Local Stabilization of Switched Affine Systems

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***Abstract*—This technical note considers the local stabilization problem for the class of switched affine systems. The main idea is to use an alternative representation of the switched affine system as a nonlinear system with input constraints. Switching laws can be derived by emulating locally classical controllers. It is shown that by restricting to local stabilization, the classical constraint on the existence of constant stable convex combinations may be easily avoided. The approach may be interpreted as a generalization where convex combinations defined as functions of the system state are being used. Constructive methods for deriving switching laws are proposed.**

***Index Terms*—Local stabilization, switched affine systems, switching control.**

I. Introduction

The design of switching laws represents an important problem for hybrid control system [4], [5], [16], [22]. Generic results on this topic may be found in the book [17] and the survey papers [18], [27]. This technical note aims at providing new results on the synthesis of stabilizing switching laws for the case of switched affine systems [3], [7], [24]. Such systems are widely used in power electronics [7], [9], [20], [28]. We will focus on designing state dependent switching laws. This problem is challenging since the different subsystems do not necessarily share a common equilibrium. Notable design methodologies for global stabilization have been proposed based on the existence of Hurwitz convex combination [3], [7], on multiple Lyapunov functions [30] or on the use of sliding modes techniques [28]. Numerical techniques for optimal control design have been proposed in [2], [21], [23], [24], [26]. The control implementation under finite switching frequency has been considered in [10]–[13]. See also [31]–[33] for related issues considering sliding dynamics and delays.

When dealing with the stabilization problem, the existing articles treat the global stabilization case. However, one may encounter switched affine systems that may be stabilized only locally. Consider system characterized by two vector fields, $f_1(x) = 3x + 1$, $f_2(x) = 2x - 1$. While global stabilization is not possible, local stabilization at the origin is possible for initial conditions in the ball $|x| < 1/3$, by choosing $f_1(x)$ for $x \leq 0$, and $f_2(x)$, whenever $x \geq 0$. Such systems cannot be considered using the existing methodology.

This article proposes constructive methods for the derivation of state dependent switching laws that ensure local stabilization of switched affine systems at the origin. The main idea is to reformulate the stabilization of switched affine systems as a classical stabilization problem for nonlinear systems affine in the input. The method derives state dependent switching laws by embedding, locally, the behavior of a continuous controller. The classical restriction concerning the existence of a Hurwitz convex combination may be easily avoided. With respect to the existing results, the proposed methodology can be interpreted as an approach that uses convex combinations that depend on the system state.

The rest of the article is structured as follows: Section II mathematically formalizes the problem under study. Section III presents an alternative nonlinear system model, affine in the input, equivalent to the original switched affine system. Generic stabilization results are given in Section IV. Section V provides simple numerical methods for the synthesis of switching laws using Linear Matrix Inequalities (LMIs). Conclusions are given in Section VII.

Notations: M^T denotes the transpose of a matrix M. $\mathbf{I}$ (or $\mathbf{0}$) denotes the identity (or the null) matrix. $|\cdot|$ denotes a vector norm. For a vector x we use $|x|_2$ to denote the euclidean norm. For a square matrix M, $|M|_2$ denotes the matrix norm induced by the euclidean vector norm. For a square symmetric matrix $P \in \mathbb{R}^{n\times n}$, $P \succ 0$ ($P \prec 0$) indicates that P is positive (negative) definite. For a given set $\mathcal{S}$, the symbol $\overline{\text{conv}}\{\mathcal{S}\}$ denotes the closed convex hull of the set. For $P \in \mathbb{R}^{n\times n}$ with $P \succ 0$ and a positive scalar c we denote $\mathcal{E}(P, c) = \{x \in \mathbb{R}^n : x^T P x < c\}$. The n dimensional open ball in $\mathbb{R}^n$ centred on x with radius $c > 0$ is denoted $\mathcal{B}(x, c) := \{y \in \mathbb{R}^n : |x - y| < c\}$. Given a convex polytope $\mathcal{S}$ we denote by $\text{vert}(\mathcal{S})$ its set of vertices. For a positive integer N, we denote by $\mathcal{I}_N$ the set $\{1, 2, \ldots, N\}$. By Δ_N we denote the unit simplex

$$\Delta_N = \left\{ \delta = (\delta_1, \ldots, \delta_N)^T \in \mathbb{R}^N : \sum_{i=1}^{N} \delta_i = 1,\ \delta_i \geq 0, i \in \mathcal{I}_N \right\}.$$

By $\psi_1, \psi_2, \ldots, \psi_N$ we denote the vertices of Δ_N.

II. Preliminaries and Problem Formulation

Let a set of couples $(\tilde{A}_i, \tilde{b}_i)$, $i \in \mathcal{I}_N$ where $\tilde{A}_i \in \mathbb{R}^{n\times n}$, $\tilde{b}_i \in \mathbb{R}^n$, for positive integers n, N. Consider the system

$$\dot{z} = \tilde{X}(z) = \tilde{A}_{s(z)} z + \tilde{b}_{s(z)} \tag{1}$$

where z represents the system state and $s : \mathbb{R}^n \to \mathcal{I}_N$ is a measurable function called *switching law*, to be designed. The system under study represents a differential equation with discontinuous right-hand side. For a tutorial on discontinuous dynamical systems, presenting the specific concepts of system solutions, equilibrium points and stability properties, we point to the work of Cortes [6]. In order to describe the system behaviour and take into account the discontinuity of the vector field and sliding dynamics, Filippov solutions can be defined using differential inclusions [8].

Definition 1: Consider the system $\dot{z} = \tilde{X}(z)$ with $\tilde{X}$ a locally bounded discontinuous vector field. A *Filippov solution* of the system on the interval $I = [t_a, t_b] \subset [0, \infty)$ is an absolutely continuous map $\phi : [t_a, t_b] \to \mathbb{R}^n$ such that the differential inclusion $\dot{\phi}(t) \in \mathcal{F}[\tilde{X}](\phi(t))$ is satisfied for almost every $t \in [t_a, t_b]$, with

$$\mathcal{F}[\tilde{X}](z) = \cap_{\epsilon>0} \cap_{\mu(\mathcal{S})=0} \overline{\text{conv}} \left\{ \tilde{X}(\bar{z}) : \bar{z} \in \mathcal{B}(z, \epsilon) \setminus \mathcal{S} \right\}$$

where μ represents the usual Lebesgue measure on $\mathbb{R}^n$. A *maximal solution* is a solution $\phi(\cdot)$ on some interval I such that there exists

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no solution on an interval that contains strictly I and that is equal to ϕ on I.

The existence of a least one solution starting from each initial condition is guaranteed if $\mathcal{F}[\tilde{X}]$ is Lebesgue measurable, upper-semicontinuous, compact, non-empty and convex, which is the case for (1) under the hypothesis of measurable switching laws $s(z)$ (see [1]). For system (1)

$$\mathcal{F}[\tilde{X}](z)=\cap_{\epsilon>0}\cap_{\mu(\mathcal{S})=0}\overline{\mathrm{conv}}\left\{A_{s(\bar{z})}\bar{z}+b_{s(\bar{z})}:\bar{z}\in\mathcal{B}(z,\epsilon)\setminus\mathcal{S}\right\}.$$

By construction, $\mathcal{F}[\tilde{X}](z)$, enriches the set of admissible velocities by including vectors in the convex hull of $A_i x + b_i, i \in \mathcal{I}_N$, which allows to describe potential sliding modes of (1). A point $z^* \in \mathbb{R}^n$ is an *equilibrium* of the differential inclusion $\dot{z} \in \mathcal{F}[\tilde{X}](z)$ if $0 \in \mathcal{F}[\tilde{X}](z^*)$ [6]. Using the definition of $\mathcal{F}[\tilde{X}](z)$ for system (1), $0 \in \mathcal{F}[\tilde{X}](z^*)$ implies that there necessarily exists δ^* in Δ_N such that $\sum_{i=1}^N(\tilde{A}_i z^* + \tilde{b}_i)\delta_i^* = 0$. This formalism allows to describe both the "natural" equilibria of system (1) (corresponding to $\delta^* \in \mathrm{vert}(\Delta_N)$) and the equilibria generated by the switching law, via sliding dynamics (corresponding to $\sum_{i=1}^N(\tilde{A}_i z^* + \tilde{b}_i)\delta_i^* = 0$ for some $\delta^* \in \Delta_N$, $\delta^* \notin \mathrm{vert}(\Delta_N)$). Roughly speaking, the set of controlled equilibria of system (1) corresponds to the one of the bilinear system [19] $\dot{z} = \tilde{F}(z,\delta) = \sum_{i=1}^N(\tilde{A}_i z + \tilde{b}_i)\delta_i$ with input $\delta \in \Delta_N$. Given a pair $(z^*, \delta^*) \in \mathbb{R}^n \times \Delta_N$ such that $(\tilde{A}_i z^* + \tilde{b}_i)\delta_i^* = 0$ and considering the switching law design problem for the equilibrium z^*, it is convenient to use a coordinate transformation $x = z - z^*$ and to study the stabilization properties at the point $x = 0$ for the equivalent switched affine model

$$\dot{x} = X(x) = A_{\sigma(x)}x + b_{\sigma(x)} \tag{2}$$

with $A_i = \tilde{A}_i$, $b_i = \tilde{A}_i z^* + \tilde{b}_i$, $i \in \mathcal{I}_N$, and switching law $\sigma(x) = i \Leftrightarrow s(z) = i$. Let us remark that for (2) there exists $\delta = \delta^* \in \Delta_N$ such that $\sum_{i=1}^N \delta_i b_i = 0$. This is a necessary conditions for the existence of $\sigma(x)$ such that $0 \in \mathcal{F}[X](0)$.

Definition 2: Consider a discontinuous system $\dot{x} = X(x)$ with X locally bounded. We say that the system is *locally asymptotically stable at the origin* if

1) for each $\epsilon > 0$ there exists $\lambda > 0$ such that any solution $x(\cdot)$ in the Filippov sense, which is defined at $t = 0$ and is maximal, is defined on $[0,\infty)$ if $|x(0)| < \lambda$ and moreover $|x(0)| < \lambda \Rightarrow |x(t)| < \epsilon, \forall\, t \in [0,\infty)$;
2) there exists $\eta \in (0,\infty]$ such that any solution $x(\cdot)$ in the Filippov sense, which is defined at $t = 0$ and is maximal, is defined on $[0,\infty)$ if $|x(0)| < \eta$ and moreover $|x(0)| < \eta \Rightarrow \lim_{t\to\infty}|x(t)| = 0$.

When the system is locally asymptotically stable at the origin we call *domain of attraction* the set of points $x_0 \in \mathbb{R}^n$ such that any solution $x(\cdot)$ in the Filippov sense, which is defined at $t = 0$ and is maximal, is defined on $[0,\infty)$ if $x(0) = x_0$ and moreover $x(0) = x_0 \Rightarrow \lim_{t\to\infty}|x(t)| = 0$.

We recall that sufficient conditions for the local stability of a discontinuous differential equation $\dot{x} = X(x)$, with X locally bounded, are given by the existence of a strict Lyapunov function $V(x)$, $V(0) = 0$, $V(x) > 0$, $\forall\, x \neq 0$, such that $\max_{y\in\mathcal{F}[X](x)}\langle\nabla V(x), y\rangle < 0$, $\forall\, x \in \mathcal{B}(0,\gamma)$ for some positive scalar γ (see [8], Ch. 15, Theorem 1, pag. 153).

The problems under study are formulated as follows:

Problem 1: is there a (measurable) switching law $\sigma : \mathbb{R}^n \to \mathcal{I}_N$ such that the system (2) is locally asymptotically stable at the origin?

Problem 2: given a positive scalar η provide sufficient conditions for the design of a (measurable) switching law $\sigma : \mathbb{R}^n \to \mathcal{I}_N$ such that the system (2) is locally asymptotically stable at the origin and the domain of attraction includes the ball $\mathcal{B}(0,\eta)$.

The idea of the work is to re-formulate the switched affine system (2) in a classical nonlinear affine form $\dot{x} = f(x) + G(x)u$ interconnected with a discontinuous control law $u = k(x)$ that is constrained to take values in a finite set of vectors $\mathcal{V}(\delta^*) = \{v_1, v_2, \ldots, v_N\} \subset \mathbb{R}^{N-1}$. We propose such a system re-formulation in Section III. Furthermore, in Section IV, we show that the obtained nonlinear affine system has nice properties: if there exists a classical continuous feedback $k^c(x)$ such that the system $\dot{x} = f(x) + G(x)k^c(x)$ is (locally or globally) stable, then there exists also a local discontinuous stabilizer, $k(x)$, taking values in $\mathcal{V}(\delta^*)$, and in extenso, a switching law σ for the switched affine system (2).

III. System Re-Formulation

In this section, system (2) is rewritten in a classical nonlinear affine form interconnected with a discontinuous control law.

Proposition 1: Consider system (2), $\delta^* \in \Delta_N$ such that $\sum_{j=1}^N \delta_j^* b_j = 0$, the notations $m = N-1$, $v_i = M(\psi_i - \delta^*)$, $i \in \mathcal{I}_N$, where $M = (\mathbf{I}_{m\times m}\ \mathbf{0}_{m\times 1}) \in \mathbb{R}^{m\times N}$, and the set

$$\mathcal{V}(\delta^*) = \{v_i,\ i \in \mathcal{I}_N\}.$$

The switched affine system (2) is equivalent to the interconnection between the nonlinear affine system

$$\dot{x} = H(x,u) = f(x) + G(x)u, \quad u \in \mathbb{R}^m \tag{3}$$

and the control law

$$u = k(x), \quad k : \mathbb{R}^n \to \mathcal{V}(\delta^*) \tag{4}$$

with $f(x) = A(\delta^*)x = \sum_{j=1}^N \delta_j^* A_j x$, $G(x) = (g_1(x)\quad g_2(x) \ldots g_m(x))$, $g_j(x) = (A_j - A_N)x + (b_j - b_N)$, $j \in \mathcal{I}_m$ and

$$k(x) = v_{\sigma(x)}. \tag{5}$$

Proof: First, system (2) can be reformulated as

$$\begin{cases} \dot{x} = \sum_{j=1}^N (A_j x + b_j)\varphi_j(x) \\ \varphi(x) = \psi_{\sigma(x)}. \end{cases} \tag{6}$$

However, designing a control $\varphi(x) \in \mathrm{vert}(\Delta_N)$, such that $\sum_{j=1}^N \varphi_j(x) = 1$, is not very practical. To express the control problem in a standard form, we use a translation of the system control by δ^* and a projection to $\mathbb{R}^{N-1}$. System (6) can be re-written as

$$\begin{cases} \dot{x} = A(\delta^*)x + \sum_{j=1}^N (A_j x + b_j)\tilde{\varphi}_j(x) \\ \tilde{\varphi}(x) = \varphi(x) - \delta^* \end{cases} \tag{7}$$

that is, a system where we control the "error" with respect to δ^*. Furthermore, since both $\varphi(x)$ and δ^* belong to Δ_N we have $\tilde{\varphi}_N(x) = -\sum_{j=1}^{N-1}\tilde{\varphi}_j(x)$. We obtain the equivalent system

$$\dot{x} = A(\delta^*)x + \sum_{j=1}^{N-1} g_j(x)\tilde{\varphi}_j(x). \tag{8}$$

Note that system (8) depends only on the first $N-1$ elements of $\tilde{\varphi}(x)$. To end the proof, remark that the control $k(x)$ in (5) is defined as these $N-1$ elements of $\tilde{\varphi}(x)$, i.e.,

$$k_j(x) = \tilde{\varphi}_j(x) = \varphi_j(x) - \delta_j^*, \quad j \in \mathcal{I}_m. \tag{9}$$

■

We may remark that designing a switching law σ leads to finding a discontinuous control law $k : \mathbb{R}^n \to \mathcal{V}(\delta^*)$, such that system (3) with the control $u = k(x)$, is locally asymptotically stable.

IV. Generic Results

As follows, we show how the existence of a continuous control $u = k^c(x)$ for system (3) can be used in order to derive a switching law $\sigma(x)$ for system (2) (or equivalently a discontinuous control (5) for the interconnection (3), (4)).

Definition 3: System (3) is said to be *continuously (locally or globally) stabilizable at the origin* if there exists a continuous feedback $u = k^c(x)$ such that the closed-loop system $\dot{x} = H(x, k^c(x))$ is (locally or globally) asymptotically stable at the origin.

Theorem 1: Consider the switched affine system (2) and the affine model (3). Assume that:

1) there exists $\delta^* = (\delta_1^*\ \delta_2^*\ \ldots, \delta_N^*)^T \in \Delta_N$ with $\delta_i^* > 0$, $i \in \mathcal{I}_N$ such that $\sum_{i=1}^N \delta_i^* b_i = 0$;
2) system (3) is continuously locally stabilizable at the origin by $u = k^c(x)$, with $k^c(0) = 0$.

Then there exists a $\mathcal{C}^\infty$ function $V(x)$ defined on some ball $\mathcal{B}(0,\eta)$, $\eta > 0$, $V(0) = 0$, $V(x) > 0$, $\forall\, x \neq 0$, a positive scalar $\gamma \in (0, \eta]$ and a measurable switching law

$$\sigma(x) \in \arg\min_{i \in \mathcal{I}_N} \langle \nabla V(x), A_i x + b_i \rangle \tag{10}$$

such that

$$\max_{y \in \mathcal{F}[X](x)} \langle \nabla V(x), y \rangle < 0, \quad \forall\, x \in \mathcal{B}(0,\gamma) \setminus \{0\} \tag{11}$$

that is system (2), (10) (or equivalently (3), (4) with $k(x)$ as in (5), (10)) is locally asymptotically stable at the origin.

Proof: If a (locally) continuous stabilizer $k^c(x)$ is known, then the closed-loop system $\dot{x} = H(x, k^c(x))$ has a continuous right-hand side. According to Kurzweil's Converse Theorem (see Theorem 2.4, page 31 in [1]), for any locally asymptotically stable system with continuous right-hand side there exists a $\mathcal{C}^\infty$ strict Lyapunov function. Then, for the closed-loop system $\dot{x} = H(x, k^c(x))$ there exists a function $V(x)$ defined on some ball $\mathcal{B}(0,\eta)$, $\eta > 0$, such that $V(0) = 0$, $V(x) > 0$, $\forall\, x \neq 0$ and

$$\langle \nabla V(x), f(x) + G(x) k^c(x) \rangle < 0, \quad \forall\, x \in \mathcal{B}(0,\eta) \setminus \{0\}. \tag{12}$$

The main idea of the proof is to show that, for a sufficiently small neighbourhood of the origin, the decay of the function V can be ensured by switching among the elements of $\mathcal{V}(\delta^*)$. This will be shown in the following steps: first we show that there exists a sufficiently small neighbourhood of the origin where $k^c(x)$ belongs in the closed convex hull of $\mathcal{V}(\delta^*)$ (Step 1); next we use arguments inspired by the classical method of stable convex combinations in order to design partitions of the state-space where the decay of V is ensured by switching among the elements of $\mathcal{V}(\delta^*)$ (Step 2); a particular attention will be given to the stability of potential sliding dynamics at switching surfaces (in Step 3).

Step 1) We show next that there exists a sufficiently small $\gamma > 0$, such that

$$x \in \mathcal{B}(0,\gamma) \Rightarrow k^c(x) \in \overline{\mathrm{conv}}\, \{\mathcal{V}(\delta^*)\}. \tag{13}$$

First we prove that the set $\overline{\mathrm{conv}}\{\mathcal{V}(\delta^*)\}$ has a non-empty interior by showing that a sufficiently small ball in $\mathbb{R}^m$, centred on the origin, may be placed inside. Intuitively speaking, this fact is important since it shows that the finite set of control vectors $\mathcal{V}(\delta^*)$ can "generate" any direction in $\mathbb{R}^m$. Next we use arguments based on continuity of $k^c(x)$ at the origin.

Let us remark that

$$\sum_{i=1}^N \delta_i^* v_i = M \left(\sum_{i=1}^N \delta_i^* \psi_i - \delta^* \right) = 0 \tag{14}$$

that is 0 belongs to the relative interior of $\overline{\mathrm{conv}}\{\mathcal{V}(\delta^*)\}$ (since $\delta_i^* > 0$, $i \in \mathcal{I}_N$). Let $e_1, e_2, \ldots, e_m$ denote the vectors of the elementary basis on $\mathbb{R}^m$ and $\bar{\delta}^* = M\delta^*$. Note that $e_j = M\psi_j$, for any $j \in \mathcal{I}_m$. Using $v_j = M(\psi_j - \delta^*)$, $j \in \mathcal{I}_N$, we have that $e_j = v_j + \bar{\delta}^*$. Using the fact that $v_N = -\bar{\delta}^*$ and (14), this leads to $\sum_{i=1}^N \delta_i^* v_i = \sum_{i=1}^m \delta_i^* v_i - \delta_N^* \bar{\delta}^* = 0$. Hence

$$e_j = v_j + (\delta_N^*)^{-1} \sum_{i=1}^m \delta_i^* v_i, \quad \forall\, j \in \mathcal{I}_m$$

that is any elementary vector e_j can be represented as a conic combination of v_i, $i \in \mathcal{I}_m$, $\delta_N^* e_j = \sum_{i=1}^m \lambda_i^j v_i$, with $\lambda_i^j = \delta_i^* > 0$, for $i \neq j$ and $\lambda_j^j = \delta_j^* + \delta_N^* > 0$. Since $\delta^* \in \Delta_N$, we have that $\sum_{i=1}^m \lambda_i^j = 1$, that is there exists $\beta^+ = \delta_N^* > 0$ such that $\beta^+ e_j \in \overline{\mathrm{conv}}\{\mathcal{V}(\delta^*)\}$ for all $j \in \mathcal{I}_m$. Similarly, $-\beta^+ e_j = -\sum_{i=1}^m \lambda_i^j v_i$. Since $\sum_{i=1}^N \delta_i^* v_i = 0$,

$$v_i = -(\delta_i^*)^{-1} \sum_{l \in \mathcal{I}_N \setminus \{i\}} \delta_l^* v_l.$$

Hence,

$$-\beta^+ e_j = \sum_{i=1}^m \lambda_i^j \cdot (\delta_i^*)^{-1} \cdot \sum_{l \in \mathcal{I}_N \setminus \{i\}} \delta_l^* v_l.$$

Let us remark that $\beta^+ > 0$, $\lambda_i^j > 0$ and $\delta_i^* > 0$, $\forall\, (i,j) \in \mathcal{I}_N \times \mathcal{I}_m$. Then there exists $\mu_i^j > 0$, $(i,j) \in \mathcal{I}_N \times \mathcal{I}_m$ (functions of β^+, λ_i^j and δ_i^*) such that

$$-e_j = \sum_{i=1}^N \mu_i^j v_i$$

which is sufficient for the existence of a $\beta_j^- > 0$ such that $-\beta_j^- e_j \in \overline{\mathrm{conv}}\{\mathcal{V}(\delta^*)\}$ for all $j \in \mathcal{I}_m$. Therefore, there exists $\beta_{\min} > 0$ such that $\pm\beta_{\min} e_j \in \overline{\mathrm{conv}}\{\mathcal{V}(\delta^*)\}$ for all $j \in \mathcal{I}_m$. Since for any $u \in \mathbb{R}^m$, $u = \sum_{j=1}^m u_j e_j$, there exists a sufficient small $\epsilon > 0$ such that the ball $\{u \in \mathbb{R}^m : |u| < \epsilon\}$ is included in the set $\overline{\mathrm{conv}}\{\mathcal{V}(\delta^*)\}$. Furthermore, the vector $u = 0$ belongs to the interior of $\overline{\mathrm{conv}}\{\mathcal{V}(\delta^*)\}$. Since k^c is continuous with $k^c(0) = 0$, there exists a sufficiently small $\gamma > 0$, such that (13) holds.

Step 2) As follows we indicate the construction of switching surfaces.

Using (13) we have that for all $x \in \mathcal{B}(0,\gamma)$ there exists $\alpha(x) = (\alpha_1(x)\ \alpha_2(x)\ \ldots\ \alpha_N(x))^T \in \Delta_N$ such that $k^c(x) = \sum_{i=1}^N \alpha_i(x) v_i$. Since we can always choose γ such that $\mathcal{B}(0,\gamma) \subseteq \mathcal{B}(0,\eta)$, the relation (12) leads to

$$\sum_{i=1}^N \alpha_i(x) \langle \nabla V(x), f(x) + G(x) v_i \rangle < 0, \quad \forall\, x \in \mathcal{B}(0,\gamma) \setminus \{0\}. \tag{15}$$

Note that $\alpha_i(x) \geq 0$, $\forall\, i \in \mathcal{I}_N$, $x \in \mathcal{B}(0,\gamma)$. Hence for any $x \in \mathcal{B}(0,\gamma) \setminus \{0\}$ there exists at least one index $i^*(x) \in \mathcal{I}_N$ such that $\langle \nabla V(x), f(x) + G(x) v_{i^*(x)} \rangle < 0$. Therefore $\min_{i \in \mathcal{I}_N} \langle \nabla V(x), f(x) + G(x) v_i \rangle < 0$. For each $x \in \mathcal{B}(0,\gamma)$, consider the subset of index $\mathcal{J}_{\min}(x) \subseteq \mathcal{I}_N$ minimizing the expression $\langle \nabla V(x), f(x) + G(x) v_i \rangle$

$$\mathcal{J}_{\min}(x) = \{ i \in \mathcal{I}_N : \langle \nabla V(x), G(x)(v_i - v_j) \rangle \leq 0,\ \forall\, j \in \mathcal{I}_N \}.$$

This set can be written as

$$\mathcal{J}_{\min}(x) = \arg\min_{i \in \mathcal{I}_N} \langle \nabla V(x), G(x) v_i \rangle. \tag{16}$$

Then for all $i \in \mathcal{J}_{\min}(x)$ we have that

$$\langle \nabla V(x), f(x) + G(x) v_i \rangle < 0, \quad \forall\, x \in \mathcal{B}(0,\gamma) \setminus \{0\}. \tag{17}$$

Remark that $f(x) + G(x)v_i = A_i x + b_i$, $\forall\, i \in \mathcal{I}_N$. Then (10) is equivalent to $\sigma(x) \in \mathcal{J}_{\min}(x)$. When $\mathcal{J}_{\min}(x)$ is a singleton, that is $J_{\min}(x) = \{i\}$, for some $i \in \mathcal{I}_N$, we have that $\mathcal{F}[X](x) = \{f(x) + G(x)v_i\}$. Therefore the partition of the state space defined by (10) ensures that $\max_{y \in \mathcal{F}[X](x)} \langle \nabla V(x), y \rangle < 0$ at any point in $\mathcal{B}(0,\gamma)$ where $\mathcal{J}_{\min}(x)$ has one element.

Step 3) Here we show that when sliding dynamics occur the decay of V is ensured. This corresponds to the case where x is on a switching surface and $\mathcal{J}_{\min}(x)$ has more than one element.

For the construction of the differential inclusion describing the system solution, we need to show that on a switching surface $\mathcal{F}[X](x)$ reduces to the convex polyhedron

$$\mathcal{F}[X](x) = \overline{\text{conv}}\left\{f(x) + G(x)v_i, i \in \mathcal{J}_{\min}(x)\right\}. \tag{18}$$

This is true if there exists $\epsilon > 0$ such that for all $y \in \mathcal{B}(0,\epsilon)\mathcal{J}_{\min}(y) \subseteq \mathcal{J}_{\min}(x)$. The fact is shown by contradiction using continuity arguments. Suppose $\forall\, \epsilon > 0$ there exists $y \in \mathcal{B}(0,\epsilon)$ such that $\exists\ i \in \mathcal{J}_{\min}(y)$, $i \notin \mathcal{J}_{\min}(x)$. Then there exist $\{y_l\}_{l\in\mathbb{N}}$, $y_l \to x$ as $l \to \infty$, such that $\exists\ i_l \in \mathcal{J}_{\min}(y_l)$, $i_l \notin \mathcal{J}_{\min}(x)$. Since the cardinal of $\mathcal{I}_N$ is finite, there exists a subsequence $\{y_{l_q}\}_{q\in\mathbb{N}}$ and an index $p^* \in \mathcal{J}_{\min}(y_{l_q})$, $\forall\, q \in \mathbb{N}$, such that $p^* \notin \mathcal{J}_{\min}(x)$. From the definition of $\mathcal{J}_{\min}(y_{l_q})$,

$$\left\langle \nabla V\left(y_{l_q}\right), G\left(y_{l_q}\right)(v_{p^*} - v_j)\right\rangle \le 0, \quad \forall\, j \in \mathcal{I}_N,\ q \in \mathbb{N}. \tag{19}$$

Let us remark that $\nabla V(x)$, $G(x)$ are continuous functions and $y_{l_q} \to x$ as $q \to \infty$. Then $\langle \nabla V(x), G(x)(v_{p^*} - v_j)\rangle \le 0$, $\forall\, j \in \mathcal{I}_N$, which implies that $p^* \in \mathcal{J}_{\min}(x)$ and contradicts the hypothesis. Then (18) holds. Using (17) yields

$$\langle \nabla V(x), y)\rangle \le 0, \quad \forall\, x \in \mathcal{B}(0,\gamma) \setminus \{0\} \tag{20}$$

and all $y \in \overline{\text{conv}}\{f(x) + G(x)v_i, i \in \mathcal{J}_{\min}(x)\}$. Then (11) holds, which ends the proof. ■

Remark 1: The proof of Theorem 1 is constructive in the sense that if the affine nonlinear system (3) is stabilized by a controller k^c and admits a (local) Lyapunov function V, then the original switched system (2) can be (locally) stabilized by a switching law of the form (10) obtained based on the same Lyapunov function V. With respect to the classical convex combination approach [3], [17] the method that we propose can be interpreted as an extension where we look for a locally stable state dependent convex combination, with barycentric coordinates defined by $\delta_i(x) = \delta_i^* + k_i^c(x)$, $i \in \mathcal{I}_{N-1}$, $\delta_N(x) = 1 - \sum_{i=1}^{N-1} \delta_i(x)$, instead of a constant convex combination, with constant barycentric coordinates δ^* (as in [3], [7]).

Theorem 1 requires the existence of a parameter $\delta^* \in \Delta_N$ such that $\delta_i^* > 0$ for all $i \in \mathcal{I}_N$, which guarantees that for the equivalent system (3) the null control lies in the interior of $\overline{\text{conv}}\{\mathcal{V}(\delta^*)\}$. For the case when δ^* lies on the boundary of Δ_N, a similar construction may be used by considering only the sub-systems $A_i x + b_i$ of system (2) for which $\delta_i^* \neq 0$.

Example 1: Numerical Illustration (Adapted From [29], Example 3.19, p. 85): Consider a system (2) described by the following matrices:

$$A_1 = A_3 = \begin{pmatrix} 0 & 2 \\ 2 & -66 \end{pmatrix}, \quad b_1 = b_2 = \begin{pmatrix} -360 \\ 0 \end{pmatrix} \tag{21}$$

$$A_2 = A_4 = \begin{pmatrix} 0 & 2 \\ 2 & 54 \end{pmatrix}, \quad b_3 = b_4 = \begin{pmatrix} 360 \\ 0 \end{pmatrix}. \tag{22}$$

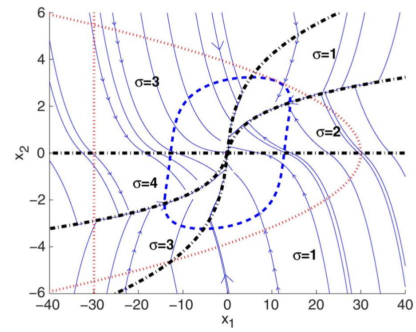

Fig. 1. Illustration of the phase plane for the closed-loop switched affine system in Example 1: dotted line—border of the set where $k^c(x) \in \overline{\text{conv}}\{\mathcal{V}(\delta^*)\}$; dashed line—a level set of $V(x)$; dash-dotted lines—switching surfaces $\langle \nabla V(x), (A_i - A_j)x + b_i - b_j)\rangle = 0, i, j \in \mathcal{I}_N$.

For $\delta_i^* = 1/4$, $i \in \mathcal{I}_4$, we have $\sum_{i=1}^4 \delta_i^* b_i = 0$. The obtained system (3) is described by

$$A(\delta^*) = \begin{pmatrix} 0 & 2 \\ 2 & -6 \end{pmatrix}, \quad g_1(x) = g_2(x) + g_3(x) \tag{23}$$

$$g_2(x) = \begin{pmatrix} -720 \\ 0 \end{pmatrix}, \quad g_3(x) = \begin{pmatrix} 0 \\ -120x_2 \end{pmatrix}. \tag{24}$$

The matrix $A(\delta^*)$ is not Hurwitz. Let $k^c(x) = 1/120(0\ x_1\ 2x_2^2)^T$. The obtained closed-loop system, $H(x, k^c(x))$, has the form

$$\dot{x}_1 = -6x_1 + 2x_2 \tag{25}$$

$$\dot{x}_2 = 2x_1 - 6x_2 - 2x_2^3. \tag{26}$$

The stability of the closed-loop system can be shown using Krasovskii's method (see [29], Theorem 3.7, p. 84). The method consists in using $V(x) = H^T(x, k^c(x))H(x, k^c(x)) = (-6x_1 + 2x_2)^2 + (2x_1 - 6x_2 - 2x_2^3)^2$ as a candidates Lyapunov function and checking whether the Jacobian matrix $J(x) = \partial H(x, k^c(x))/\partial x$ satisfies the relation $J^T(x) + J(x) \prec 0$ in some neighborhood of the origin. For system (25)

$$J^T(x) + J(x) = \begin{pmatrix} -12 & 4 \\ 4 & -12 - 12x^2 \end{pmatrix} \prec 0 \tag{27}$$

for all $x \in \mathbb{R}^n$. Then the closed-loop system $\dot{x} = H(x, k^c(x))$ is asymptotically stable. Since the conditions of Theorem 1 are satisfied, the function $V(x)$ can be used for constructing a switching law (10) that ensures the local stabilization of the switched affine system. An illustration of the phase plane for the closed-loop switched affine system is provided in Fig. 1.

Remark 2: The main advantage of the proposed method is the fact that the difficult problem of existence of a stabilizing switching law for the switched system (2) (Problem 1) is reduced to the classical stabilization problem of a nonlinear affine system (3), on which a very large variety of control design methods are possible.

Example 2: Stabilization Based on the Linearized Model: As follows, simple stabilization conditions are given using the local linearized model of system (3). Consider the notation

$$B = (b_1 - b_N \quad b_2 - b_N \quad \ldots \quad b_{N-1} - b_N). \tag{28}$$

System (3) can be re-expressed as

$$\dot{x} = A(\delta^*)x + Bu + w(x,u), \tag{29}$$

$$w(x,u) = D(u)x \tag{30}$$

where $w(x,u)$ is obtained from $w(x,u) = (G(x) - B)u$ and

$$D(u) = \sum_{i=1}^{N-1} (A_i - A_N)u_i. \tag{31}$$

Assume that the pair $(A(\delta^*), B)$ is stabilizable for some $\delta^* = (\delta_1^*\ \delta_2^*\ \ldots,\ \delta_N^*)^T \in \Delta_N$ with $\delta_i^* > 0$, $i \in \mathcal{I}_N$. Then there exists a gain matrix K and functions $V(x) = x^T P x$, $W(x) = x^T Q x$, $P, Q \succ 0$, such that

$$\langle \nabla V(x), (A(\delta^*) + BK)\, x \rangle < -W(x). \tag{32}$$

The derivative of the function V along (29) satisfies

$$\langle \nabla V(x), (A(\delta^*) + BK)\, x + w(x, Kx) \rangle < -W(x) + 2x^T P w(x, Kx). \tag{33}$$

Let us remark that for any $\rho > 0$ there exists $r > 0$ such that $|w(x, Kx)|_2 < \rho |x|_2$ for any $|x|_2 < r$. Then $x^T P w(x, Kx) < \rho |P|_2 |x|_2, \forall |x|_2 < r$, which leads to

$$\langle \nabla V(x), (A(\delta^*) + BK)\, x + w(x, Kx) \rangle < -(\mathrm{eig}_{\min}(Q) - 2\rho |P|_2)\, |x|_2 \tag{34}$$

for all $|x|_2 < r$, that is the state feedback $u = Kx$ ensures local stabilization of system (29) for ρ chosen such that $\rho < 1/2\mathrm{eig}_{\min}(Q)/|P|_2$, where $\mathrm{eig}_{\min}(Q)$ denotes the minimum eigenvalue of Q. Applying Theorem 1, one can conclude that the switched system (2) can be locally stabilized. The obtained switching law has the form

$$\sigma(x) \in \arg\min_{i \in \mathcal{I}_N} x^T P (A_i x + b_i). \tag{35}$$

However, differently from [3], [7], $A(\delta^*)$ is not required to be a Hurwitz matrix. For local stabilization we only need the pair $(A(\delta^*), B)$ to be stabilizable.

Remark 3: The existence of a continuous stabilizing feedback k^c for system (3) is not very restrictive. In fact, for nonlinear affine systems such as (3), when the system can be stabilizable at the origin (in the sense of Filippov solutions) by means of a locally bounded, measurable feedback $u = k^b(x)$ such that $\lim_{\epsilon \to 0} \operatorname{ess\,sup}_{|x| < \epsilon} |k^b(x)| = 0$, there exists also a continuous stabilizer $u = k^c(x)$ for the same system (see [1, p. 61]). Furthermore, the non-existence of a stabilizing feedback for system (3) can be expressed as a certain topological obstruction. For the necessity of existence of continuous stabilizer k^c we point to the classical Brockett test. For the more general case of locally bounded, measurable stabilizers k, necessary condition may be found in [25]. Since for each subset $\mathcal{U} \subset \mathbb{R}^m$ and each $x \in \mathbb{R}^n$, system (3) satisfies

$$H\,(x, \mathrm{conv}(\mathcal{U})) = \mathrm{conv}\,(H(x, \mathcal{U})) \tag{36}$$

a necessary condition for the existence of a locally bounded, measurable feedback $u = k(x)$ which stabilizes the system (in the sense of Filippov) is that for each $\epsilon > 0$ there exists $\lambda > 0$ such that

$$\forall\, y \in \mathcal{B}(0, \lambda), \quad \exists\, x \in \mathcal{B}(0, \epsilon), \exists\, u \in \mathbb{R}^m \text{ such that } y = H(x, u).$$

This may be useful to determine the existence of stabilizing switching laws for the original switched affine system (2). This implies, for example, that switched affine systems for which $A(\delta^*) = 0$ whenever $\sum_{i=1}^N \delta_N^* b_i = 0$ and $rank(G(x)) = m < n$ cannot be stabilized by a static switching law $\sigma(x)$ if solutions are understood in the sense of Filippov.

Example 3: Stabilization Obstruction for Switched Affine System: Consider a system (2) with

$$A_1 = -A_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0.5 & -0.5 & 0 \end{pmatrix},\ b_1 = -b_4 = \begin{pmatrix} 0.5 \\ 0.5 \\ 0 \end{pmatrix} \tag{37}$$

$$A_2 = -A_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0.5 & 0.5 & 0 \end{pmatrix},\ b_2 = -b_3 = \begin{pmatrix} -0.5 \\ 0.5 \\ 0 \end{pmatrix}. \tag{38}$$

For any $\delta^* \in \Delta_N$ such that $\sum_{i=1}^4 b_i \delta_i^* = 0$ we have $A(\delta^*) = 0$. The model (3) is characterized by $g_1(x) = g_2(x) + g_3(x)$, $g_2(x) = (0\ 1\ x_1)^T$, $g_3(x) = (1\ 0\ -x_2)^T$. This leads to

$$\dot{x}_1 = u_1 + u_3 \tag{39}$$
$$\dot{x}_2 = u_1 + u_2 \tag{40}$$
$$\dot{x}_3 = (u_1 + u_2)x_1 - (u_1 + u_3)x_2 \tag{41}$$

where the reader may recognize a classical non-holonomic integrator (see [1, p. 55]) for which no point x of the form $x = (0\ 0\ \epsilon)^T$, $\epsilon \neq 0$, belongs to the image of H. We conclude that there is no switching law $\sigma(x)$ which makes the origin of the switched affine system locally asymptotically stable (in the sense of Filippov solutions).

V. Numerical Issues

In practical applications, it is of interest to provide numerical tools for the design of switching laws. For the system under study, we may be interested in optimizing the domain of attraction, the speed of convergence, etc. Here, we present simple LMI based criteria for the design of a stabilizing switching law which optimizes an ellipsoidal estimation of the domain of attraction for given decay rate.

Consider the set of allowable control values $\mathcal{V}(\delta^*)$. The set $\overline{\mathrm{conv}}\{\mathcal{V}(\delta^*)\}$ is a convex polytope. It can be described by a finite number N_r of vectors $r_i \in \mathbb{R}^m$, $i \in \mathcal{I}_{N_r}$, such that

$$\overline{\mathrm{conv}}\,\{\mathcal{V}(\delta^*)\} = \left\{ u \in \mathbb{R}^N : r_i^T u \le 1, i \in \mathcal{I}_{N_r} \right\}. \tag{42}$$

Proposition 2: Consider the switched system (2), the equivalent representation (29) with controls u restricted to the set $\mathcal{V}(\delta^*)$ and the polytope (42). Assume that $\delta_i^* > 0$, $i \in \mathcal{I}_N$. Given tuning parameters $\chi, c > 0$ assume that there exists $Q \succ 0, \theta > 0$ such that

$$(A(\delta^*) + D(v_i))\, Q + Q\, (A(\delta^*) + D(v_i))^T - \theta B B^T \prec -2\chi Q, \quad i \in \mathcal{I}_N \tag{43}$$

$$\begin{pmatrix} cI & I \\ I & Q \end{pmatrix} \succ 0 \tag{44}$$

and

$$\begin{pmatrix} 1 & \frac{\theta}{2} r_j^T B^T \\ \frac{\theta}{2} B r_j & Q \end{pmatrix} \succ 0, \quad j \in \mathcal{I}_{N_r}. \tag{45}$$

Then the switched system (2) with the switching law

$$\sigma(x) \in \arg\min_{i \in \mathcal{I}_N} x^T Q^{-1} (A_i x + b_i) \tag{46}$$

is locally asymptotically stable at the origin. Furthermore, the domain of attraction includes the ball $\mathcal{B}(0, 1/\sqrt{c})$ and there exists a positive scalar κ such that $|x(t)|^2 \le \kappa e^{-2\chi t} |x(0)|^2$ for any $x(0) \in \mathcal{B}(0, 1/\sqrt{c})$.

Proof: The proof uses arguments inspired from [14], [15]. The set of LMIs (43) guarantees that the function $V(x) = x^T Q^{-1} x$ satisfies the relation

$$\langle \nabla V(x), (A(\delta^*) + D(\tilde{u}) + BK)\, x \rangle < -2\chi V(x) \tag{47}$$

for any $x \neq 0$, $\tilde{u} \in \overline{\mathrm{conv}}\{\mathcal{V}(\delta^*)\}$ with $K = -(\theta/2) B^T Q^{-1}$. Applying the Schur complement, the set of LMIs (45) ensures that

$$1 - r_j^T K Q K^T r_j > 0, \quad j \in \mathcal{I}_{N_r}. \tag{48}$$

Let us remark that the minimum of V along the hyperplane $r_j^T K x = 1$ is given by $\min_{r_j^T K x = 1} x^T Q^{-1} x = (r_j^T K Q K^T r_j)^{-1}$. From (48) we obtain that for any $x \in \mathcal{E}(Q^{-1}, 1)$ we have $r_j^T K x \le 1$, $j \in \mathcal{I}_{N_r}$, i.e., the relation $Kx \in \overline{\mathrm{conv}}\{\mathcal{V}(\delta^*)\}$ holds. With (47), we have that

$$\langle \nabla V(x), (A(\delta^*) + D(Kx) + BK)\, x \rangle < -2\chi V(x) \tag{49}$$

for any $x \in \mathcal{E}(Q^{-1}, 1) \setminus \{0\}$. Using the relation between systems (29) and (3), this leads to

$$\langle \nabla V(x), f(x) + G(x) K x \rangle < -2\chi V(x) \tag{50}$$

for any $x \in \mathcal{E}(Q^{-1}, 1) \setminus \{0\}$. Recall that $Kx \in \overline{\text{conv}}\{\mathcal{V}(\delta^*)\}$ for $x \in \mathcal{E}(Q^{-1}, 1)$. Following the same arguments as in the proof of Theorem 1, one may show that

$$\max_{y \in \mathcal{F}[X](x)} \langle \nabla V(x), y \rangle < -2\chi V(x), \quad \forall x \in \mathcal{E}(Q^{-1}, 1) \setminus \{0\} \quad (51)$$

for the switching law σ defined by (46). To end the proof let us remark that the condition (44) guarantees that $\mathcal{B}(0, 1/\sqrt{c}) \subset \mathcal{E}(Q^{-1}, 1)$. □

Remark 4: The feasibility of the LMIs (43)–(45) guarantees that any system solution originating in the ball $\mathcal{B}(0, 1/\sqrt{c})$ converges to the origin with a decay rate χ. The size of the domain of attraction can be optimized by considering the optimization problem

$$\inf c \text{ under the constraints } (43), (44), (45) \quad (52)$$

which is a standard optimization problem. The LMI criteria (43)–(45) represent sufficient condition for local stabilization in a domain that includes a prescribed ball $\mathcal{B}(0, 1/\sqrt{c})$ (solution to Problem 2 with $\eta = 1/\sqrt{c}$). The set of LMIs implies that the local linearised model at $x = 0$ is stabilizable. The method is based on robust control arguments, in the sense that the term $w(x, u)$ in (29) is treated as a perturbation. This aspect may induce some conservatism in the design. Additional conservatism in the estimation of the domain of attraction may also stem from the choice of quadratic candidate Lyapunov functions. In terms of computational complexity, the approach requires solving $N + N_r + 3$ LMIs involving $0.5(n^2 + n) + 2$ variables.

Example 4: LMI Stabilization: Consider a switched affine system described by the matrices:

$$A_1 = \begin{pmatrix} -3 & 0 \\ 0 & 12 \end{pmatrix}, \ b_1 = \begin{pmatrix} 0 \\ 7 \end{pmatrix}, \ A_2 = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \ b_2 = \begin{pmatrix} 0 \\ -14 \end{pmatrix}.$$

For $\delta^* = (2/3 \ \ 1/3)^T$, $\sum_{i=1}^{2} \delta_i^* b_i = 0$. However, $A(\delta^*)$ is not Hurwitz therefore the example cannot be treated using the global stabilization approaches in [3], [7], [30]. Using the formulation (29) and solving the optimization problem (52) for $\chi = 0.25$ leads to a switching law of the form (46) with

$$Q = \begin{pmatrix} 2.87 & -3.62 \\ -3.62 & 17.45 \end{pmatrix} \times 10^{-2} \quad (53)$$

which guarantees local stabilization $\forall\, x(0) \in \mathcal{B}(0, 0.14)$.

VI. Conclusion

This note has presented results for the local stabilization of switched affine systems. An alternative representation of the switched affine system as a nonlinear system with input constraints has been proposed. The new model allows to derive state dependent switching laws by emulating locally classical controllers. The approach allows to avoid the classical obstruction on the existence of Hurwitz convex combinations by restricting to local stabilization. Examples illustrate the feasibility and limitations of the proposed methodology. In the future, the extension to the case of dynamic, time-dependent switching laws, represents an interesting perspective.

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