLS):

$$\dot{x}(t) = A_{\sigma(t)}x(t), \quad \sigma(t) \in \mathcal{Q} \triangleq \{1, \dots, M\} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ denotes the continuous state of the system, $\sigma(t)$ denotes the switching control signal that determines the active subsystem at time $t \in \mathbb{R}_+$, and $\{A_i\}_{i\in\mathcal{Q}}$ are constant matrices. Note that for any measurable switching signal $\sigma : \mathbb{R}_+ \to \mathcal{Q}$, the overall switched vector field, $f(t, x(t)) \triangleq A_{\sigma(t)}x(t)$, is time-varying and continuous in state $x(t)$, for which a Caratheodory solution always exists [6, Proposition S1]. We

Manuscript received May 19, 2015; revised October 20, 2015; accepted December 28, 2015. Date of publication January 5, 2016; date of current version October 25, 2016. This work was supported in part by the National Science Foundation under Grant ECCS-1309569. Recommended by Associate Editor M. Margaliot.

The authors are with the Department of Electrical and Computer Engineering, Ohio State University, Columbus, OH 43210 USA (e-mail: lu.692@osu.edu; zhang.491@osu.edu).

Digital Object Identifier 10.1109/TAC.2016.2514844

denote $x(\cdot; z, \sigma) : \mathbb{R}_+ \to \mathbb{R}^n$ as a Caratheodory solution of system (1) under a measurable switching signal σ with initial state $z \in \mathbb{R}^n$.

The study of switching stabilizability depends crucially on the assumptions on the switching input. The switching input can be restricted to certain class of time-domain signals, or can be generated by certain class of state-feedback laws. We will consider both cases. Let $\mathcal{S}_m$ be the set of measurable switching signals, $\mathcal{S}_p$ be the set of piecewise constant switching signals. Denoted by $\mathcal{S}_p[\tau_D]$ the set of switching signals with interval between consecutive discontinuities no smaller than τ_D. Let $\mathcal{S}_p^+ \triangleq \cup_{\tau_D \in \mathbb{R}_+} \mathcal{S}_p[\tau_D]$. The most general definition of switching stabilizability is defined on the set of measurable switching signals $\mathcal{S}_m$.

Definition 1 (Switching Stabilizability): System (1) is called *switching stabilizable* if for each $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $\|z\| < \delta$, there exists a measurable $\sigma \in \mathcal{S}_m$ under which the state trajectory $x(\cdot; z, \sigma)$ satisfies $\|x(t; z, \sigma)\| < \epsilon$, for all $t \in \mathbb{R}_+$ and $x(t; z, \sigma) \to 0$ as $t \to \infty$.

Definition 1 is very general in the sense that it considers all measurable switching signals. If we focus on state-feedback switching laws, the definition of switching stabilizability depends further on the adopted solution notion of the closed-loop system. Assume that the state $x(t)$ is available at all time $t \in \mathbb{R}_+$, and the switching control is determined through a state-feedback switching law $\nu : \mathbb{R}^n \to \mathcal{Q}$. Then the corresponding closed-loop system can be written as

$$\dot{x}(t) = A_{\nu(x(t))} x(t). \tag{2}$$

Although each subsystem vector field is continuous, the switching law ν may introduce discontinuities in the closed-loop vector field. In general, the differential equation (2) may not have a classical or Caratheodory solution [6]. Filippov solution notion [8] is often adopted to handle the discontinuities on the right hand side of (2). We denote $x(\cdot; z, \nu) : \mathbb{R}_+ \to \mathbb{R}^n$ as a Filippov solution [6, p.13–14] of the closed-loop system (2) under a measurable switching law ν with initial state $z \in \mathbb{R}^n$. Switching stabilizability can also be defined as the existence of a switching law under which the closed-loop system is asymptotically stable in the Filippov sense.

Definition 2 (Feedback Stabilizability in Filippov Sense): System (1) is called *feedback stabilizable in Filippov sense* if there exists a measurable switching law $\nu : \mathbb{R}^n \to \mathcal{Q}$ such that for each $\epsilon > 0$, there exists a $\delta > 0$ for which whenever $\|z\| < \delta$, any closed-loop Filippov trajectory $x(\cdot; z, \nu)$ satisfies that $\|x(t; z, \nu)\| < \epsilon$, $\forall\, t \in \mathbb{R}_+$, $x(t; z, \nu) \to 0$ as $t \to \infty$.

Definition 2 is very useful for switching stabilization problems due to the crucial importance of Filippov solution to switched systems. It includes trajectories with sliding motions, which are elegant abstractions of trajectories of the nonsmooth closed-loop system. In fact, most existing studies on switching stabilization adopt Definition 2 to derive various sufficient conditions for switching stabilizability. Sample-and-hold (abbrev. S-H) solution (or π-solution) is another widely used solution notion for discontinuous dynamical systems [6, p.22]. We denote $x_\pi(\cdot; z, \nu)$ as the π-solution of the closed-loop system (2) under a measurable switching law ν with initial state $z \in \mathbb{R}^n$. One may interpret S-H solution as representing the behavior of sampling under a fixed feedback law. The feedback control is evaluated only at sampling times with the values being held until the next sampling time. Feedback stabilizability in the context of S-H solution means asymptotic stability of the sampled closed-loop system, which in general may involve an unbounded sampling rate as the trajectory approaches to the origin. In this technical note, we are interested in the case where asymptotic stability can be obtained by sampling with bounded rate (i.e., nonvanishing intersampling time).

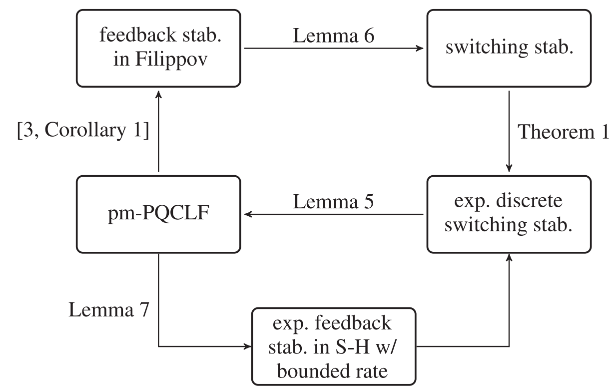

Fig. 1. "stab."—stabilizability; "exp."—exponential; pm-PQCLF: Definition 6.

Definition 3 (Feedback Stabilizability in S-H Sense With Bounded Sampling Rate): System (1) is called *feedback stabilizable in S-H sense with bounded sampling rate* if there exists a feedback law $\nu : \mathbb{R}^n \to \mathcal{Q}$ and a constant $h_0 > 0$ such that whenever $d(\pi) < h_0$, the closed-loop π-trajectory $x_\pi(\cdot; z, \nu)$ satisfies $\forall\, \epsilon > 0, \exists \delta > 0$ such that whenever $\|z\| < \delta$, $\|x_\pi(t; z, \nu)\| < \epsilon, \forall\, t \in \mathbb{R}_+$ and $x_\pi(t; z, \nu) \to 0$ as $t \to \infty$.

Switching stabilizability defined in Definition 3 clearly implies the existence of a piecewise constant stabilizing signal $\sigma \in \mathcal{S}_p[h]$ for all $h \in (0, h_0)$. This is different, but closely related to the discrete switching stabilizability defined below.

Definition 4 (Discrete Switching Stabilizability): System (1) is called *discrete switching stabilizable* if there exists a constant $h_0 > 0$ such that for any $h \in (0, h_0)$, there exists a $\sigma : \mathbb{R}_+ \to \mathcal{Q}$ with $\sigma(t) = \sigma_k \in \mathcal{Q}, \forall\, t \in [kh, (k+1)h), \forall\, k \in \mathbb{N}$ under which the state trajectory $x(\cdot; z, \sigma)$ satisfies $\forall\, \epsilon > 0, \exists \delta > 0$ such that $\|z\| < \delta$ implies that $\|x(t; z, \sigma)\| < \epsilon, \forall\, t \in \mathbb{R}_+$ and $x(t; z, \sigma) \to 0$ as $t \to \infty$.

We call the stabilizability in Definition 1, 2, 3 and 4 *exponential* if $\exists C, \gamma > 0$ so that the solution $x(\cdot) : \mathbb{R}_+ \to \mathbb{R}^n$ with $x(0) = z$ satisfies $\|x(t)\| \le C e^{-\gamma t} \|z\|, \forall\, t \in \mathbb{R}_+, \forall\, z \in \mathbb{R}^n$.

The goal of this technical note is to show the four switching stabilizability definitions are all equivalent to the existence of a piecewise quadratic control-Lyapunov function (CLF). Furthermore, such a CLF can be expressed as the pointwise minimum of a finite number of quadratic functions. To better unfold the proof of the main result, we first give a preview of the logical flow of all the arguments we will establish. The rest of the technical note is then dedicated to show the diagram in Fig. 1 commutes, from which the equivalence of all the key concepts can be concluded.

III. Connection to Discrete Switching Stabilizability

The goal of this section is to show that the general switching stabilizability defined in Definition 1 implies exponential discrete switching stabilizability (Definition 4).

It is well known that asymptotic controllability implies feedback stabilizability in S-H sense for general nonlinear control systems [9]. However, such a result cannot be directly applied to switched systems as the open-loop vector field is required to be continuous in control. In fact, even if we have such a result for switched systems, it still does not imply discrete switching stabilizability due to the possibly unbounded growth of sampling rate close to the origin. As a result, sampling interval will vanish and the corresponding discrete-time system is not well defined. Therefore, nonvanishing intersampling time is essential for establishing the connection to discrete-time switching stabilization problems.

In general, intersampling time has to tend to zero to stabilize the sampled closed-loop system. One exception is homogeneous system whose open-loop vector field satisfies $g(ax,u) = ag(x,u), \forall a \geq 0$. For such systems, it is shown in [10] that asymptotic controllability implies feedback stabilizability in S-H sense with bounded sampling rate. However, the result cannot be directly applied here as g is required to be continuous in both x and u in [10], while the open-loop vector field of system (1) is not continuous in σ. To deal with the discontinuities due to the switching control σ, we consider a relaxed system that is continuous in control.

The technique of embedding switched system into a larger family of nonlinear systems with relaxed continuous control has been used to solve switched optimal control problems [11]. It is shown by the so-called chattering lemma that trajectories of the relaxed system can be approximated by those of the switched system with error bound of arbitrary accuracy. Our derivation of the connection between continuous-time and discrete-time switching stabilizability is also based on such embedding technique. It turns out that we require an error bound that is much stronger than the one provided by the chattering lemma in [11]. Next, we will prove a new chattering lemma for switching stabilization problems.

Denote $\mathcal{U}_p \triangleq \{\alpha \in \{0,1\}^M : \sum_{i=1}^M \alpha_i = 1\}$ and $\mathcal{U}_r \triangleq \{\alpha \in [0,1]^M : \sum_{i=1}^M \alpha_i = 1\}$. We refer to system (1) as a pure system $(\mathcal{P})$, which can be equivalently written as

$$(\mathcal{P}) : \dot{x}(t) = \sum_{i \in \mathcal{Q}} \alpha_i(t) A_i x(t), \quad \alpha(t) \in \mathcal{U}_p.$$

Define the corresponding relaxed system $(\mathcal{R})$ as

$$(\mathcal{R}) : \dot{x}(t) = \sum_{i \in \mathcal{Q}} \alpha_i(t) A_i x(t), \quad \alpha(t) \in \mathcal{U}_r.$$

Let $x(\cdot; z, \alpha^p) : \mathbb{R}_+ \to \mathbb{R}^n$ be the state trajectory of $(\mathcal{P})$ under a pure control signal $\alpha^p : \mathbb{R}_+ \to \mathcal{U}_p$ and $x(\cdot; z, \alpha^r) : \mathbb{R}_+ \to \mathbb{R}^n$ be the state trajectory of $(\mathcal{R})$ under a relaxed control signal $\alpha^r : \mathbb{R}_+ \to \mathcal{U}_r$. We call a relaxed control signal $\alpha^r : \mathbb{R}_+ \to \mathcal{U}_r$ *exponentially stabilizing* if $\exists C, \gamma > 0$ s.t. $\|x(t; z, \alpha^r)\| \leq Ce^{-\gamma t}\|z\|, \forall t \in \mathbb{R}_+, \forall z \in \mathbb{R}^n$. The new chattering lemma proves an error bound proportional to the norm of initial state.

Lemma 1: For any exponentially stabilizing relaxed control signal $\alpha^r : [0,T] \to \mathcal{U}_r$ and any $\epsilon > 0$, there exists a pure control signal $\alpha^p : [0,T] \to \mathcal{U}_p$ where $\alpha^p \in \mathcal{S}_p^+$ such that $\|x(t; z, \alpha^p) - x(t; z, \alpha^r)\| < \epsilon\|z\|, \forall t \in [0,T], \forall z \in \mathbb{R}^n$.

Proof: Denote $\phi(t) \triangleq x(t; z, \alpha^p)$ and $\tilde{\phi}(t) \triangleq x(t; z, \alpha^r)$. Given relaxed control signal $\alpha^r : [0,T] \to \mathcal{U}_r, \epsilon > 0$ and initial state $z \in \mathbb{R}^n$, the goal is to construct a pure control signal $\alpha^p : [0,T] \to \mathcal{U}_p$ where $\alpha^p \in \S_p^+$ such that $\|\phi(t) - \tilde{\phi}(t)\| < \epsilon\|z\|, \forall t \in [0,T]$. We first partition $[0,T]$ into equal length subintervals and then apply the following construction strategy for each subinterval. Let $h > 0$ be the length of subinterval (we will decide its upper bound later). On each subinterval $[kh, (k+1)h), k \in \mathbb{N}$, α^p sequentially takes value from the set $\mathcal{U}_p$ of M elements, i.e.,

$$\alpha_i^p(t) = \begin{cases} 1, & t \in [t_{k,i-1}, t_{k,i}) \\ 0, & \text{otherwise} \end{cases} \quad \forall i = 1, \cdots, M \tag{3}$$

where $t_{k,0} = kh$ and $t_{k,i}$ are defined recursively by

$$t_{k,i} = t_{k,i-1} + \int_{kh}^{(k+1)h} \alpha_i^r(\tau) d\tau, \forall i = 1, \cdots, M. \tag{4}$$

By construction, $\Delta t_{k,i} \triangleq t_{k,i} - t_{k,i-1} > 0, \forall k \in \mathbb{N}, i \in \mathcal{Q}$ and thus $\alpha^p \in \S_p^+$. Similar as the proof in [11], the error can be divided into two terms, i.e., $\|\phi(t) - \tilde{\phi}(t)\| \leq E_1 + E_2$, where

$$E_1 \triangleq \left\| \int_0^t \sum_{i=1}^M \alpha_i^p(\tau) A_i \big(\phi(\tau) - \tilde{\phi}(\tau)\big) d\tau \right\|$$

$$E_2 \triangleq \left\| \int_0^t \sum_{i=1}^M \big(\alpha_i^p(\tau) - \alpha_i^r(\tau)\big) A_i \tilde{\phi}(\tau) d\tau \right\|.$$

Next, we derive the upper bounds for E_1 and E_2. By matrix norm inequality and $\alpha^p \in \mathcal{U}_p$,

$$\begin{aligned} E_1 &\leq \int_0^t \sum_{i=1}^M \|\alpha_i^p(\tau) A_i \big(\phi(\tau) - \tilde{\phi}(\tau)\big)\| d\tau \\ &\leq \int_0^t \sum_{i=1}^M \alpha_i^p(\tau) \|A_i\| \|\phi(\tau) - \tilde{\phi}(\tau)\| d\tau \\ &\leq L_1 \int_0^t \|\phi(\tau) - \tilde{\phi}(\tau)\| d\tau, \text{ where } L_1 \triangleq \max_{i \in \mathcal{Q}} \|A_i\|. \end{aligned}$$

Due to the construction of α^p in (3), we have i) $\int_{kh}^{(k+1)h} \sum_{i=1}^M \alpha_i^p(\tau) A_i \tilde{\phi}(\tau) d\tau = \sum_{i=1}^M \int_{t_{k,i-1}}^{t_{k,i}} A_i \tilde{\phi}(\tau) d\tau$. It follows from (4) that ii) $\int_{[t_{k,i-1}, t_{k,i})} (1 - \alpha_i^r(\tau)) d\tau = \int_{[kh,(k+1)h) \setminus [t_{k,i-1}, t_{k,i})} \alpha_i^r(\tau) d\tau$. Let $\tilde{\phi}^{\Delta}(t) \triangleq \tilde{\phi}(t) - \tilde{\phi}(t_k)$. Since α^r is exponentially stabilizing, $\exists\ C > 0$ s.t. $\|\tilde{\phi}(t)\| \leq C\|z\|, \forall t \in \mathbb{R}_+$ and thus iii) $\|\tilde{\phi}^{\Delta}(t)\| \leq hL_1 C\|z\|, \forall t \in [kh, (k+1)h)$. Based on i), ii) and iii),

$$\begin{aligned} E_2 &\leq \sum_k \left\| \int_{kh}^{(k+1)h} \sum_{i=1}^M \big(\alpha_i^p(\tau) A_i \tilde{\phi}(\tau) - \alpha_i^r(\tau) A_i \tilde{\phi}(\tau)\big) d\tau \right\| \\ &\leq \sum_k \sum_{i=1}^M \left\| \int_{[t_{k,i-1}, t_{k,i})} (1 - \alpha_i^r(\tau)) A_i \tilde{\phi}(\tau) d\tau - \right. \\ &\qquad \left. \int_{[kh,(k+1)h) \setminus [t_{k,i-1}, t_{k,i})} \alpha_i^r(\tau) A_i \tilde{\phi}(\tau) d\tau \right\| \\ &\leq \sum_k \sum_{i=1}^M \int_{kh}^{(k+1)h} \|A_i\| \|\tilde{\phi}^{\Delta}(\tau)\| d\tau \leq \frac{T}{h} M h^2 L_1^2 C \|z\|. \end{aligned}$$

Let $\kappa \triangleq TML_1^2 C$. By choosing $h < \frac{\epsilon}{\kappa} e^{-L_1 T}$, the rest of the proof follows from Gronwall inequality. □

Remark 1: The new chattering lemma differs from the original version in the following aspects: i) The error bound is any desired accuracy times the norm of initial state rather than just the desired accuracy; ii) The choice of switching signals is from the set $\S_p^+$ rather than the set $\S_m$; iii) It is under the assumption of relaxed control signal being exponentially stabilizing. In fact, the above three properties play important roles in establishing the connection to exponential discrete switching stabilizability.

The relaxed system $(\mathcal{R})$ is a homogeneous system, whose vector field is continuous with respect to both state and the control input α^r. It is proved in [10, Proposition 4.4] that asymptotic controllability of homogeneous system implies exponential stability of sampled closed-loop system with sufficiently small but nonvanishing intersampling time. The existence of exponentially stable trajectories of $(\mathcal{R})$ allows us to construct exponentially stabilizing switching signals from the set $\mathcal{S}_p^+$ based on Lemma 1.

Lemma 2: If system (1) is switching stabilizable, then it is exponentially switching stabilizable under a switching signal $\sigma : \mathbb{R}_+ \to \mathcal{Q}$ where $\sigma \in \mathcal{S}_p^+$.

Proof: Consider the pure system $(\mathcal{P})$ and the relaxed system $(\mathcal{R})$ defined before. Obviously, $(\mathcal{R})$ is asymptotically controllable given $(\mathcal{P})$ is switching stabilizable. Furthermore, $(\mathcal{R})$ is exponentially feedback stabilizable in S-H sense with bounded sampling rate [10, Proposition 4.4]. Let $\nu:\mathbb{R}^n\to\mathcal{U}_r$ be the stabilizing feedback law of $(\mathcal{R})$. The goal is to find an exponentially stabilizing signal $\sigma\in\mathcal{S}_p^+$ of $(\mathcal{P})$. We now fix a nonvanishing sampling schedule $\pi=\{t_k\}_{k\in\mathbb{N}}$ and consider a relaxed control signal defined as $\alpha^r(t)=\nu(x_\pi(t_k;z,\nu))$, $\forall\, t\in[t_k,t_{k+1}),\forall\, k\in\mathbb{N}$. As α^r is exponentially stabilizing, there exists $C>0,\gamma>0$ such that $\|x(t;z,\alpha^r)\|\le Ce^{-\gamma t}\|z\|,\forall\, z\in\mathbb{R}^n$, $\forall\, t\in\mathbb{R}_+$. Let the finite horizon $T>(2\log(2C)/\gamma)$ and $\epsilon=Ce^{-\gamma T}$. By Lemma 1, we can construct a pure control signal $\alpha^{p,0}:[0,T]\to\mathcal{U}_p$ where $\alpha^{p,0}\in\mathcal{S}_p^+$ such that $\|x(t;z,\alpha^{p,0})-x(t;z,\alpha^r)\|<\epsilon\|z\|$, $\forall\, t\in[0,T],\forall\, z\in\mathbb{R}^n$. Then, the state trajectory of $(\mathcal{P})$ under $\alpha^{p,0}$ satisfies $\|x(t;z,\alpha^{p,0})\|\le 2Ce^{-\gamma t}\|z\|,\forall\, t\in[0,T]$. We next iteratively apply the bound on intervals of length T to obtain the exponential convergence on $\mathbb{R}_+$. Let $\alpha^p:\mathbb{R}_+\to\mathcal{U}_p$ be the concatenation of $\alpha^{p,k}:[kT,(k+1)T)\to\mathcal{U}_p,k\in\mathbb{N}$. For $t\in[kT,(k+1)T)$, $\|x(t;z,\alpha^p)\|\le(2C)^{k+1}e^{-\gamma kT}\|z\| < e^{-((k/(k+1))\gamma-(\log(2C)/T))t}\|z\|$. In general, for any $t\in\mathbb{R}_+$, $\|x(t;z,\alpha^p)\|<e^{-\gamma' t}\|z\|$ where $\gamma'=(\gamma/2)-(\log(2C)/T)\in(0,\gamma)$. The stabilizing signal $\sigma:\mathbb{R}_+\to\mathcal{Q}$ can be obtained from α^p as follows: $\sigma(t)=i$, if $\alpha_i^p(t)=1,\forall\, t\in\mathbb{R}_+$. As $\alpha^{p,k}\in\mathcal{S}_p^+,\forall\, k\in\mathbb{N}$, we have $\alpha^p\in\mathcal{S}_p^+$ and thus $\sigma\in\mathcal{S}_p^+$. □

Now we have found a switching signal $\sigma\in\mathcal{S}_p^+$ that exponentially stabilizes the system. However, the stabilizing switching signal σ may not have a uniform intersampling time. It remains to show that if we sample the signal with a fixed intersampling time that is sufficiently small and hold the signal until the next sampling, the corresponding state trajectory is also exponentially stable. This will then imply discrete switching stabilizability.

Theorem 1: If system (1) is switching stabilizable, then it is exponentially discrete switching stabilizable.

Proof: By Lemma 2, there exists a switching signal $\sigma_0:\mathbb{R}_+\to\mathcal{Q}$ where $\sigma_0\in\mathcal{S}_p^+$ under which the state trajectory $x(\cdot;z,\sigma_0)$ is exponentially stable, i.e., $\exists C_0>0,\gamma>0$, s.t. $\|x(t;z,\sigma_0)\|\le C_0e^{-\gamma t}\|z\|,\forall\, t\in\mathbb{R}_+,\forall\, z\in\mathbb{R}^n$. Let $\sigma_h:\mathbb{R}_+\to\mathcal{Q}$ be the sampled signal of σ_0 with sampling intervals of uniform length h, i.e., $\sigma_h(t)\triangleq\sigma_0(kh),\forall\, t\in[kh,(k+1)h),\forall\, k\in\mathbb{N}$. Let $\phi_0(t)\triangleq x(t;z,\sigma_0),\phi_h(t)\triangleq x(t;z,\sigma_h)$. The rest of the proof has two ingredients: i) the exponential convergence of the error between ϕ_0 and ϕ_h on a finite horizon and ii) the extension of the exponential convergence of ϕ_h from a finite horizon to $\mathbb{R}_+$.

To show i), one can follow the proof of Lemma 1 by dividing the error into two terms and bounding the first term with integral of the error and the second term with constant times $h\|z\|$. We briefly discuss the second term here. Since $\sigma_0\in\mathcal{S}_p^+$, there are at most $N<\infty$ switches on a finite interval and thus i.1) σ_h and σ_0 differ on intervals of length at most Nh. As σ_0 is exponentially stabilizing, i.2) $\|\phi_0(t)\|\le C_0\|z\|,\forall\, t\in\mathbb{R}_+$. Based on i.1) and i.2), $E_2\triangleq\int_0^t\|(A_{\sigma_h(\tau)}-A_{\sigma_0(\tau)})\phi_0(\tau)\|d\tau\le L_2NhC_0\|z\|$.

To show ii), one can follow the proof of Lemma 2 by iteratively applying the bound on intervals of length T. By i), for sufficiently small h, $\|\phi_h(t)\|\le 2C_0e^{-\gamma t}\|z\|,\forall\, t\in[0,T],z\in\mathbb{R}^n$. By choosing $T>(2\log(2C_0)/\gamma)$, $\|\phi_h(t)\|<e^{-\gamma' t}\|z\|,\forall\, t\in\mathbb{R}_+,\forall\, z\in\mathbb{R}^n$ where $\gamma'=(\gamma/2)-(\log(2C_0)/T)\in(0,\gamma)$. □

The above theorem indicates that switching stabilizability implies exponential switching stabilizability of discrete-time systems obtained by sampling the original system with sufficiently small and fixed intersampling time. Although such a result appears to be natural, its proof is highly nontrivial due to the possibility of wild behaviors of a measurable stabilizing switching signal $\sigma\in\mathcal{S}_m$ and the discontinuity of the switched vector field with respect to the switching input σ. In fact, the result does not hold for general switched nonlinear systems, for which the existence of a stabilizing switching signal $\sigma\in\mathcal{S}_m$ does not imply the existence of a $\sigma\in\mathcal{S}_p^+$ with switching intervals of uniform length.

IV. Converse Control-Lyapunov Function Theorem for Switched Linear Systems

In this section, we will develop a converse CLF theorem for the switching stabilizability in Definition 1 where the switching control σ is only required to be measurable. This is more general than the definition used in many other works [13] for SLSs that require σ to be piecewise constant.

A. Control-Lyapunov Function

Control-Lyapunov function (CLF) is an important tool to study stabilization problems. This technical note focuses on an important class of nonsmooth CLFs, namely, pointwise minimum piecewise quadratic CLFs.

Definition 5 (Pointwise Minimum Piecewise Quadratic Function (pm-PQF)): Let $P_j,j\in\mathbb{N}_m$ be symmetric matrices, i.e., $P_j^T=P_j,\forall\, j\in\mathbb{N}_m$. The function defined by

$$V(x)\triangleq\min_{j\in\mathbb{N}_m}x^TP_jx,\quad x\in\mathbb{R}^n \tag{5}$$

is called a pm-PQF if $\Omega_j\neq\emptyset$, $\forall\, j\in\mathbb{N}_m$, where $\Omega_j\triangleq\{x\in\mathbb{R}^n: x^TP_jx<x^TP_kx,\forall\, k\neq j\}$.

A pm-PQF is clearly a piecewise smooth function, for which directional derivative exists everywhere [14, p.43].

Lemma 3 ([14]): For any pm-PQF $g:\mathbb{R}^n\to\mathbb{R}$, the limit $Dg(x;\eta)\triangleq\lim_{\delta\downarrow 0}(1/\delta)(g(x+\delta\eta)-g(x))$ exists, $\forall\, x,\eta\in\mathbb{R}^n$.

We are now ready to define CLFs based on pm-PQFs where conditions are given in terms of directional derivative.

Definition 6 (Pointwise Minimum Piecewise Quadratic Control-Lyapunov Function (pm-PQCLF)): A pm-PQF $V:\mathbb{R}^n\to\mathbb{R}_+$ is called a pm-PQCLF if there exists a continuous function $W:\mathbb{R}^n\to\mathbb{R}_+$ such that the following conditions hold:

$$V(x)>0,W(x)>0,\forall\, x\neq 0,\quad V(0)=0 \tag{6}$$

$$\mathcal{L}_\beta=\{x: V(x)\le\beta\}\text{ is bounded for each }\beta \tag{7}$$

$$\min_{i\in\mathcal{Q}}DV\left(x;f_i(x)\right)\le -W(x),\quad\forall\, x\in\mathbb{R}^n. \tag{8}$$

For a discrete-time SLS, it has been shown that switching stabilizability implies the existence of a pm-PQCLF [15]. We will prove a similar converse pm-PQCLF theorem for continuous-time switching stabilizability. The proof relies on the connection between continuous-time and discrete-time switching stabilizability established in the previous section. Consider the discrete-time switched linear system (DTSLS) obtained by sampling system (1) with intervals of length h:

$$x(k+1)=e^{A_{\sigma(k)}h}x(k),\quad\sigma(k)\in\mathcal{Q},k\in\mathbb{N}. \tag{9}$$

Denote $x^h(\cdot;z,\sigma):\mathbb{N}\to\mathbb{R}^n$ as the solution of DTSLS (9) under a switching sequence $\sigma:\mathbb{N}\to\mathcal{Q}$ with initial state $z\in\mathbb{R}^n$. As shown in [16], pm-PQCLFs for DTSLSs can be constructed from finite-horizon value function defined below.

Definition 7 (Value Function): Denoted by $J_N^h(z,\sigma)\triangleq\sum_{k=0}^N\|x^h(k;z,\sigma)\|^2$ the N-horizon cost function of system (9) with initial state z under switching sequence $\sigma=\{\sigma_k\}_{k=0}^N$. The N-horizon value function of system (9) is defined as $V_N^h(z)=\min_\sigma J_N^h(z,\sigma)$.

It can be easily shown that the value function is a pm-PQF.

Lemma 4 ([16]): The N-horizon value function of system (9) takes the form of $V_N^h(z) = \min_{P \in \mathcal{H}_N} z^T P z$ where $\mathcal{H}_N$ is a finite set of positive definite matrices.

The converse result for switching stabilizability of DTSLSs is developed in terms of finite-horizon value functions. It suggests that the finite-horizon value function V_N^h will eventually become a pm-PQCLF as the horizon N increases.

Theorem 2 ([15]): If system (9) is exponentially switching stabilizable, there exist constants $N_0 < \infty, \kappa > 0$ such that for any $N \geq N_0$, the N-horizon value function V_N^h satisfies

$$\min_{i \in \mathcal{Q}} \left\{ V_N^h\left(e^{A_i h} z\right) - V_N^h(z) \right\} \leq -\kappa \|z\|, \quad \forall z \in \mathbb{R}^n. \tag{10}$$

Note that condition (10) can be considered as a discrete-time version of condition (8). As we will show next, the former implies the latter by proper choice of h and N.

B. Converse pm-PQCLF Theorem

We now develop a converse CLF result for the most general switching stabilizability (Definition 1). According to Theorem 1, switching stabilizability implies exponential switching stabilizability of a collection of DTSLSs (9) with sufficiently small h. Then, Theorem 2 ensures that the finite-horizon value function V_N^h is a pm-PQCLF for DTSLS (9). We want to show that V_N^h is also a pm-PQCLF for system (1). The main challenge lies in the dependency of V_N^h on h.

Lemma 5: If system (1) is exponentially discrete switching stabilizable, it admits the finite-horizon value function V_N^h with sufficiently small h and sufficiently large N as a pm-PQCLF.

Proof: Obviously, V_N^h satisfies condition (6) and (7). We are left to show that it also satisfies the decreasing condition (8). By the assumption of exponential discrete switching stabilizability, there exist constants $h_0 > 0, C > 0, \gamma > 0, \kappa > 0$ such that for any DTSLS (9) with $h \in (0, h_0)$, there exists a switching sequence σ under which the state trajectory satisfies $\|x^h(k; z, \sigma)\| \leq Ce^{-\gamma h k}\|z\|, \forall z \in \mathbb{R}^n, \forall k \in \mathbb{N}$. Furthermore, there exists a $N < \infty$ such that $\min_{i \in \mathcal{Q}}\{V_N^h(e^{A_i h} z) - V_N^h(z)\} \leq -\kappa h \|z\|, \forall z \in \mathbb{R}^n$. Since $V_N^h = \min_{P \in \mathcal{H}_N} z^T P z$

$$\min_{i \in \mathcal{Q}} \left\{ z^T \left(e^{A_i h}\right)^T P' e^{A_i h} z - z^T P z \right\} \leq -\kappa h \|z\|, \text{ where}$$

$$P \triangleq \underset{P \in \mathcal{H}_N}{\arg\min}\, z^T P z, P' \triangleq \underset{P \in \mathcal{H}_N}{\arg\min}\, z^T \left(e^{A_i h}\right)^T P e^{A_i h} z.$$

By Taylor expansion, $e^{A_i h} = I + A_i h + o(h^2)$, which gives

$$\min_{i \in \mathcal{Q}} \left\{ z^T (P' - P) z + h z^T \left(A_i^T P' + P' A_i\right) z \right\} \leq -\kappa h \|z\|.$$

Note that $DV_N^h(z; A_i z) = z^T (A_i^T P + P A_i) z$. Then

$$\min_{i \in \mathcal{Q}} DV_N^h(z; A_i z) + \min_{i \in \mathcal{Q}} \left\{ \frac{1}{h} z^T (P' - P) z \right.$$
$$\left. + z^T \left(A_i^T (P' - P) + (P' - P) A_i\right) z \right\} \leq -\kappa \|z\|. \tag{11}$$

Let $\Delta P \triangleq P' - P$. We next discuss the order of $\|P\|$ and $\|\Delta P\|$ for their dependency on h. We claim that i) $\|P\| = O((1/h))$ and ii) $\|\Delta P\| = O(1)$. The proof goes as follows: i) By monotonicity of value function in terms of horizon, $V_N^h(z) = z^T P z = O(1/(1 - e^{-\gamma h})) \cdot \|z\|$. Thus, $\|P\| = O(1/(1 - e^{-\gamma h})) \to O(1/h)$ as $h \to 0$. ii) Again by monotonicity property, $|V_N^h(e^{A_i h} z) - V_N^h(z)| = |z^T (e^{A_i h})^T P' e^{A_i h} z - z^T P z| = O(1/(1 - e^{-\gamma h})) \cdot \|e^{A_i h} z - z\|$. It then gives $\|(e^{A_i h})^T P' e^{A_i h} - P\| = O(1/(1 - e^{-\gamma h})) \cdot O(h)$. Note that $\|(e^{A_i h})^T P' e^{A_i h} - P\| = \|\Delta P + h(A_i^T P' + P' A_i) + o(h^2)\| \geq \|\Delta P\| - h\|A_i^T P' + P' A_i\|$, where the inequality is due to matrix norm triangle inequality. By reorganizing terms, we have $\|\Delta P\| \leq O(1/(1 - e^{-\gamma h})) \cdot O(h) + h\|A_i^T P' + P' A_i\| = O(1/(1 - e^{-\gamma h})) \cdot O(h) + O(h) \cdot O(1/(1 - e^{-\gamma h})) \to O(1)$ as $h \to 0$, where the equality is due to the order of $\|P\|$ discussed in i). Based on the property of pointwise minimum that $z^T \Delta P z \geq 0$ and the fact that $\|\Delta P\| = O(1)$ we just proved, there exists a sufficiently small $h > 0$ such that $z^T((1/h)\Delta P + A_i^T \Delta P + \Delta P A_i) z > 0$, $\forall z \in \mathbb{R}^n, \forall i \in \mathcal{Q}$. In other words, $\min_{i \in \mathcal{Q}}\{(1/h) z^T \Delta P z + z^T (A_i^T \Delta P + \Delta P A_i) z\} \geq 0$. Together with (11) we have $\min_{i \in \mathcal{Q}} DV_N^h(z; A_i z) \leq -\kappa \|z\|, \forall z \in \mathbb{R}^n$, which completes the proof. □

Theorem 3 (Converse pm-PQCLF Theorem): If system (1) is switching stabilizable, then it admits a pm-PQCLF.

Proof: If system (1) is switching stabilizable, then it is exponentially discrete switching stabilizable (Theorem 1). Furthermore, the V_N^h with sufficiently small h and sufficiently large N is a pm-PQCLF for system (1) (Lemma 5). □

Theorem 3 provides a formal justification for many existing works that have adopted quadratic or piecewise quadratic CLFs for simplicity or heuristic reasons [1]–[4], [7], [13], [17]. It allows us to only focus on pm-PQCLFs in the study of switching stabilizability for continuous-time SLSs.

V. Equivalent Characterizations for Switching Stabilizability

The goal of this section is to prove the equivalence of the four switching stabilizability definitions and their sufficient and necessary condition as the existence of a pm-PQCLF. We first introduce several lemmas to show some key pairwise relations among them.

Lemma 6: If system (1) is feedback stabilizable in Filippov sense, then it is switching stabilizable.

Proof: Assume system (1) is feedback stabilizable in Filippov sense. There exist a stabilizing feedback law $\nu : \mathbb{R}^n \to \mathcal{Q}$ and a constant $T > 0$ such that $\|x(t; z, \nu)\| \leq (1/2)\|z\|, \forall t \geq T, \forall z \in \mathbb{R}^n$. We now fix the finite time horizon T and construct a stabilizing switching signal $\sigma : \mathbb{R}_+ \to \mathcal{Q}$ recursively on intervals of length T. Let $\phi(\cdot) \triangleq x(\cdot; z, \sigma) : \mathbb{R}_+ \to \mathbb{R}^n$ be the state trajectory of system (1) under σ. Since the velocity of a Filippov solution can always be written as the convex combination of subsystem vector fields, i.e., $\dot{x}(t; z, \nu) = \sum_{i \in \mathcal{Q}} \alpha_i(t) A_i x(t; z, \nu)$, where $\sum_{i \in \mathcal{Q}} \alpha_i(t) = 1, \forall t \in \mathbb{R}_+$, we can think of $x(\cdot; z, \nu) : \mathbb{R}_+ \to \mathbb{R}^n$ as a stabilizing trajectory of the relaxed system $(\mathcal{R})$. By Lemma 1, $\forall z \in \mathbb{R}^n, \epsilon > 0$, $\exists \sigma(z, \epsilon, \nu) \in \mathcal{S}_p^+$ s.t. $\|x(t; z, \sigma(z, \epsilon, \nu)) - x(t; z, \nu)\| \leq \epsilon \|z\|, \forall t \in [0, T]$, where the parenthesis in $\sigma(z, \epsilon, \nu)$ is used to emphasize the dependency of σ on z, ϵ, ν. Let $\sigma_k \triangleq \sigma|_{[kT, (k+1)T]} : [0, T] \to \mathcal{Q}$ be the restriction of σ on $[kT, (k+1)T]$. Consider the Filippov solution (also a relaxed trajectory) starting from the end point of the trajectory under σ on the last interval, i.e. $x(\cdot; \phi(kT), \nu)$. By assumption, $\|x(T; \phi(kT), \nu)\| \leq (1/2)\|\phi(kT)\|$. By Lemma 1, $\exists \sigma_k : [0, T] \to \mathcal{Q}$ where $\sigma_k \in \mathcal{S}_p^+$ s.t. $\|x(t; \phi(kT), \sigma_k) - x(t; \phi(kT), \nu)\| \leq (1/2^{k+1})\|\phi(kT)\|, \forall t \in [0, T]$. Thus, $\|\phi((k+1)T)\| \leq ((1/2) + (1/2^{k+1}))\|\phi(kT)\|, \forall k \in \mathbb{N}$ where $\phi(0) = z$. By construction, the state trajectory under σ satisfies $\|\phi(t)\| \leq \Pi_{k=0}^{\lfloor t/T \rfloor - 1}((1/2) + (1/2^{k+1}))\|z\| \leq (3/4)^{\lfloor t/T \rfloor - 1}\|z\| \to 0$ as $t \to \infty$. Since $\sigma_k \in \mathcal{S}_p^+, \forall k \in \mathbb{N}$, we verified that $\sigma \in \mathcal{S}_p^+$. □

One sufficient condition for feedback stabilizability in Filippov sense is the existence of a pm-PQCLF provided in [3, Corollary 1]. Combined with the converse pm-PQCLF theorem and the above lemma, we can claim the equivalence of switching stabilizability, feedback stabilizability in Filippov sense and the existence of a pm-PQCLF. It remains to establish their relation to exponential feedback stabilizability in S-H sense with bounded sampling rate.

Lemma 7: If system (1) admits a pm-PQCLF, then it is exponentially feedback stabilizable in sample-and-hold sense with bounded sampling rate.

Proof: Let V be a pm-PQCLF. There exists $0 < C_V^- < C_V^+ < \infty$ such that $C_V^- \|z\|^2 \leq V(z) \leq C_V^+ \|z\|^2, \forall z \in \mathbb{R}^n$. According to condition (8), there exists $\kappa > 0$ such that $\min_{i \in \mathcal{Q}} DV(z; A_i z) \leq -3\kappa V(z), \forall z \in \mathbb{R}^n$. For each $\kappa > 0$, we can find an h_0 such that $0 < h_0 \leq \kappa C_V^- / \max_{i \in \mathcal{Q}, k \in \mathbb{N}_m} \quad \|A_i^T (A_i^T P_k + P_k A_i) + (A_i^T P_k + P_k A_i) A_i\|$ and $1 - 2\kappa h_0 \leq e^{-2\kappa h_0}$. Let the switching law $\nu : \mathbb{R}^n \to \mathcal{Q}$ be $\nu(z) = \arg\min_{i \in \mathcal{Q}} DV(z; A_i z), \forall z \in \mathbb{R}^n$. Consider a sampling schedule $\pi = \{t_k\}_{k \in \mathbb{N}}$ with $d(\pi) < h_0$. It follows from the definition of sample-and-hold solution that for any $\tau \in (0, h_0)$, $V(x_\pi(\tau; z, \nu)) = V(z) + \int_0^\tau DV(e^{A_{\nu(z)} t} z; A_{\nu(z)} e^{A_{\nu(z)} t} z) \ dt = V(z) + \tau DV(e^{A_{\nu(z)} t} z; A_{\nu(z)} e^{A_{\nu(z)} t} z)$ for some $t \in (0, \tau)$, where the last equality is due to Mean Value Theorem. For $0 < t < \tau < h_0$, the directional derivative in the last equation can be bounded by $DV(e^{A_{\nu(z)} t} z; A_{\nu(z)} e^{A_{\nu(z)} t} z) \leq DV(z; A_{\nu(z)} z) + t \cdot V(z)/C_V^- \cdot \|A_{\nu(z)}^T (A_{\nu(z)}^T P_k + P_k A_{\nu(z)}) + (A_{\nu(z)}^T P_k + P_k A_{\nu(z)}) A_{\nu(z)}\| \leq -2\kappa V(z)$. Thus, the value of V along closed-loop π-solution satisfies $V(x_\pi(\tau; z, \nu)) \leq (1 - 2\kappa\tau) V(z) \leq e^{-2\kappa\tau} V(z), \forall z \in \mathbb{R}^n, \forall \tau \in (0, h_0)$. By iteratively applying the above inequality on intervals $[t_k, t_{k+1}], k \in \mathbb{N}$ of length less than h_0, we have $V(x_\pi(t; z, \nu)) \leq e^{-2\kappa t} V(z), \forall t \in \mathbb{R}_+, \forall z \in \mathbb{R}^n$. By the bound of V, $\|x_\pi(t; z, \nu)\| \leq C e^{-\kappa t} \|z\|, \forall t \in \mathbb{R}_+, \forall z \in \mathbb{R}^n$, where $C \triangleq (C_V^+ / C_V^-)^{(1/2)}$. □

We are now ready to state the main result of this technical note, namely, the equivalence among all the four switching stabilizability definitions and the existence of a pm-PQCLF. The proof of the main result is illustrated by the diagram in Fig. 1.

Theorem 4: The following statements are equivalent for continuous-time switched linear system (1):

i) It is switching stabilizable;
ii) It is feedback stabilizable in Filippov sense;
iii) It is exponentially feedback stabilizable in sample-and-hold sense with bounded sampling rate;
iv) It is exponentially discrete switching stabilizable;
v) It admits a pm-PQCLF.

Proof: It suffices to show the diagram in Fig. 1 commutes. All the links have been established. ii) ⇒ i): Lemma 6; i) ⇒ iv): Theorem 1; iv) ⇒ v): Lemma 5; v) ⇒ ii): [3, Corollary 1]; v) ⇒ iii): Lemma 7; iii) ⇒ iv): It trivially holds by choosing sampling schedule π with intersampling time of uniform length. □

VI. Conclusion

This technical note studies switching stabilization problems for continuous-time switched linear systems. We show the equivalence of the four switching stabilizability definitions and the existence of a pm-PQCLF. Such a result unifies the study of switching stabilizability under different assumptions on the switching control input. It also justifies many existing stabilization results that have used piecewise quadratic CLF for simplicity or heuristic reasons. Future work will focus on developing efficient algorithms to construct the proposed pm-PQCLF and the corresponding stabilizing feedback switching law.

References

[1] E. Skafidas, R. J. Evans, A. V. Savkin, and I. R. Petersen, "Stability results for switched controller systems," *Automatica*, vol. 35, no. 4, pp. 553–564, 1999.

[2] S. Pettersson, "Synthesis of switched linear systems," in *Proc. 42nd IEEE Conf. Decision Control*, Maui, HI, USA, 2003, vol. 5, pp. 5283–5288.

[3] T. Hu, L. Ma, and Z. Lin, "Stabilization of switched systems via composite quadratic functions," *IEEE Trans. Autom. Control*, vol. 53, no. 11, pp. 2571–2585, Nov. 2008.

[4] R. A. DeCarlo, M. S. Branicky, S. Pettersson, and B. Lennartson, "Perspectives and results on the stability and stabilizability of hybrid systems," *Proc. IEEE*, vol. 88, no. 7, pp. 1069–1082, Jul. 2000.

[5] D. Liberzon and A. S. Morse, "Basic problems in stability and design of switched systems," *IEEE Control Syst. Mag.*, vol. 19, no. 5, pp. 59–70, Oct. 1999.

[6] J. Cortes, "Discontinuous dynamical systems," *IEEE Control Syst. Mag.*, vol. 28, no. 3, pp. 36–73, Jun. 2008.

[7] H. Lin and P. J. Antsaklis, "Stability and stabilizability of switched linear systems: A survey of recent results," *IEEE Trans. Autom. Control*, vol. 54, no. 2, pp. 308–322, Feb. 2009.

[8] A. F. Filippov and F. M. Arscott, *Differential Equations With Discontinuous Righthand Sides: Control Systems*. New York, NY, USA: Springer, 1988, vol. 18.

[9] F. H. Clarke, Y. S. Ledyaev, E. D. Sontag, and A. I. Subbotin, "Asymptotic controllability implies feedback stabilization," *IEEE Trans. Autom. Control*, vol. 42, no. 10, pp. 1394–1407, Oct. 1997.

[10] L. Grüne, "Homogeneous state feedback stabilization of homogenous systems," *SIAM Control Optimiz.*, vol. 38, no. 4, pp. 1288–1308, 2000.

[11] S. C. Bengea and R. A. DeCarlo, "Optimal control of switching systems," *Automatica*, vol. 41, no. 1, pp. 11–27, 2005.

[12] Y. Lu, and W. Zhang, On Switching Stabilizability for Continuous-Time Switched Linear Systems. [Online]. Available: http://arxiv.org/abs/1506.04194

[13] D. Liberzon, *Switching in Systems and Control*. New York, NY, USA: Springer, 2003.

[14] A. I. Subbotin, *Generalized Solutions of First Order PDEs: The Dynamical Optimization Perspective*. Boston, MA, USA: Birkhauser, 1995.

[15] W. Zhang, A. Abate, J. Hu, and M. P. Vitus, "Exponential stabilization of discrete-time switched linear systems," *Automatica*, vol. 45, no. 11, pp. 2526–2536, 2009.

[16] W. Zhang, J. Hu, and A. Abate, "Infinite horizon switched lqr problems in discrete time: A suboptimal algorithm with performance analysis," *IEEE Trans. Autom. Control*, vol. 57, no. 7, pp. 1815–1821, Jul. 2012.

[17] M. A. Wicks, P. Peleties, and R. A. DeCarlo, "Switched controller synthesis for the quadratic stabilisation of a pair of unstable linear systems," *Eur. J. Control*, vol. 4, no. 2, pp. 140–147, 1998.