A linear state feedback switching rule for global stabilization of switched nonlinear systems about a nonequilibrium point

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Abstract

A switched equilibrium of a switched system of two subsystems is a such a point where the vector fields of the two subsystems point strictly towards one another. Using the concept of stable convex combination that was developed by Wicks-Peleties-DeCarlo (1998) for linear systems, Bolzern-Spinelli (2004) offered a design of a state feedback switching rule that is capable to stabilize an affine switched system to any switched equilibrium. The state feedback switching rule of Bolzern-Spinelli gives a nonlinear (quadratic) switching threshold passing through the switched equilibrium. In this paper we prove that the switching threshold (i.e. the associated switching rule) can be chosen linear, if each of the subsystems of the switched system under consideration are stable.

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1. Introduction

Using the concept of stable convex combination that was developed by Wicks et al [12] for linear systems, Bolzern-Spinelli [2] offered a design of a state feedback switching rule that is capable to stabilize an affine switched system[1]

$$\dot{x} = A^\sigma x + b^\sigma, \quad x \in \mathbb{R}^n, \ \sigma \in \{-1, 1\} \tag{1}$$

to any point x_0 (called *switched equilibrium*) that satisfies

$$\lambda \left(A^+ x_0 + b^+ \right) + (1 - \lambda) \left(A^- x_0 + b^- \right) = 0, \tag{2}$$

for some $\lambda \in [0, 1]$. If the matrix $\lambda A^+ + (1 - \lambda)A^-$ is Hurwitz, then, according to Bolzern-Spinelli [2], the switching signal $\sigma(x)$ can be defined as

$$\sigma(x) = \arg \min_{i \in \{-1,1\}} \{V'(x)(A^i x + b^i)\} = \\ = \text{sign} \left(V'(x)(A^- x + b^-) - V'(x)(A^+ x + b^+) \right), \tag{3}$$

where V is the quadratic Lyapunov function of the linear system

$$\dot{x} = \lambda \left(A^+ x + b^+ \right) + (1 - \lambda) \left(A^- x + b^- \right).$$

When $A^- = A^+$, the rule (3) reduces to

$$\sigma(x) = \text{sign} \left(V'(x)b^- - V'(x)b^+ \right), \tag{4}$$

whose switching threshold $\{x \in \mathbb{R}^n : V'(x)b^- - V'(x)b^+\} \ni x_0$ is a hyperplane, but in general the state feedback switching rule (3) gives a nonlinear switching threshold (quadratic surface) passing through the switched equilibrium x_0.

In this paper we provide a wider class of switched systems (1) that can be stabilized to a switched equilibrium by a linear switching rule. Specifically, we show that the nonlinear switching rule (3) can always be replaced with the linear one

$$\sigma(x) = \text{sign} \left\langle x - x_0, [V''(x_0)(A^-(x_0) + b^-)]^T \right\rangle, \tag{5}$$

when the subsystems $\dot{x} = A^+ x$ and $\dot{x} = A^- x$ admit a common quadratic Lyapunov function. Here $V''(x_0)$ doesn't depend on x_0 because V is assumed quadratic. We also note that (5) coincides with (4) when $A^- = A^+$.

The paper is organized as follows. In the next section of the paper we discuss the main idea behind the switching rule (3), which is based on construction of suitable sets Ω^- and Ω^+, such that any switching rule $\sigma(x)$ with the property

$$\sigma(x) = \begin{cases} -1 & \text{if} \quad x \in \Omega^-, \\ 1 & \text{if} \quad x \in \Omega^+, \end{cases}$$

stabilizes (1) to x_0. In section 3 we prove our main result (Theorem 3.1), which offers a linear state feedback switching rule to stabilize a nonlinear switched system

$$\dot{x} = f^\sigma(x), \quad x \in \mathbb{R}^n, \ \sigma \in \{-1, 1\}, \tag{6}$$

to a switched equilibrium x_0. We recall that, according to Demidovich [3, Ch. IV, §281], nonlinear systems (6) admit a common quadratic Lyapunov function, if the simmetrized derivative

$$f_x^\sigma(x) + [f_x^\sigma(x)]^T$$

is uniformly negative definite uniformly in $x \in \mathbb{R}^n$, and $\sigma \in \{-1, 1\}$, see also Pavlov et al [9]. The switching rule (14) proposed in Theorem 3.1 takes the form (5) when switched system (6) is affine. The main discovery used in Theorem 3.1 is that,

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for subsystems of (6) that admit a common quadratic Lyapunov function, the boundaries of Ω^- and Ω^+ are contained in ellipsoids that touch one another at the point x_0, see Fig. 2. The proof uses a standard Lyapunov stability theorem that is also implicitly used in Bolzern-Spinelli [2]. Specifically, we use a Lyapunov stability theorem for Filippov systems with smooth Lyapunov functions, which is a particular case of more general results available e.g. in Shevitz-Paden [11] or M.-Aguilara-Garcia [7]. But since deriving the required Lyapunov theorem (Theorem 3.2) from [7, 11] is not very straightforward (and since we didn't find the exact required theorem elsewhere in the literature), we added a proof for completeness, that we placed in the Appendix section.

In section 4 we consider an application of Theorem 3.1 to a model of boost converter and, for illustration purposes, also implement the Bolzern-Spinelli rule (3) for the same model. Some further discussion on when the switching rule (5) coincides with (3) is carried out in the conclusions section.

2. The idea of Wicks et al [12] and Bolzern-Spinelli [2]

Recall that x_0 is a switched equilibrium for the nonlinear switched system (6), if there exists $\lambda_0 \in [0, 1]$ such that

$$\lambda_0 f^-(x_0) + (1 - \lambda_0)f^+(x_0) = 0. \tag{7}$$

Assume that the equilibrium x_0 of the convex combination

$$\dot{x} = \lambda_0 f^-(x) + (1 - \lambda_0)f^+(x). \tag{8}$$

is asymptotically stable and let V be the respective Lyapunov function satisfying

$$V'(x)\left(\lambda_0 f^-(x) + (1 - \lambda_0)f^+(x)\right) < 0 \quad \text{for all } x \neq x_0. \tag{9}$$

The fundamental idea of Bolzern-Spinelli [2] (who extended Wicks et al [12] to affine linear systems) is that for (6) to stabilize to x_0, the switching rule $\sigma(x)$ must take the value $\sigma(x) = -1$ in the region

$$\Omega^- = \{x : V'(x)f^-(x) < 0\} \tag{10}$$

and the value $\sigma(x) = +1$ in the region

$$\Omega^+ = \{x : V'(x)f^+(x) < 0\}. \tag{11}$$

The following lemma discusses the geometry of the intersection $\Omega^- \cap \Omega^+$, in particular it clarifies that there are situations where one cannot draw a hyperplane in $\Omega^- \cap \Omega^+$ passing through x_0 (Fig. 1a) and there are situations when one can (Fig. 1b). The existence of a hyperplane in $\Omega^- \cap \Omega^+$ passing through x_0 corresponds to the existence of a linear switching rule $\sigma(x)$ that stabilizes (6) to x_0. Therefore, what this paper will really prove in Section 3 is that it is Fig. 1b which takes place when both of the subsystems of (6) are stable.

Lemma 2.1. (ideas of [12, 2]) Consider f^-, $f^+ \in C^1(\mathbb{R}^n, \mathbb{R}^n)$. Let x_0 be a switched equilibrium for the vector fields f^- and f^+, i.e. (7) holds. Assume that the equilibrium x_0 of system (8) is asymptotically stable and the respective Lyapunov function $V \in C^1(\mathbb{R}^n, \mathbb{R})$ satisfies (9). Then, the sets Ω^- and Ω^+ satisfy the properties:
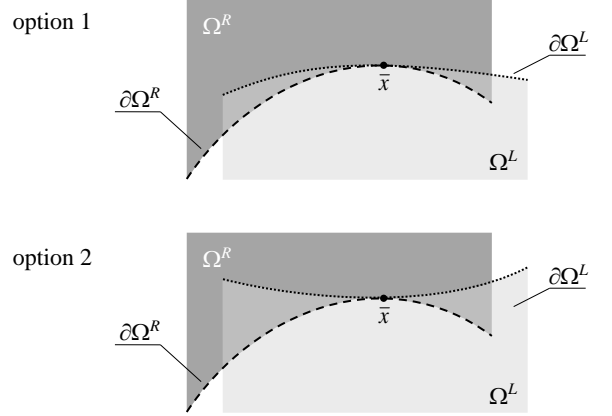

Figure 1: Relative locations of sets Ω^L and Ω^R.

1) $\Omega^- \cup \Omega^+ \cup \{x_0\} = \mathbb{R}^n$, $\overline{\Omega^-} \cup \overline{\Omega^+} = \mathbb{R}^n$,

2) $\partial\Omega^- \backslash \{x_0\} \subset \Omega^+$, $\partial\Omega^+ \backslash \{x_0\} \subset \Omega^-$,

3) $x_0 \in \partial\Omega^-$, $x_0 \in \partial\Omega^+$.

Proof. Part 1. Follows directly from (9).

Part 2. Consider $x \in \partial\Omega^-$. Then $x \notin \Omega^-$ because Ω^- is open. Then $x \in \overline{\Omega^+}$ by Part 1. The property $\partial\Omega^+ \subset \overline{\Omega^-}$ can be proved by analogy.

Part 3. It is sufficient to show that $V'(x_0) = 0$. To observe this, fix an arbitrary $j \in \overline{1, n}$ and consider the vector $\xi^j \in \mathbb{R}^n$ defined as $\xi_i^j = 0$, $i \neq j$, and $\xi_j^j = 1$. Since $V(x) > 0$, $x \neq x_0$, we have

$$0 < V(x_0 + k\xi^j) - V(x_0) = V'(x_0 + k_*\xi^j)\xi^j \cdot k =$$
$$= \frac{\partial V}{\partial x_j}(x_0 + k_*\xi^j)k,$$
$$0 < V(x_0 - k\xi^j) - V(x_0) = -V'(x_0 - k_{**}\xi^j)\xi^j \cdot k =$$
$$= -\frac{\partial V}{\partial x_j}(x_0 - k_{**}\xi^j)k,$$

for any $k > 0$ and for some $k_*, k_{**} \in [0, k]$ (that depend on k). Passing to the limit as $k \to 0$, one gets $\frac{\partial V}{\partial x_j}(x_0) = 0$.

The proof of the lemma is complete. $\qquad\square$

3. The main result

In this section we assume that the switched equilibrium x_0 admits a common quadratic Lyapunov function

$$V(x) = (x - x_0)^T P(x - x_0)$$

with respect to each of the two systems

$$\dot{x} = f^-(x) - f^-(x_0) \quad \text{and} \quad \dot{x} = f^+(x) - f^+(x_0), \tag{12}$$

where P is an $n \times n$ symmetric matrix and the following standard properties hold:

$$V'(x)\left(f^-(x) - f^-(x_0)\right) \leq -\alpha\|x - x_0\|^2,$$
$$V'(x)\left(f^+(x) - f^+(x_0)\right) \leq -\alpha\|x - x_0\|^2, \tag{13}$$

for some fixed constant $\alpha > 0$.

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Theorem 3.1. *Consider f^-, $f^+ \in C^1(\mathbb{R}^n, \mathbb{R}^n)$. Let x_0 be a switched equilibrium for the vector fields f^+ and f^-, i.e. (7) holds. Assume that the systems of (12) admit a common quadratic Lyapunov function $V \in C^2(\mathbb{R}^n, \mathbb{R})$ that satisfies (13). Then the switching signal*

$$\sigma(x) = \operatorname{sign}\left\langle x - x_0, [V''(x_0)f^-(x_0)]^T \right\rangle \qquad (14)$$

makes x_0 quadratically globally stable switched equilibrium of switched system (6).

Note that rule (14) takes the form (5) when the nonlinear switched system (6) takes the form (1). Also, using (7) the switching rule (14) can be rewritten as

$$\sigma(x) = \operatorname{sign}\left\langle x - x_0, [V''(x_0)(f^-(x_0) - f^+(x_0))]^T \right\rangle.$$

In order to prove the theorem, we introduce two sets

$$\Omega_\alpha^- = \left\{ x \in \mathbb{R}^n : -\alpha\|x - x_0\|^2 + V'(x)f^-(x_0) < 0 \right\},$$
$$\Omega_\alpha^+ = \left\{ x \in \mathbb{R}^n : -\alpha\|x - x_0\|^2 + V'(x)f^+(x_0) < 0 \right\}$$

and establish the following lemma about the relative properties of the sets Ω_α^i and Ω^i as introduced in (10)-(11).

Lemma 3.1. Assume that the conditions of Theorem 3.1 hold. Then Ω_α^- and Ω_α^+ verify the following properties:

1) $\Omega^- \supset \Omega_\alpha^-$, $\Omega^+ \supset \Omega_\alpha^+$,

2) $x_0 \in \partial\Omega_\alpha^-$, $x_0 \in \partial\Omega_\alpha^+$,

3) both $\partial\Omega_\alpha^-$ and $\partial\Omega_\alpha^+$ are ellipsoids,

4) hyperplane $\sigma(x) = 0$ is tangent to both Ω_α^- and Ω_α^+ at x_0,

5) $\Omega_\alpha^- \subset \{x : \sigma(x) < 0\}$, $\Omega_\alpha^+ \subset \{x : \sigma(x) > 0\}$.

The notations and statements of Lemma 3.1 are illustrated at Fig. 2.

Proof. Part 1. Let $x \in \Omega_\alpha^-$. Then

$$V'(x)f^-(x) = V'(x)(f^-(x) - f^-(x_0)) + V'(x)f^-(x_0) \leq$$
$$\leq -\alpha\|x - x_0\|^2 + V'(x)f^-(x_0) < 0.$$

Therefore, $x \in \Omega^-$. The proof for Ω_α^+ and Ω_α^+ is analogous.

Part 2. Follows from $V'(x_0) = 0$ established in the proof of Part 3 of Lemma 2.1.

Part 3. We execute the proof for $x_0 = 0$. The proof in the general case doesn't differ. The change of the coordinates $y = x - \Delta$ transforms the equation

$$-\alpha\|x - x_0\|^2 + V'(x)f^-(x_0) = 0$$

into

$$-\alpha\|y\|^2 - 2\alpha\langle\Delta, y\rangle + 2\langle Pf^-(0), y\rangle - \alpha\|\Delta\|^2 + 2\langle\Delta, Pf^-(0)\rangle = 0.$$

If $\Delta = \frac{Pf^-(0)}{\alpha}$, then we further get

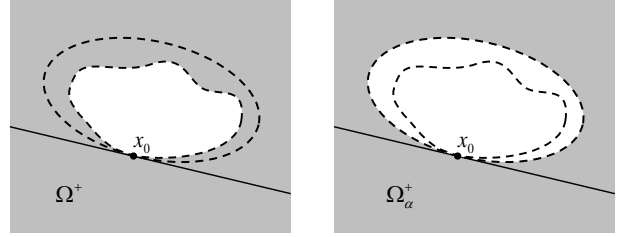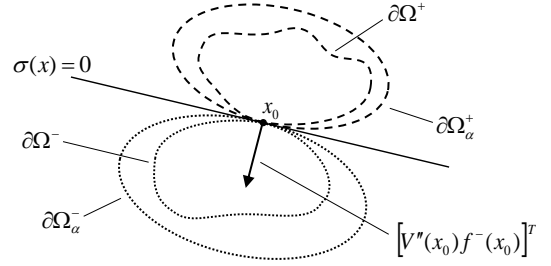$$-\alpha\|y\|^2 - \frac{1}{\alpha}\|Pf^-(0)\|^2 + \frac{2}{\alpha}\|Pf^-(0)\|^2 = 0,$$

Figure 2: Top figure: Locations of the boundaries of Ω^+, Ω_α^+, Ω^-, Ω_α^- with respect to the hyperplane $\sigma(x) = 0$ and with respect to each other. Bottom figures: The sets Ω^+ and Ω_α^+ (grey regions).

which is the equation of ellipsoid centered at 0 and radius $\frac{1}{\alpha^2}\|Pf^-(0)\|^2$.

The proof for $\partial\Omega_\alpha^+$ is analogous.

Part 4. This follows from the equality

$$\frac{d}{dx}\left(-\alpha\|x - x_0\|^2 + V'(x)f^-(x_0)\right)\Big|_{x=x_0} = V''(x_0)f^-(x_0).$$

and the property (7) of switched equilibrium.

Part 5. Let $H(x) = -\alpha\|x - x_0\|^2 + V'(x)f^-(x_0)$. The interior of the ellipsoid $\partial\Omega_\alpha^-$ corresponds to $H(x) > 0$. Therefore, the exterior of the ellipsoid $\partial\Omega_\alpha^-$ (which, by definition, coincides with the set Ω_α^-) corresponds to $H(x) < 0$. This proves the statement of Part 5 for Ω_α^-. Since $(1 - \lambda_0)f^+(x_0) = -\lambda f^-(x_0)$ by (7), the proof for Ω_α^+ follows same lines.

The proof of the lemma is complete. $\qquad\square$

The proof of our main result uses the following Lyapunov stability theorem for discontinuous systems with smooth Lyapunov functions, which is implicitly used in [12, 2].

Theorem 3.2. (Lyapunov stability theorem for discontinuous systems with smooth Lyapunov functions) (similar to [11, Theorem 3.1], [7, Theorem 2.3]) Consider a system of differential equations with discontinuous right-hand side

$$\dot{x} = g(x), \quad \text{with } g(x) = \begin{cases} g^+(x), & \text{if } H(x) > 0, \\ g^-(x), & \text{if } H(x) < 0, \end{cases} \quad x \in \mathbb{R}^n, \quad (15)$$

where g^-, g^+, and H are C^1-functions. Consider $x_0 \in \mathbb{R}^n$ satisfying $H(x_0) = 0$. Let V be a C^1-smooth Lyapunov function with $V(x_0) = 0$ and $V(x) \neq 0$ for $x \neq x_0$. Consider a piecewise continuous strictly positive for $x \neq x_0$ scalar function $x \mapsto w(x)$ such that for any $\rho > 0$ there exists $\varepsilon > 0$ for which $w(x) \geq \varepsilon$ as
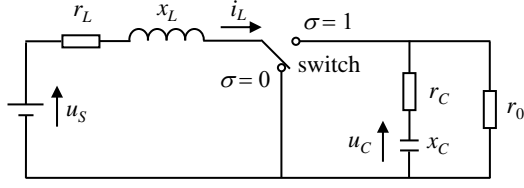
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Figure 3: Boost converter from Fribourg-Soulat [4] and Beccuti et al [1].

long as $\|x - x_0\| \geq \rho$. If

$$V'(x)\xi \leq -w(x) \quad \text{for any } \xi \in K[g](x), \text{ and any } x \neq x_0,$$

then x_0 is an asymptotically globally stable stationary point of (15). Here $K[g](x)$ stays for convexification of the discontinuous function g at x, see e.g. Shevitz-Paden [11].

The proof of Theorem 3.2 is given in Appendix.

Proof of Theorem 3.1. We will show that the conditions of Theorem 3.2 hold with

$$w(x) = \begin{cases} -V'(x)f^-(x), & \sigma(x) < 0, \\ -\max\{V'(x)f^-(x), V'(x)f^+(x)\}, & \sigma(x) = 0, \\ -V'(x)f^+(x), & \sigma(x) > 0. \end{cases}$$

If $x \in \overline{D^-}\backslash\{x_0\}$, then $x \in \Omega_\alpha^- \subset \Omega^-$ by statements 5 and 1 of Lemma 3.1, which implies $w(x) > 0$. Analogously, $w(x) > 0$, if $x \in \overline{D^+}\backslash\{x_0\}$. This implies that $\max\limits_{x:\|x-x_0\|=\rho} w(x)$ is a positive function of ρ that approaches 0 as $\rho \to 0$.

Since $K[f](x) = \{f^-(x)\}$, when $\sigma(x) < 0$, and $K[f](x) = \{f^+(x)\}$, when $\sigma(x) > 0$, then condition $V'(x)\xi \leq -w(x)$ of Theorem 3.2 holds for $\sigma(x) \neq 0$.

Consider $\sigma(x) = 0$. Then each $\xi \in K[f](x)$ has the form $\xi = \lambda f^-(x) + (1 - \lambda)f^+(x)$, where λ is a constant from the interval $[0, 1]$. We have

$$\begin{aligned} V'(x)\xi &= \lambda V'(x)f^-(x) + (1 - \lambda)V'(x)f^+(x) \leq \\ &\leq \max\{V'(x)f^-(x), V'(x)f^+(x)\} = -w(x), \end{aligned}$$

that completes the proof of the theorem. $\square$

4. Application to a model of boost converter

Consider a dc-dc boost converter of Fig. 3 with a switching feedback $\sigma(x)$. Denoting the inductor current i_L by x_1 and the capacitor voltage u_C by x_2, the differential equations of the converter read as (see e.g. Fribourg-Soulat [4], Beccuti et al [1])

$$\dot{x} = \begin{pmatrix} -\frac{r_L}{x_L} & -\frac{r_0}{x_L(r_0+r_C)}\sigma \\ \frac{r_0}{x_C(r_0+r_C)}\sigma & -\frac{1}{x_C(r_0+r_C)} \end{pmatrix} x + \begin{pmatrix} \frac{u_s}{x_L} \\ 0 \end{pmatrix}, \quad \sigma \in \{0, 1\}, \quad (16)$$

Let us view the right-hand-side of (16) with $\sigma = 0$ and $\sigma = 1$ as $f^-(x)$ and $f^+(x)$ respectively. The equation (7) for switched equilibrium x_0 yields

$$\begin{aligned} -r_L x_{01} + u_s - (1 - \lambda_0)\frac{r_0 r_C}{r_0+r_C}x_{01} - (1 - \lambda_0)\frac{r_0}{r_0+r_C}x_{02} &= 0, \\ -x_{02} + (1 - \lambda_0)r_0 x_{01} &= 0, \end{aligned} \quad (17)$$
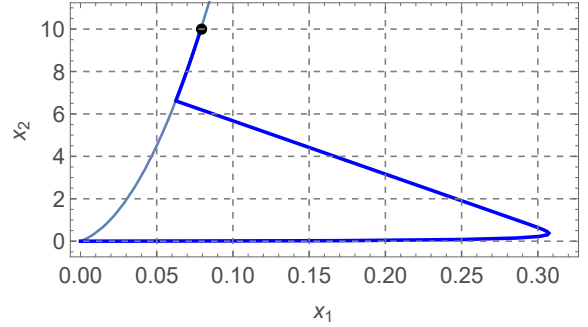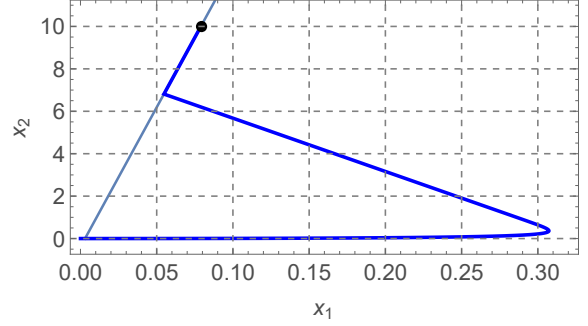

Figure 4: The solution (bold curve) of switched system (17) with the initial condition $x(0) = 0$, the parameters (19), and the switching signal $\sigma(x)$ given by (18) (top figure) and by (20) (bottom figure). The thin curve is the switching manifold $\sigma(x) = 0$ and the bold point is the switched equilibrium x_0.

which can be solved for (x_{01}, λ_0) when the reference voltage x_{02} is fixed. The conditions of Theorem 3.1 hold with the Lyapunov function

$$V(x) = \frac{1}{2x_C}(x_1 - x_{01})^2 + \frac{1}{2x_L}(x_2 - x_{02})^2.$$

Therefore,

$$V''(x_0)f^-(x_0) = \left(\frac{1}{x_C}\left(-\frac{r_L}{x_L}x_{01} + \frac{v_s}{x_L}\right), \frac{1}{x_L}\left(-\frac{1}{x_C(r_0+r_C)}x_{02}\right)\right),$$

which transpose will be denoted by n. Plugging n into (14), we conclude that any point x_0 that satisfies the switched equilibrium condition (17) with $\lambda_0 \in (0, 1)$, can be stabilized using the switching rule

$$\sigma(x) = \begin{cases} 1, & \text{if} \quad (x - x_d)n > 0, \\ 0, & \text{if} \quad (x - x_d)n < 0. \end{cases} \quad (18)$$

An implementation of switching rule (18) with the parameters

$$r_L = 20, \ r_C = 5, \ x_L = 600, \ x_C = 70, \ r_0 = 200, \ u_s = 8, \quad (19)$$

and the reference voltage $x_{02} = 10$ (which, when plugged into (17), yields $x_{01} = 0.79$ and $\lambda_0 = 0.367$ as one of the two possible solutions) is given in Fig. 4 (top).

For comparison, Fig. 4 (bottom) shows stabilization of (17) to the switched equilibrium $x_0 = (0.79, 10)$ using the switching rule (3), that can be shown to simplify to

$$\sigma(x) = \text{sign}\left(r_L r_C x_1^2 - (x_{01}r_L r_C - x_{02})x_1 - x_{01}x_2\right). \quad (20)$$

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The parameters (19) are slightly artificial, but similar to Fig. 4 simulations are achieved in the case of more realistic parameters e.g. taken from [1], [4], or [5]. The parameters (19) are chosen in such a way that the nonlinear behavior of the Bolzern-Spinelli rule (20) is clearly seen in Fig. 4 (bottom). The top and bottom figures of Fig. 4 turn out to be indistinguishable (on the screen) for the parameters from [1, 4, 5].

5. Conclusions

In this paper we showed that the switching rule (3) of Bolzern-Spinelli [2] for quadratic stabilization of a switched equilibrium x_0 of switched system (1) can be replaced by a linear switching rule (5) when the subsystems of (1) admit a common quadratic Lyapunov function. Moreover, our main result (Theorem 3.1) applies to nonlinear switched systems (6) complimenting the work by Mastellone et al [8] that proposes a nonlinear extension of Bolzern-Spinelli [2] in the case where the subsystems of (6) are shifts of one another (at the same time, the work [8] addresses the case of an arbitrary number of subsystems, while the present paper focuses on just two subsystems).

We would like to note that seemingly nonlinear switching rule (3) of Bolzern-Spinelli [2] simplifies to linear in wide classes of particular applications, e.g. in applications to buck converters (see e.g. Lu et al [6]), where $A^+ = A^-$ in (1), or in applications to boost converters of Fig. 3 with neglected resistance r_C of the capacitor (see e.g. Schild et al [10]). Still, the switching rule (3) stays nonlinear in some other classes of applications, e.g. in more general boost converters such as the one of Fig. 3 or its further extensions (see Gupta-Patra [5] and references therein). In these classes of applications the linear switching rules (5) and (14) proposed in this paper may simplify the engineering implementation of the feedback control.

6. Appendix: Lyapunov stability theorem for discontinuous systems with smooth Lyapunov functions

Proof of Theorem 3.2. Let x be a Filippov solution of (15), see e.g. Shevitz-Paden [11]. We pick $\rho > 0$ and prove that $x(t) \in \overline{W_\rho}$ beginning some $t = t_\rho$, where $W_\rho = \{x \in \mathbb{R}^n : V(x) < \rho\}$.

Step 1. Let $r > 0$ be such a constant that $x(0) \in \partial W_r$. We claim that $x(t) \in W_r$ for all $t > 0$. We prove by contradiction, i.e. assume that $x(\tau) \notin W_r$ for some $\tau > 0$. Without loss of generality we can assume that $x([0, \tau]) \subset W$, where W is an open neighborhood of $\overline{W_r}$, such that $w(x)$ is strictly positive in $W \setminus \{x_0\}$. For the function $v(t) = V(x(t))$ we have

$$v(0) = r \quad \text{and} \quad v(\tau) \geq r. \tag{21}$$

Step 1.1 We claim that $v(t) > r/2$ for all $t \in [0, \tau]$. Indeed, if the latter is wrong, then defining $s = \max\{t \in [0, \tau] : v(t) \leq r/2\}$, one gets

$$v(s) = r/2, \; v(\tau) = r, \; v(t) \in [r/2, r], \; \text{for any } t \in [s, \tau]. \tag{22}$$

In particular, $x(t) \neq x_0$ for all $t \in [s, \tau]$ and, therefore,

$$v'(t) = V'(x(t))\xi < 0,$$

for some $\xi \in K[f](x(t))$ and almost any $t \in [s, \tau]$. This contradicts (22) and proves that $v(t) > r/2$ for all $t \in [0, \tau]$.

Step 1.2 Step 1.1 implies that $x(t) \neq 0$, for any $t \in [0, \tau]$, and, as a consequence,

$$v'(t) < 0, \; \text{for any } t \in [0, \tau],$$

which contradicts (21) and completes the proof of the fact that $x(t) \in W_r$ for all $t > 0$.

Step 2. Let us show that $x(t)$ reaches $\overline{W_\rho}$ at some time moment. Assume that $x(t)$ never reaches $\overline{W_\rho}$. Then

$$v'(t) = V'(x(t))\xi < -w(x(t)),$$

for some $\xi \in K[f](x(t))$ and almost any $t > 0$. The definition of function w implies that $w_{\min} = \min\{w(x), \; x \in \overline{W_r} \setminus W_\rho\} > 0$. Therefore,

$$v(t) = v(0) + \int_0^t v'(t)dt < v(0) - w_{\min}t$$

and $v(t)$ becomes negative, if $x(t)$ never reaches $\overline{W_\rho}$. Since $\rho \in (0, r)$ was chosen arbitrary, our conclusion implies that $x(t) \to x_0$ as $t \to \infty$.

The proof of the theorem is complete. $\square$

Acknowledgements

7. References

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