Stability analysis of a class of uncertain switched systems on time scale using Lyapunov functions

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\section*{A B S T R A C T}

This paper deals with the stability analysis of a class of uncertain switched systems on non-uniform time domains. The considered class consists of dynamical systems which commute between an uncertain continuous-time subsystem and an uncertain discrete-time subsystem during a certain period of time. The theory of dynamic equations on time scale is used to study the stability of these systems on non-uniform time domains formed by a union of disjoint intervals with variable length and variable gap. Using the concept of common Lyapunov function, sufficient conditions are derived to guarantee the asymptotic stability of this class of systems on time scale with bounded graininess function. The proposed scheme is used to study the leader–follower consensus problem under intermittent information transmissions.

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1. Introduction

The time scale theory was found promising because it demonstrates the interplay between the theories of continuous-time and discrete-time systems [1,2]. It leads to a new understanding and analyzing of dynamical systems on any non-uniform time domains that are closed subsets of $\mathbb{R}$. Time scale dynamic equations reduce to standard continuous differential equations (resp. standard difference equations) when the time scale is the continuous line, i.e. $\mathbb{R}$ (resp. homogeneous discrete domain). Besides these two extreme cases, there are many interesting examples considering nonhomogeneous time scales.

Stability of linear systems on time scales has been studied in [3–5]. Time-varying dynamic equations [6] and nonlinear uncertain systems [7] have also been investigated. Recently, dynamic equations with bounded perturbations have been considered [8,9]. In these works, some conditions on the bound of the nonlinear part have been derived using the explicit solution of the linear system. However, this scheme cannot be easily extended to the class of switched systems.

This paper deals with uncertain switched systems on non-uniform time domains. During the last two decades, many works have investigated this class of systems due to its broad range of applications in various areas [10–12]. Stability and stabilization have been widely studied for switched systems. They can be categorized into two separated directions depending on whether each subsystem is continuous-time [13,14] or discrete-time [15,16]. Recently, adaptive control laws are proposed to asymptotically track the states of a multimodal piecewise affine reference model in continuous-time [17,18] or discrete-time [19].

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In this paper, the objective is to study the stability for a class of uncertain switched systems where the dynamical system commutes between a continuous-time uncertain subsystem and a discrete-time uncertain subsystem during a certain period of time. This class can describe a wide range of physical and engineering systems. For instance, impulsive systems (which are a relevant class of switched systems, in which the state jumps occur only at some time instances [20]) with non-instantaneous state jumps are one example. Indeed, their temporal nature cannot be represented by the continuous line (i.e. \( \mathbb{R} \)) or the discrete line (i.e. \( \mathbb{Z} \)). Cooperative control over network [21–23] has also attracted a great deal of attention in the last few years due to its broad range of applications in many areas (flocking, consensus, etc.). Usually, the derived controller assumes that local information is transmitted continuously or at some moments with an identical step size. However, local information is only exchanged over some disconnected time intervals due to communication obstacles or sensor failures. In this case, the time domain is neither continuous nor uniformly discrete due to the intermittent information transmissions.

Stability has been studied for switched normal linear systems where the dynamical system commutes between a continuous-time subsystem and a discrete-time subsystem with fixed sampling periods in [24]. The extension to a larger class of systems evolving on a non-uniform time domain and considering uncertainties is not straightforward. To solve this issue, the dynamical systems theory on an arbitrary time scale seems to be appropriate. Recently, the stability of switched linear systems, where each individual subsystem is asymptotically stable, has been studied using common quadratic Lyapunov function [2,25,26]. However, the conditions to derive the quadratic Lyapunov function are not easily satisfied. In [27], the case of individual unstable subsystem has been discussed using the explicit solution of the linear subsystems. It is worthy of noting that these schemes do not consider uncertainties. When considering systems whose temporal nature is represented by the continuous line (i.e. \( \mathbb{R} \)), the asymptotic stability of the linearized system implies asymptotic stability of the corresponding uncertain system. However, it should be highlighted that this is not true when considering systems on nonhomogeneous time scales (see [4]). Hence, the extension to uncertain switched systems is not straightforward.

In this paper, we are interested in extending the existing results for a nonuniform time domain formed by a union of disjoint intervals with variable length and variable gap. The considered class consists of a set of uncertain continuous-time and uncertain discrete-time subsystems during a certain period of time (which may correspond to the variable delays on the state jumps). Since uncertainties will be considered, stability cannot be easily deduced using the explicit solution of the switched system. It will be shown that, using the linear growth conditions on uncertainties, one can derive some conditions to guarantee the asymptotic stability of the switched uncertain systems on time scale with bounded graininess function. The proposed scheme will be used to study the leader–follower consensus problem under intermittent information transmissions.

The outline of this paper is as follows. Section 2 contains basic information about calculus on time scale and states the problem. In Section 3, first, the stability for the linear switched system, without uncertainty, on non-uniform time domains is discussed. Then, sufficient conditions are derived to guarantee the asymptotic stability of the switched system where uncertainties on the continuous-time subsystem and the discrete-time subsystem are considered. Section 4 formulates the leader–follower consensus problem under intermittent information transmissions and shows the viability of the proposed scheme.

2. Problem statement

In this paper, the time scale theory is introduced to study the stability of a special class of uncertain switched systems where the dynamical system commutes between a continuous-time subsystem and a discrete-time subsystem during a certain period of time (which may correspond to the time needed for the state jump or the interruption of the information transmissions for instance). Before describing the studied class of systems, let us recall some basic notations, definitions and properties of this theory [1].

2.1. Preliminaries

A time scale \( \mathbb{T} \) is an arbitrary nonempty closed subset of \( \mathbb{R} \).

For \( t \in \mathbb{T} \), the forward jump operator \( \sigma(t) : \mathbb{T} \to \mathbb{T} \) is defined by

\[
\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}. \tag{1}
\]

The mapping \( \mu : \mathbb{T} \to \mathbb{R}^+ \), called the graininess function, is defined by

\[
\mu(t) := \sigma(t) - t. \tag{2}
\]

A point \( t \in \mathbb{T} \) is called right-scattered if \( \sigma(t) > t \) and right-dense if \( \sigma(t) = t \).

The set \( \mathbb{T}^\kappa \) is defined as follows: if \( \mathbb{T} \) has a left-scattered maximum \( m \), then \( \mathbb{T}^\kappa = \mathbb{T} - \{m\} \); otherwise \( \mathbb{T}^\kappa = \mathbb{T} \).

These necessary definitions are required to define the differential operator for functions with time scale domains.

Let \( f : \mathbb{T} \to \mathbb{R} \) be \( \Delta \)-differentiable on \( \mathbb{T}^\kappa \). The \( \Delta \)-derivative of \( f \) at \( t \in \mathbb{T}^\kappa \) is defined as

\[
f^\Delta(t) := \lim_{s \to t} \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s}.
\]

One can notice that if \( \mathbb{T} = \mathbb{R} \), then \( f^\Delta(t) = \dot{f}(t) \), which is the euclidean derivative of \( f \); and if \( \mathbb{T} = h\mathbb{Z} \), then \( f^\Delta(t) = \frac{f(t+h) - f(t)}{h} \).

Hence, using the time scale theory, the theory of both differential and difference equations is unified.
2.2. Problem statement

A function \( f : \mathbb{T} \rightarrow \mathbb{R} \) is said to be right-dense continuous or rd-continuous, if it is continuous (in the usual sense) over any right-dense interval within \( \mathbb{T} \).

A function \( f : \mathbb{R} \rightarrow \mathbb{R} \) is regressive if \( 1 + \mu(t)f(t) \neq 0 \), \( \forall t \in \mathbb{T}^+ \). We denote the set of all regressive and rd-continuous functions by \( \mathcal{R} \) and by \( \mathcal{R}^+ \) if they satisfy \( 1 + \mu(t)f(t) > 0 \), \( \forall t \in \mathbb{T}^+ \). Similarly, a matrix \( A : \mathbb{T} \rightarrow M_n(\mathbb{R}) \) is called regressive, if \( \forall t \in \mathbb{T}^+, I + \mu(t)A(t) \) is invertible, where \( I \) is the identity matrix. Equivalently, matrix \( A \) is regressive if and only if all its eigenvalues are regressive.

Let \( \{t_0, t_1, t_2, t_3, \ldots \} \) be a monotonically increasing sequence of times without finite accumulation points. \( t_0 = 0 \) is the initial time and \( t_{k+1} \) \( (k \in \mathbb{N}) \) are the switching times. Let us consider a particular time scale \( \mathbb{T} \) defined as

\[
\mathbb{T} = \mathbb{P}_{\{\sigma(t_k), t_{k+1}\}} = \bigcup_{k=0}^{\infty} [\sigma(t_k), t_{k+1}].
\]

The graininess function is such that \( I(t_k) = \sigma(t_k) - t_k \) and \( \sigma(t_0) = t_0 = 0 \).

The studied uncertain switched system on time scale \( \mathbb{T} = \mathbb{P}_{\{\sigma(t_k), t_{k+1}\}} \) can be written as

\[
x^A(t) = \begin{cases} 
A_c x(t) + g_c(x(t)) & \text{for } t \in \bigcup_{k=0}^{\infty} [\sigma(t_k), t_{k+1}] \\
A_d x(t) + g_d(x(t)) & \text{for } t \in \bigcup_{k=0}^{\infty} [t_{k+1}] 
\end{cases}
\]

where \( x(t) \in \mathbb{R}^n \) is the state of the system \( x(0) \in \mathbb{R}^n \) is the initial state), \( A_c \in \mathbb{R}^{n \times n} \) and \( A_d \in \mathbb{R}^{n \times n} \) are constant regressive matrices. Uncertainties act both on the continuous-time and discrete-time dynamics and are characterized by functions \( g_c : \mathbb{R}^n \rightarrow \mathbb{R}^n \) and \( g_d : \mathbb{R}^n \rightarrow \mathbb{R}^n \). The first equation of (4) describes the uncertain continuous-time dynamics of the system and the second one can be seen as the non instantaneous state jumps (see Fig. 1). One should notice that the initial state condition for each continuous time interval is directly given by the discrete-time subsystem. The dynamical system commutes between a continuous-time uncertain subsystem and an uncertain discrete-time subsystem during a certain period of time (which may correspond to the time needed for the state jump). It could be also seen as an extension of impulsive systems where state jumps are not instantaneous and depend on the graininess function.

3. Stability analysis

Stability of switched systems requires other conditions that each subsystem being stable [11]. One method to analyze the stability of continuous-time switched or discrete-time systems is based on the existence of a common Lyapunov function for family of stable subsystems. Here, we will generalize this well-known result to switched systems on non-uniform time domains. Then, some sufficient conditions are derived to guarantee the asymptotic stability of system (4) where uncertainties on the continuous-time subsystem and the discrete-time subsystem are considered.

First, the stability for the linear switched system, without uncertainty, on non-uniform time domains is discussed. Let us consider the switched linear system (4) without uncertainty (i.e. \( g_c(x(t)) = 0 \) and \( g_d(x(t)) = 0 \))

\[
x^A(t) = \begin{cases} 
A_c x(t) & \text{for } t \in \bigcup_{k=0}^{\infty} [\sigma(t_k), t_{k+1}] \\
A_d x(t) & \text{for } t \in \bigcup_{k=0}^{\infty} [t_{k+1}] 
\end{cases}
\]
**Lemma 1** ([26]). The equilibrium of (5) is asymptotically stable if there exists a common quadratic Lyapunov function \( V : \mathbb{R}^n \rightarrow \mathbb{R}^+ \) of the form
\[
V(x) = x^T P x
\]
where \( P = P^T \) is positive definite such that
\[
V(A x) < 0
\]
for all nonzero \( x \in \mathbb{R}^n \) where the time derivative is taken along solutions of (5).

**Lemma 2.** Let us consider system (5) with bounded grainingness function, i.e. \( 0 < \mu(t) \leq \mu_{\text{max}} \). If there exists a positive definite matrix \( P \) such that the following inequalities are simultaneously fulfilled
\[
\begin{align*}
A_c^T P + PA_c &< 0 \\
A_d^T P + PA_d + \mu_{\text{max}} A_d^T PA_d &< 0.
\end{align*}
\]
Then, the candidate function (6) is a common quadratic Lyapunov function associated with system (5). Therefore, the equilibrium of (5) is asymptotically stable

**Proof.** From inequality (7), the time derivative of (6) along the trajectories of the continuous-time subsystem is
\[
\dot{V}(x) = x^T (A_c^T P + PA_c) x < 0.
\]
Hence, (6) is a quadratic Lyapunov function for the continuous-time subsystem.

Inequality (8) yields
\[
A_d^T P + PA_d < -\mu_{\text{max}} A_d^T PA_d.
\]
It implies that for all \( \mu(t) \leq \mu_{\text{max}} \)
\[
A_d^T P + PA_d + (\mu(t) - \mu_{\text{max}}) A_d^T PA_d < 0.
\]
Using time scale dynamic Lyapunov theory, the \( \Delta \)-derivative of (6) along the trajectories of the discrete-time subsystem is
\[
\dot{V}(x) = x^T (A_d^T P + PA_d + \mu(t) A_d^T PA_d) x < 0.
\]
This concludes the proof. ■

Considering a switched system over \( \mathbb{R} \) with pairwise commuting asymptotically stable subsystems, it is well-known that a common quadratic Lyapunov function can be designed [11]. In the following lemma, we will extend this result to switched systems whose temporal nature cannot be represented by the continuous line or the discrete line. An explicit design of a common quadratic Lyapunov function, which will be useful to study the uncertain switched system, is proposed.

**Lemma 3.** Let us consider system (5) with bounded grainingness function, i.e. \( 0 < \mu(t) \leq \mu_{\text{max}} \). Furthermore, it is assumed that matrices \( A_c \) and \( A_d \) are pairwise commuting and Hilger stable with respect to time scale \( \mathbb{P}_{[\sigma(t_k), t_{k+1}]} \) (i.e. the eigenvalues of \( A_c \) and \( A_d \) lie strictly within the Hilger circle). Then, a common quadratic Lyapunov function associated with system (5) exists and can be designed.

**Proof.** Since matrices \( A_c \) and \( A_d \) are Hilger stable with respect to time scale \( \mathbb{P}_{[\sigma(t_k), t_{k+1}]} \), there exist \( Q(t) \) an arbitrary positive definite matrix and unique positive definite solutions \( P_d \) and \( P_c \) to the algebraic Lyapunov equations
\[
\begin{align*}
A_c^T P_c + P_c A_c &= -P_d \\
-P_d + (I + \mu(t) A_d^T) P_d (I + \mu(t) A_d) &= -\mu(t) Q(t).
\end{align*}
\]
Let us consider the candidate Lyapunov function
\[
V(x) = x^T P_c x.
\]
Inequality (7) holds from (9) with \( P = P_c \).
Replacing \( P_d \) in (9) into (10) yields
\[
A_c^T (A_c^T P_c + P_c A_c) + (A_d^T P_c + P_c A_d) A_d + \mu(t) A_d^T (A_c^T P_c + P_c A_c) A_d = Q(t).
\]
Using commutativity of \( A_c \) and \( A_d \), one gets
\[
A_c^T (A_c^T P_c + P_c A_d + \mu(t) A_d^T P_c A_d) + (A_d^T P_c + P_c A_d + \mu(t) A_d^T P_c A_d) A_c = Q(t).
\]
Since all eigenvalues of \( A_c \) lie strictly within the Hilger circle and \( Q(t) \) is definite positive, inequality (8) holds.

Using Lemma 2, one can conclude that (11) is a common quadratic Lyapunov function associated with system (5). ■
Based on the above preliminary result, sufficient conditions are derived to guarantee the asymptotic stability of system (4) where uncertainties on the continuous-time subsystem and the discrete-time subsystem are considered.

**Theorem 1.** Consider the uncertain switched system (4). It is assumed that the following assumptions hold:

(a) There exist positive definite matrices $P$, $Q_1$ and $Q_2$ such that the inequalities

$$A^T_c P + PA_c < -Q_1$$

$$A^T_d P + PA_d + \mu_{\text{max}} A^T_d P A_d < -Q_2$$

are simultaneously fulfilled.

(b) The time derivative of (17) along the trajectories of the uncertain continuous-time subsystem is

$$\dot{V}(x) = x^T (A^T_c P + PA_c) x + (g_c^T (x)) P x + x^T P g_c (x)$$

$$\leq -\lambda_{\text{min}}(Q_1) \|x\|^2 + 2\|P\| \|g_c (x)\| \|x\|$$

$$\leq -\lambda_{\text{min}}(Q_1) \|x\|^2 + 2L_1 \lambda_{\text{max}}(P) \|x\|^2$$

$$= [-\lambda_{\text{min}}(Q_1) + 2L_1 \lambda_{\text{max}}(P)] \|x\|^2.$$  

Since constant $L_1$ is bounded according to Eq. (15), function (17) is a quadratic Lyapunov function for the continuous-time subsystem of (4).

The $\Delta$-derivative of $V(x)$ along the trajectories of the uncertain discrete-time subsystem is

$$V^\Delta (x) = (x^T)^{\Delta} P x (\sigma (t)) + x^T P x^{\Delta}$$

$$= (x^T A_{d}^{\Delta} + g_d (x)^T) P ((I + \mu(t) A_d) x + \mu(t) g_d (x)) + x^T P (A_d x + g_d (x))$$

$$= x^T (A_d^T P + PA_d + \mu(t) A_d^T P A_d) x + g_d (x)^T P ((I + \mu(t) A_d) x + \mu(t) g_d (x)) + x^T (\mu(t) A_d^T + I) P g_d (x)$$

$$= -x^T Q_d x + 2x^T (\mu(t) A_d^T + I) P g_d (x) + \mu(t) g_d (x)^T P g_d (x)$$

where $I$ is the identity matrix with appropriate dimensions.

It yields

$$V^\Delta (x) \leq [-\lambda_{\text{min}}(Q_2) + 2L_2 (1 + \mu_{\text{max}} \|A_d\|) \lambda_{\text{max}}(P) + \mu_{\text{max}} L_2^2 \lambda_{\text{max}}(P)] \|x\|^2.$$  

Since the nonlinear term $g_d (x)$ is bounded according to Eq. (16), function (17) is a quadratic Lyapunov function for the discrete-time subsystem of (4). One can conclude that function (17) is a common Lyapunov function for the switched uncertain system (4). Therefore, the switched uncertain system (4) is asymptotically stable. ■

**Corollary 1.** Consider the uncertain switched system (4). It is assumed that the following assumptions hold

(a) Matrices $A_c$ and $A_d$ are pairwise commuting and Hilger stable with respect to time scale $P(\sigma (t_k), t_{k+1})$. Hence, there exist positive definite matrices $P_c$, $P_d$ and $Q$ which satisfy the inequalities (9)–(10).

(b) The graininess function is bounded, i.e. $0 < \mu(t) \leq \mu_{\text{max}}$, $\forall t \in P(\sigma (t_k), t_{k+1})$.
(c) The perturbations satisfy (14) with
\[
L_1 < \frac{\lambda_{\text{min}}(P_0)}{2\lambda_{\text{max}}(P_e)}
\]  
and
\[
2L_2 (1 + \mu_{\text{max}}\|A_d\|) \lambda_{\text{max}}(P_e) + \mu_{\text{max}} L_2^2 \lambda_{\text{max}}(P_e) < \lambda_{\text{min}}(S).
\]  
The positive definite matrix S is as follows:
\[
S = -A_d^T P_e - P_e A_d - \mu_{\text{max}} A_d^T P_e A_d
\]  
\[
\lambda_{\text{min}}(P_t), \lambda_{\text{min}}(S) \text{ are the smallest eigenvalues of } P_t \text{ and } S \text{ respectively, and } \lambda_{\text{max}}(P_e) \text{ is the largest eigenvalue of } P_e.
\]  
Under these conditions, the uncertain switched system (4) is asymptotically stable.

**Example 1.** Let us consider the time scale
\[
T = \mathbb{P}_{[\sigma(t_k), t_{k+1}]} = \bigcup_{k=0}^{\infty} \left[ k + \frac{k}{k+1}, k+1 \right]
\]  
with for \(k = 1, \ldots, \infty\)
\[
\begin{align*}
t_k &= k \\
\sigma(t_k) &= k + \frac{k}{k+1}.
\end{align*}
\]
Let us study the uncertain switched system (4) on this time scale. It commutes between a stable continuous-time linear subsystem with \(A_c = \begin{pmatrix} -1 & 0.1 \\ 0 & -2 \end{pmatrix}\) and a stable linear discrete-time subsystem \(A_d = \begin{pmatrix} 1 & 5 \\ -3 & 4 \end{pmatrix}\). It is worthy of noting that matrices \(A_c\) and \(A_d\) are not pairwise commuting. The uncertain terms are described by
\[
\begin{align*}
g_c(x) &= 0.03 \sin(x) \\
g_d(x) &= 0.002 \sin(x).
\end{align*}
\]
Assumption (a) of Theorem 1 is verified using the definite positive matrices \(P = \begin{pmatrix} 0.219 & 0.128 \\ 0.128 & 0.09 \end{pmatrix}, Q_1 = \begin{pmatrix} 0.437 & 0.363 \\ 0.363 & 0.334 \end{pmatrix} \) and \(Q_2 = \begin{pmatrix} 0.076 & 0.038 \\ 0.038 & 0.031 \end{pmatrix}\). Assumption (b) holds since the graininess function is bounded, i.e. \(1/2 \leq \mu(t_k) = \sigma(t_k) - t_k = \frac{k}{k+1} \leq 1\). The perturbations \(g_c(x)\) and \(g_d(x)\) are upper bounded by \(L_1\) as follows:
\[
L_1 = 0.03 < \frac{\lambda_{\text{min}}(Q_1)}{2\lambda_{\text{max}}(P)} = \frac{0.0185}{2 \times 0.298} = 0.031
\]  
and by \(L_2 = 0.002\) which satisfies the following inequality:
\[
2L_2 (1 + \mu_{\text{max}}\|A_d\|) + \mu_{\text{max}} L_2^2 = 0.0221 < \frac{\lambda_{\text{min}}(Q_2)}{\lambda_{\text{max}}(P)} = 0.0316.
\]  
Hence, the assumptions of Theorem 1 are satisfied. Therefore, the uncertain switched system (4) is asymptotically stable.

The uncertain switched system trajectories with initial state \(x(0) = [-0.2, 1]^T\) are depicted in Fig. 2. It is worth pointing out that the proposed common Lyapunov function shown in Fig. 3 is decreasing on time scale \(T = \mathbb{P}_{[\sigma(t_k), t_{k+1}]}\).

### 4. Illustrative example

In order to illustrate the viability of the proposed scheme, we investigate the consensus problem for multi-agent system with intermittent information transmissions using the time scale theory.

#### 4.1. System description

Consider a multi-agent system consisting of a leader and followers. The dynamics of each follower and of the leader agent is given by
\[
\begin{align*}
\dot{x}_i &= Ax_i + Bu_i + f(x_i), & i \in \{1, \ldots, N\} \\
\dot{x}_0 &= Ax_0 + f(x_0)
\end{align*}
\]  
where \(x_0 \in \mathbb{R}^n\) is the state of the leader, \(x_i \in \mathbb{R}^n\) is the state of agent \(i\) and \(u_i \in \mathbb{R}^m\) is the control input of agent \(i\). \(A\) and \(B\) are constant real matrices with appropriatedimensions. Moreover, \(f\) is an uncertain dynamics. Since \(f\) is uncertain, it is
not possible to cancel it with the control. To simplify the following derivations, the uncertainty is assumed to be linear, i.e. $f(x) = \delta A x$ where $\delta A$ is a constant real matrix with appropriate dimensions.

The communication topology among all followers is fixed and is represented by a digraph $\mathcal{G}$ which consists of a nonempty set of nodes $\mathcal{V}$ and a set of edges $\mathcal{E}$. Each node corresponds to an agent $i$, and each edge $(i, j) \in \mathcal{E}$ in the directed graph [21] corresponds to an information link from agent $i$ to agent $j$, which means that agent $j$ can receive information from agent $i$. The topology of graph $\mathcal{G}$ is represented by the weighted adjacency matrix $A = (a_{ij}) \in \mathbb{R}^{N \times N}$ given by $a_{ij} = 1$ if $(j, i) \in \mathcal{E}$ and $a_{ij} = 0$, otherwise. The Laplacian matrix of $\mathcal{G}$ is defined as $L = (l_{ij}) \in \mathbb{R}^{N \times N}$ with $l_{ii} = \sum_{j=1}^{N} a_{ij}$ and $l_{ij} = -a_{ij}$ for $i \neq j$. The digraph $\bar{\mathcal{G}}$ is fixed and describes the communication topology of all followers and the leader. It is assumed that the leader has no information from the followers. The topology of $\bar{\mathcal{G}}$ is described by the weighted matrix $H = L + D \in \mathbb{R}^{N \times N}$ where $D = \text{diag}(d_1, \ldots, d_N)$ with $d_i = 1$ if the leader state is available to follower $i$ and with $d_i = 0$ otherwise.

To solve the consensus problem under intermittent information transmissions, the following hypothesis is considered:

**Assumption 1.** It is assumed that:

- The fixed digraph $\bar{\mathcal{G}}$ has a directed spanning tree [8].
- The duration of a communication failure is bounded by a known value $\mu_{\text{max}} = b \in \mathbb{R}^+$. 
- Matrix $(A + \delta A)$ is assumed to be invertible.
- Pair $(A + \delta A, B)$ is stabilizable.

The following switched agreement control law is applied: $\forall i \in \{1, \ldots, N\}$,

$$u_i(t) = \begin{cases} Kz_i(t) & \text{if } t \in \bigcup_{k=0}^{\infty} [\sigma(t_k), t_{k+1}] \\ Kz_i(t_{k+1}) & \text{if } t \in \bigcup_{k=0}^{\infty} [t_{k+1}, \sigma(t_{k+1})] \end{cases}$$

(22)
where $K$ is an appropriate matrix that should be appropriately designed and $z_i$ is considered as local information available for agent $i$, i.e.,
\begin{equation}
    z_i = \sum_{j \in \mathcal{N}_i} (x_j - x_i) + d_i (x_0 - x_i)
\end{equation}

\[ \mathcal{N}_i = \{ j \in V : (i, j) \in \mathcal{E}, j \neq i \} \] is the set of neighbors of agent $i$, i.e. $a_{ij} = 1$.

The union of time intervals over which the agents can communicate with their neighbors is represented by $\cup_{k=0}^{\infty} [\sigma (t_k), t_{k+1}]$ with
\begin{equation}
    \begin{cases}
        \sigma (t_0) = t_0 = 0 \\
        \sigma (t_k) = t_k + b, \quad k \in \mathbb{N}^*.
    \end{cases}
\end{equation}

The remaining intervals represent the time intervals over which the feedback does not evolve due to the possible absence of local information. The time sequence $\{t_1, t_2, t_3, \ldots \}$ is monotonically increasing without finite accumulation points and characterizes the time when the communication failure occurs.

Here, the objective is to verify that matrix $K$ in the distributed control laws $u_i$, $i = 1, \ldots, N$ guarantees
\begin{equation}
    \lim_{t \to \infty} \|e_i(t)\| = 0, \quad \forall i = \{1, \ldots, N\}
\end{equation}

using the time scale theory, where the state error between agent $i$ and the leader is as follows:
\begin{equation}
    e_i = x_i - x_0.
\end{equation}

Let us define $z = (z_1^T, \ldots, z_N^T)^T$, $u = (u_1^T, \ldots, u_N^T)^T$ and the tracking error $\varepsilon = (\varepsilon_1^T, \ldots, \varepsilon_N^T)^T$. The dynamics of the state error $\varepsilon$ can be written in a compact form as:
\begin{equation}
    \dot{\varepsilon} = (I \otimes A) \varepsilon(t) + (I \otimes B) u(t) + (I \otimes \delta A) \varepsilon(t)
\end{equation}

\begin{equation}
    u(t) = \begin{cases}
        -(H \otimes K) \varepsilon(t), & \text{if } t \in \cup_{k=0}^{\infty} [\sigma (t_k), t_{k+1}] \\
        -(H \otimes \sigma (t_{k+1)}) \varepsilon(t), & \text{if } t \in \cup_{k=0}^{\infty} [\sigma (t_k+1), \sigma (t_{k+1})].
    \end{cases}
\end{equation}

The closed-loop system (27) is equivalent to:
\begin{equation}
    \dot{\varepsilon} = \begin{cases}
        [(I \otimes (A + \delta A)) - (H \otimes BK)] \varepsilon(t), & \text{if } t \in \cup_{k=0}^{\infty} [\sigma (t_k), t_{k+1}] \\
        (I \otimes (A + \delta A)) \varepsilon(t) + (I \otimes B) u_{k+1}, & \text{if } t \in \cup_{k=0}^{\infty} [\sigma (t_{k+1}), \sigma (t_{k+2})]
    \end{cases}
\end{equation}

where $u_{k+1} = -(H \otimes K) \varepsilon(t_{k+1})$ is constant on the time interval $[t_{k+1}, \sigma (t_{k+1})]$. The first equation of (28) describes the linear subsystem where the agents can communicate with their neighbors whereas the second one represents the linear subsystem where the feedback does not evolve.

To verify that the distributed switched agreement control law (22) solves the consensus problem under intermittent information transmissions, one must verify that matrix $K$ guarantees that system (28) is asymptotically stable.

### 4.2. Formulation of the consensus problem under intermittent information transmissions using time scale theory

The consensus problem with intermittent information transmissions can be stated using the time scale theory. To facilitate the analysis and the controller design, during the communication failures, only the behavior of the solution of (28) at the discrete times $\{t_{k+1}\}$ and $\{\sigma (t_{k+1})\}$ is considered. The solution of the second subsystem of (28) for $t \in [t_{k+1}, \sigma (t_{k+1})]$ in the continuous sense is given by
\begin{equation}
    \varepsilon(t) = e^{(I \otimes (A + \delta A)) t - k t_{k+1}} [\varepsilon(t_{k+1}) + (I \otimes (A + \delta A)^{-1} B) u_{k+1}] - (I \otimes (A + \delta A)^{-1} B) u_{k+1}.
\end{equation}

The $\Delta$-derivative of $\varepsilon$ on the discrete time scale is
\begin{equation}
    \varepsilon^\Delta(t) = \frac{\varepsilon(\sigma (t)) - \varepsilon (t)}{\sigma (t) - t}.
\end{equation}

At time $t = t_{k+1}$, Eq. (29) yields
\begin{equation}
    \varepsilon^\Delta(t_{k+1}) = \left( \frac{e^{(I \otimes (A + \delta A)) t_{k+1}} - I}{\mu (t_{k+1})} \right) \left[ I - (H \otimes (A + \delta A)^{-1} B K) \varepsilon (t_{k+1}).
\right]
\end{equation}

Let us consider the particular time scale $\mathbb{T} = \bigcup_{k=0}^{\infty} [\sigma (t_k), t_{k+1}]$. The graininess function is $\mu (t_k) = \sigma (t_k) - t_k = b$. Using time scale $\mathbb{T}$, the closed-loop system (28) is written as the following switched system:
\begin{equation}
    \varepsilon^\Delta = \begin{cases}
        [(I \otimes A) - (H \otimes BK)] \varepsilon(t) + (I \otimes \delta A) \varepsilon(t), & \text{if } t \in \bigcup_{k=0}^{\infty} [\sigma (t_k), t_{k+1}] \\
        \left( \frac{e^{(I \otimes A) b} - I}{b} \right) \left[ I - (H \otimes A^{-1} B K) \varepsilon(t) + \Delta A \varepsilon(t) \right], & \text{if } t \in \bigcup_{k=0}^{\infty} [t_{k+1}, t_{k+2}].
    \end{cases}
\end{equation}
where $\Delta A$ is the uncertain term which depends on matrix $\delta A$ and is as follows:

$$
\Delta A = \left( e^{(I \otimes (A + \delta A)b)} - I \right) \left[ I - (H \otimes (A + \delta A)^{-1} BK) \right] - \left( e^{(I \otimes A)b} - I \right) \left[ I - (H \otimes A^{-1} BK) \right].
$$

The switched uncertain system (30) commutes between a continuous-time subsystem and a discrete-time subsystem with uncertainties during a certain period of time (which corresponds to the interruption time of the control evolution due to the lack of information transmissions).

### 4.3. Numerical application

To illustrate the procedure given above, let us consider the consensus problem under intermittent information transmissions for a multi-agent system which consists of three robots (one leader and $N = 2$ followers).

The communication topology of all followers and the leader is shown in Fig. 4. One can see that the fixed digraph $\bar{G}$ has a directed spanning tree. It is described by the weighted matrix $H = \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}$. The dynamics of the agents is given by (21) with $A = \begin{pmatrix} -0.25 & -0.125 \\ 1.5 & -1.25 \end{pmatrix}$ and $B = \begin{pmatrix} 0.5 \\ 0 \end{pmatrix}$. The uncertainty is linear with parameter $\delta A = 10^{-4} \begin{pmatrix} -0.7 & -0.7 \\ 0.7 & 0.7 \end{pmatrix}$. It is assumed that the three agents can communicate with their neighbors only when $t \in \bigcup_{k=0}^{\infty} [\sigma(t_k), t_{k+1}]$ with:

$$
\begin{align*}
\sigma(t_0) &= t_0 = 0 \\
\sigma(t_k) &= t_k + b, \quad k \in \mathbb{N}^* \\
t_k &= 1.5(2k - 1) + 0.5 \log k, \quad k \in \mathbb{N}^*.
\end{align*}
$$

(31)

The time instants $t_k$, $k \in \mathbb{N}^*$ indicate when a communication failure occurs. The duration of communication failures is randomly generated but bounded by $b = \frac{3}{2}$ (see Fig. 5). One can notify that Assumption 1 is fulfilled.

The control gain of the switched agreement control law (22) is set as $K = \begin{pmatrix} -\frac{22}{3} & \frac{44}{13} \\ -4 & \frac{5}{7} \end{pmatrix}$. The objective is to verify that this agreement control law (22) solves the consensus problem under intermittent information transmissions.

The consensus problem for multi-agent system (21) with agreement control law (22) is equivalent to the stabilization of system (4) on time scale $T = \bigcup_{k=0}^{\infty} [\sigma(t_k); t_{k+1}]$ with

$$
A_c = \begin{pmatrix}
-0.5833 & -0.1917 & 0 & 0 \\
2.3 & -2.1167 & 0 & 0 \\
0.3333 & 0.0667 & -0.9167 & -0.2583 \\
-0.8 & 0.8667 & 3.1 & -2.9833
\end{pmatrix}
$$
inequality: without uncertainty is asymptotically stable. One can easily verify that the assumptions of Lemma 3 are verified. Therefore, the corresponding linear switched system without uncertainty is asymptotically stable.

Let us define the positive definite symmetric matrices

\[ P_c = \begin{pmatrix}
9.6712 & -0.3363 & -0.8182 & 0.3442 \\
-0.3363 & 2.6713 & 0.1095 & -0.0893 \\
-0.8182 & 0.1095 & 4.9236 & -0.84 \\
0.3442 & -0.0893 & -0.84 & 0.3919 \\
\end{pmatrix}, \]

and

\[ P_d = \begin{pmatrix}
13.9261 & -5.5499 & -4.8593 & 1.8153 \\
-5.5499 & 11.3197 & 0.8519 & -0.645 \\
-4.8593 & 0.8519 & 14.2343 & -3.2189 \\
1.8153 & -0.645 & -3.2189 & 1.9046 \\
\end{pmatrix}, \]

such that the algebraic Lyapunov equation (9) holds. Furthermore, matrix \( A_d^T P_d + P_d A_d + \mu_{\max} A_d^T P_d A_d \) is negative positive. The quadratic function \( V(x) = x^T P_c x \) is a common Lyapunov function for the linear switched system without uncertainty.

Matrix \( S = \begin{pmatrix}
3.8567 & 0.8811 & 0.3676 & 0.0119 \\
0.8811 & 1.1946 & -0.2943 & 0.0918 \\
0.3676 & -0.2043 & 2.1608 & -0.1830 \\
0.0119 & 0.0918 & -0.1830 & 0.0276 \\
\end{pmatrix} \) defined in Eq. (20) is symmetric definite positive. Using matrices \( P_c, P_d \) and \( Q \), Assumptions (a)–(c) of Corollary 1 are verified. Indeed, the perturbation \( g_c(\epsilon) = (I \otimes \delta A) \epsilon \) is upper bounded by \( L_1 \) as follows:

\[ L_1 = 1.4 \times 10^{-4} \leq \frac{\lambda_{\min}(P_d)}{2 \lambda_{\max}(P_c)} = \frac{1.079}{2 \times 9.8503} = 0.0548 \]

and the perturbation \( g_d(\epsilon) = \Delta A \epsilon \) is upper bounded by \( L_2 = 8.8525 \times 10^{-5} \). Furthermore, it satisfies the following inequality:

\[ 2L_2(1 + b\|A_d\|)\lambda_{\max}(P_c) + bL_2^2\lambda_{\max}(P_c) = 0.0064 < \lambda_{\min}(S) = 0.0072. \]

Therefore, using Corollary 1, one can conclude that the distributed switched agreement control law (22) solves the consensus problem under intermittent information transmissions. The trajectories of the tracking error \( \epsilon = [\epsilon_{11}, \epsilon_{12}, \epsilon_{21}, \epsilon_{22}]^T \) on time scale \( T = \bigcup_{k=0}^{\infty} [\sigma(t_k), t_{k+1}] \) are depicted in Figs. 6–7.

5. Conclusion

In this paper, we have studied the stability of a class of uncertain switched systems on non-uniform time domains. The considered class consists of dynamical systems which commute between an uncertain continuous-time subsystem and an uncertain discrete-time subsystem during a certain period of time. Using the concept of common Lyapunov function, sufficient conditions are derived to guarantee the asymptotic stability of this class of systems on time scales with bounded graininess function. The proposed scheme has been applied to solve the leader–follower consensus problem under intermittent information transmissions.
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