On the Mean Stability of a Class of Switched Linear Systems

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***Abstract*— This paper investigates the mean stability of a class of discrete-time stochastic switched linear systems using the L^p-norm joint spectral radius of the probability distributions governing the switched systems. First we prove a converse Lyapunov theorem that shows the equivalence between the mean stability and the existence of a homogeneous Lyapunov function. Then we show that, when p goes to ∞, the stability of the pth mean becomes equivalent to the absolute asymptotic stability of an associated deterministic switched system. Finally we study the mean stability of Markovian switched systems. Numerical examples are presented to illustrate the results.**

I. Introduction

This paper studies the discrete-time stochastic switched linear system of the form

$$\Sigma : x(k+1) = A_k x(k) \tag{1}$$

where $x(k)$ represents a finite-dimensional state vector and $\{A_k\}_{k=0}^{\infty}$ is a stochastic process taking values in the set of square matrices of an appropriate dimension. One of its most natural stability notions is almost sure stability [16], which requires that the state $x(k)$ converges to the origin with probability one as $k \to \infty$. However this stability is difficult to check in practice because it is characterized by a quantity called the top Lyapunov exponent, whose computation is in general a hard problem [22]. For example, the necessary and sufficient condition in [6] is one of the most tractable conditions but cannot necessarily be checked with finite computation.

This difficulty has motivated many authors to study another stability, called pth *mean stability*, which requires that the expected value of the pth power of the norm of the state $x(k)$ converges to 0. It is well known [10]–[12], [14], [24] that the pth mean stability can be often checked by computing the spectral radius of a matrix that is easy to compute. Recently the authors showed [18] that, if each A_k follows an identical probability distribution independently and either p is even or the distribution possess a certain invariance property, then the pth mean stability of the system (1) is characterized by the spectral radius of a matrix, generalizing the results obtained in the literature [10], [11], [14]. Also it is shown that the mean square stability is equivalent to the existence of a certain quadratic Lyapunov function. The derivation of these results depends on an extended version [18] of L^p*-norm joint spectral radius* [19].

Generalizing the above result on Lyapunov functions, in this paper we first show that, for a general even exponent p, the pth mean stability of Σ is equivalent to the existence of a homogeneous Lyapunov function of degree p. This equivalence is still true for a general p provided the distribution possesses a certain invariance property. Imposing that the value of the function decreases along trajectories only in *expectation* enables us to construct a Lyapunov function even when the system is not stable for an arbitrary switching signal, as opposed to the results in the literature [7], [8], [17], Moreover our Lyapunov function can be constructed easily by solving a linear matrix inequality or an eigenvalue problem.

Then we study a limiting behavior of the pth mean stability. It is well known [9] that, roughly speaking, the pth mean stability becomes equivalent to almost sure stability in the limit of $p \to 0$. As a counterpart of this fact we show that, in the limit of $p \to \infty$, the pth mean stability is equivalent to the stability of an associated *deterministic* switched system for an arbitrary switching. This result will be proved by showing an extension of the formulas [2], [25] that express joint spectral radius as the limit of L^p-norm joint spectral radius.

Finally we extend the results in [18] to Markovian switched systems, where A_k in (1) is not necessarily identically and independently distributed. Again assuming the invariantce property of the Markov process we will give a characterization of the pth mean stability.

This paper is organized as follows. After preparing necessary mathematical notation and conventions, in Section II we review the basic facts of the stability of discrete-time linear switched systems. Section III proves a converse Lyapunov theorem. Then in Section IV we study the limiting behavior of the pth mean stability. Finally Section V studies the mean stability of Markovian switched systems.

A. Mathematical Preliminaries

Let $\mathbb{R}_+$ denote the set of nonnegative numbers. The spectral radius of a square matrix is denoted by $\rho(\cdot)$. A subset $K \subset \mathbb{R}^d$ is called a cone if K is closed under multiplication by nonnegative scalars. The cone is said to be solid if it possesses a nonempty interior. We say that a cone is pointed if it contains no line; i.e., if $x, -x \in K$ then $x = 0$. We say that K is *proper* if it is closed, convex, solid, and pointed. For example the positive orthant $\mathbb{R}_+^d$ of $\mathbb{R}^d$ is a proper cone. The dual cone K^* is defined by

$$K^* = \{f \in \mathbb{R}^d : f^\top x \geq 0 \text{ for every } x \in K\}.$$

A matrix $M \in \mathbb{R}^{d\times d}$ is said to leave K invariant, written $M \geq^K 0$, if $MK \subset K$. A subset $\mathscr{M} \subset \mathbb{R}^{d\times d}$ is said to leave K invariant if any matrix in $\mathscr{M}$ leaves K invariant. For $M, N \in$

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$\mathbb{R}^{d\times d}$, by $M \geq^K N$ we mean $M-N \geq^K 0$. M is said to be K-positive, written $M >^K 0$, if $M(K-\{0\})$ is contained in the interior $\operatorname{int} K$ of K. The next lemma collects some elementary facts of cones and their duals.

Lemma 1.1: Let K be a proper cone and let $M >^K 0$.

1) M has a simple eigenvalue $\rho(M)$, which is greater than the magnitude of any other eigenvalue of M. Moreover the eigenvector corresponding to the eigenvalue $\rho(M)$ is in $\operatorname{int} K$ (see, e.g., [23]);
2) K^* is a proper cone and $M^\top$ is K^*-positive [1, 2.23].

A norm $\|\cdot\|$ on $\mathbb{R}^d$ is said to be cone absolute [20] with respect to a proper cone K if, for every $x \in \mathbb{R}^d$,

$$\|x\| = \inf_{\substack{v,w\in K\\ x=v-w}} \|v+w\|. \tag{2}$$

Also we say that $\|\cdot\|$ is cone linear with respect to K if there exists $f \in K^*$ such that

$$\|x\| = f^\top x \text{ for every } x \in K. \tag{3}$$

A norm that is cone absolute and cone linear with respect to a proper cone is said to be cone linear absolute. Every $f \in \operatorname{int} K^*$ yields [20] the cone linear absolute norm determined by (3) and (2), and we denote this norm by $\|\cdot\|_f$. The next lemma lists some properties of cone linear absolute norms.

Lemma 1.2 ([20]): Let $\|\cdot\|$ be a cone linear absolute norm.

1) The induced norm of $M \in \mathbb{R}^{d\times d}$, defined by $\|M\| = \sup_{x\in\mathbb{R}^d} \frac{\|Mx\|}{\|x\|}$, satisfies
$$\|M\| = \sup_{x\in K} \frac{\|Mx\|}{\|x\|}. \tag{4}$$
2) If $M \geq^K N \geq^K 0$ then $\|M\| \geq \|N\|$.
3) If $M_i \geq^K N_i \geq^K 0$ for all $i = 1,\dots,k$ then $\|M_1\cdots M_k\| \geq \|N_1\cdots N_k\|$.

Proof: The first two claims can be found in [20]. The last one immediately follows from the second one. ∎

We denote the Kronecker product (see, e.g., [4]) of matrices M and N by $M\otimes N$. For a positive integer p define the Kronecker power $M^{\otimes p}$ by $M^{\otimes 1} := M$ and $M^{\otimes (p)} = M^{\otimes (p-1)} \otimes M$ recursively for a general p. It holds [4] that

$$(MN)^{\otimes p} = M^{\otimes p} N^{\otimes p}. \tag{5}$$

Also for $\mathscr{M} \subset \mathbb{R}^{d\times d}$ define $\mathscr{M}^{\otimes p} := \{M^{\otimes p} : M \in \mathscr{M}\} \subset \mathbb{R}^{d^p\times d^p}$. The next lemma is proved in [2].

Lemma 1.3: Let $\mathscr{M} \subset \mathbb{R}^{d\times d}$. If $\mathscr{M}$ leaves a proper cone K invariant then $\mathscr{M}^{\otimes p}$ leaves the proper cone

$$\tilde{K}_p := \overline{\operatorname{conv}(K^{\otimes p})} \tag{6}$$

invariant.

Let μ be a probability distribution on $\mathbb{R}^{d\times d}$. The support of μ is denoted by $\operatorname{supp}\mu$. For a measurable function f on $\mathbb{R}^{d\times d}$ we denote the expected value of f by $E_\mu[f] = \int_{\mathbb{R}^{d\times d}} f(X)\,d\mu(X)$. The subscript μ will be omitted when it is clear from the context. We define the probability distribution $\mu^{\otimes p}$ on $\mathbb{R}^{d^p\times d^p}$ as the image of the measure μ under the mapping $(\cdot)^{\otimes p}\colon \mathbb{R}^{d\times d} \to \mathbb{R}^{d^p\times d^p} : M \mapsto M^{\otimes p}$ (for detail see, e.g., [3]).

Finally we define the operator $\operatorname{vec}\colon \mathbb{R}^{m\times d} \to \mathbb{R}^{md}$ by

$$\operatorname{vec}\begin{bmatrix} M_1 & \cdots & M_d \end{bmatrix} := \begin{bmatrix} M_1 \\ \vdots \\ M_d \end{bmatrix},\ M_1,\dots,M_d \in \mathbb{R}^m.$$

We extend this definition to the product space $(\mathbb{R}^m)^N$ as

$$\operatorname{vec}(M_1,\dots,M_N) := \operatorname{vec}\begin{bmatrix} M_1 & \cdots & M_N \end{bmatrix}.$$

II. STABILITY OF SWITCHED LINEAR SYSTEMS

This section reviews the stability of switched linear systems, namely, the relationship between mean stability and L^p-norm joint spectral radius [18], and that between absolute asymptotic stability and joint spectral radius. Throughout this paper p will denote a positive integer and μ will denote a probability distribution on $\mathbb{R}^{d\times d}$ with compact support. Unless otherwise stated, A and A_k $(k = 0,1,\dots)$ will denote random variables that follow μ independently. $\|\cdot\|$ will denote a norm on $\mathbb{R}^d$ or $\mathbb{R}^{d\times d}$.

Consider the stochastic linear switched system

$$\Sigma_\mu : x(k+1) = A_k x(k),\ A_k \text{ follows } \mu \text{ independently}.$$

The solution of Σ_μ with the initial state $x(0) = x_0$ is denoted by $x(\cdot;x_0)$.

Definition 2.1: We say that Σ_μ is

1) *exponentially stable in pth mean* (*pth mean stable* for short) if there exist $M > 0$ and $\beta > 0$ such that, for any initial state x_0,
$$E[\|x(k;x_0)\|^p] \leq Me^{-\beta k}\|x_0\|^p;$$
2) *exponentially stable in pth moment* (*pth moment stable* for short) if there exist $M, \beta > 0$ such that, for every x_0,
$$\|E[x(k;x_0)^{\otimes p}]\| \leq Me^{-\beta k}\|x_0\|^p.$$

The mean stability of Σ_μ is closely related [18] to the L^p-norm joint spectral radius of μ defined as follows.

Definition 2.2 ([18]): The *L^p-norm joint spectral radius* (*p-radius* for short) of μ is defined by

$$\rho_{p,\mu} := \lim_{k\to\infty} \left(E[\|A_k\cdots A_1\|^p]\right)^{1/pk}. \tag{7}$$

This definition extends [18] the classical L^p-norm joint spectral radius (see, e.g., [15]) for a finite set of matrices. By the compactness of the support of μ, the p-radius $\rho_{p,\mu}$ is well-defined and is finite [18]. Also, by the equivalence of the norms on $\mathbb{R}^{d\times d}$, the value of p-radius does not depend on the norm used in (7).

The next proposition summarizes the characterization of the pth mean stability obtained in [18].

Proposition 2.3 ([18]): Assume that either

a) p is even or
b) $\operatorname{supp}\mu$ leaves a proper cone invariant.

Then the following conditions are equivalent:

1) $\rho_{p,\mu} < 1$;

2) Σ_μ is pth mean stable;
3) Σ_μ is pth moment stable.

Moreover it holds that

$$\rho_{p,\mu} = \left(\rho(E[A^{\otimes p}])\right)^{1/p}.$$

We will also need the next lemma that lists basic properties of p-radius.

Lemma 2.4 ([18]):

1) $\rho_{p,\mu}$ is non-decreasing with respect to p.
2) For all p and k it holds that

$$\rho_{p,\mu} = \left[\rho_{p/k,\mu^{\otimes k}}\right]^{1/k}. \tag{8}$$

Let us also review the notion of joint spectral radius [13]. Let $\mathscr{M}$ be a subset of $\mathbb{R}^{d\times d}$. The joint spectral radius of $\mathscr{M}$ is defined by

$$\hat{\rho}(\mathscr{M}) := \limsup_{k\to\infty} \sup_{A_1,\dots,A_k\in\mathscr{M}} \|A_k\cdots A_1\|^{1/k}.$$

Again this quantity is independent of the norm $\|\cdot\|$. The joint spectral radius is known to characterize the stability of the *deterministic* switched system

$$\Sigma_{\mathscr{M}} : x(k+1) = A_k x(k),\ A_k \in \mathscr{M}.$$

$\Sigma_{\mathscr{M}}$ is said to be *absolutely asymptotically stable* if $x(k)\to 0$ as $k\to\infty$ for every possible switching pattern. The next proposition is well known (for its proof see, e.g., [21]).

Proposition 2.5: $\Sigma_{\mathscr{M}}$ is absolutely asymptotically stable if and only if $\hat{\rho}(\mathscr{M}) < 1$.

III. Converse Lyapunov Theorem

The aim of this section is to show a converse Lyapunov theorem for the switched system Σ_μ. Let us begin by defining Lyapunov functions for Σ_μ.

Definition 3.1: A continuous and positive definite function $V: \mathbb{R}^d \to \mathbb{R}$ is said to be a *Lyapunov function* for Σ_μ if there exists $0 \le \gamma < 1$ such that

$$E[V(Ax)] \le \gamma V(x) \tag{9}$$

for every $x \in \mathbb{R}^d$.

The next theorem is the main result of this section.

Theorem 3.2: Assume that either

a) p is even or
b) $\operatorname{supp}\mu$ leaves a proper cone K invariant and moreover $E[A^{\otimes p}] >^{\tilde{K}_p} 0$.

Then Σ_μ is pth mean stable if and only if it admits a homogeneous Lyapunov function of degree p.

Remark 3.3: The assumptions in Theorem 3.2 are needed because the theorem relies on Proposition 2.3. The relaxation of those assumptions is left as an open problem. A possible approach can be found in [15], where the authors proposes a method to approximately compute p-radius without such assumptions.

It is straightforward to prove sufficiency.

Proof of sufficiency in Theorem 3.2: Assume that Σ_μ admits a homogeneous Lyapunov function of degree p. Let $x_0 \in \mathbb{R}^d$ be arbitrary. Using induction we can show $E[V(x(k;x_0))] \le \gamma^k V(x_0)$. Since V is continuous, homogeneous with degree p, and positive definite, there exist constants $C_1, C_2 > 0$ such that $C_1\|x\|^p \le V(x) \le C_2\|x\|^p$ for every $x\in\mathbb{R}^d$. Therefore $E[\|x(k;x_0)\|^p] \le (C_2/C_1)\gamma^k\|x_0\|^p$. Thus Σ_μ is pth mean stable. ■

The rest of this section is devoted for the proof of necessity. The proof for the case a) depends on its special case $p=2$, which is proved in [18] and is quoted as the next proposition for ease of reference.

Proposition 3.4 ([18]): Σ_μ is mean square stable if and only if it admits a quadratic Lyapunov function of the form $x^\top Hx$, where H is a positive definite matrix. Moreover such an H can be obtained by solving a linear matrix inequality.

To prove the second case b) we will need the next proposition.

Proposition 3.5: Let $K \subset \mathbb{R}^d$ be a proper cone and assume that $M >^K 0$. Then there exists $f \in \operatorname{int}(K^*)$ that induces the cone linear absolute norm $\|\cdot\|_f$ such that $\|M\|_f = \rho(M)$.

Proof: By Lemma 1.1 the matrix M admits the Jordan canonical form $J = V^{-1}MV$ where $V\in\mathbb{R}^{d\times d}$ is an invertible matrix whose columns are the generalized eigenvectors of M and $J \in \mathbb{R}^{d\times d}$ is of the form

$$J = \begin{bmatrix} J_0 & 0 \\ 0 & \rho(A)\end{bmatrix}$$

for some upper diagonal matrix J_0. Define $f\in\mathbb{R}^d$ by

$$V^{-1} = \begin{bmatrix} * \\ f^\top \end{bmatrix}.$$

We can easily see that f is an eigenvector of $M^\top$ corresponding to the eigenvalue $\rho(M)$. Since K^* is proper, Lemma 1.1 shows $f\in\operatorname{int}(K^*)$. Thus f gives a cone linear absolute norm with respect to K. Since for every $x\in K$ we have

$$\|Mx\|_f = f^\top Mx = \rho(M) f^\top x = \rho(M)\|x\|_f,$$

the equation (4) immediately shows $\|M\|_f = \rho(M)$. ■

Now we can complete the proof of Theorem 3.2.

Proof of necessity in Theorem 3.2: First consider the case a). Assume that Σ_μ is pth mean stable for $p=2q$ where q is a positive integer. Then Proposition 2.3 gives $\rho_{p,\mu} < 1$ and therefore $\rho_{2,\mu^{\otimes q}} < 1$ by (8). Hence, by Proposition 3.4, $\Sigma_{\mu^{\otimes q}}$ admits a homogeneous Lyapunov function V of degree 2. Now define $W:\mathbb{R}^d\to\mathbb{R}$ by $W(x) := V(x^{\otimes q})$. Since V is a Lyapunov function and is homogeneous of degree 2, W is continuous, positive definite, and is homogeneous of degree $2q = p$. Moreover, by (5), if B follows $\mu^{\otimes q}$ then

$$\begin{aligned} E[W(Ax)] &= E[V(A^{\otimes q}x^{\otimes q})] \\ &= E[V(Bx^{\otimes q})] \\ &\le \gamma V(x^{\otimes q}) \\ &= \gamma W(x) \end{aligned}$$

for some $\gamma < 1$ since V is a Lyapunov function for $\Sigma_{\mu^{\otimes q}}$. This shows that W is a Lyapunov function for Σ_μ.

Then let us turn to the second case b). We only consider the special case $p=1$. Notice that $\tilde{K}_1 = K$. Assume that

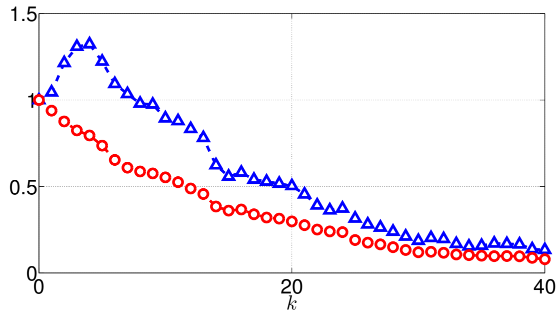

Fig. 1. The sample means of the Lyapunov function (circle) and the Euclidean norm (triangle)

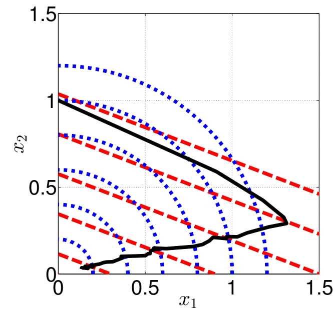

Fig. 2. The averaged sample path (solid) and the level plots of the Lyapunov function (dashed) and the Euclidean norm (dotted)

$\operatorname{supp}\mu$ leaves a proper cone K invariant and $E[A] >^K 0$. If Σ_μ is (first) mean stable then $\gamma := \rho(E[A]) < 1$ by Proposition 2.3. Also, by Proposition 3.5, there exists a cone linear absolute norm $\|\cdot\|$ on $\mathbb{R}^d$ such that $\|E[A]\| = \gamma$. Now we define $V(x) := \|x\|$. We need to show (9). Let $x \in \mathbb{R}^d$ and $\varepsilon > 0$ be arbitrary. Since $\|\cdot\|$ is cone absolute there exist $x_1, x_2 \in K$ such that $x = x_1 - x_2$ and $\|x_1 + x_2\| \le \|x\| + \varepsilon$. Notice that, by the linearity of $\|\cdot\|$ on K, we have $E[\|Ax_i\|] = \|E[A]x_i\| \le \gamma\|x_i\|$. Therefore

$$\begin{aligned} E[V(Ax)] &= E[\|Ax_1 - Ax_2\|] \\ &\le E[\|Ax_1\|] + E[\|Ax_2\|] \\ &\le \gamma(\|x_1\| + \|x_2\|) \\ &\le \gamma(\|x\| + \varepsilon) \end{aligned}$$

again by the linearity of $\|\cdot\|$ on K. Since $\varepsilon > 0$ was arbitrary the inequality (9) actually holds and this finishes the proof for $p = 1$. The case for a general p can be proved in the same way as the first half of this proof with Lemma 1.3. ■

Example 3.6: Consider the probability distribution

$$\mu = \begin{bmatrix} [0, 1.5] & [0, 1.8] \\ [0, 0.15] & [0, 1.2] \end{bmatrix}$$

where each closed interval indicate that the corresponding element of μ is the uniform distribution on the interval. Clearly $\operatorname{supp}\mu$ leaves the proper cone $\mathbb{R}^2_+$ invariant. Also the mean $E[A]$ has only positive entries so that it is $\mathbb{R}^2_+$-positive. Therefore Theorem 3.2 shows that Σ_μ admits a homogeneous Lyapunov function of degree 1. From the proof of the theorem, the Lyapunov function is given as a cone linear absolute norm $\|x\|_f$ and we can obtain the f by following the proof of Proposition 3.5 as $f = \begin{bmatrix} 0.3838 & 1 \end{bmatrix}^\top$.

We generate 200 sample paths of Σ_μ with the initial state $x_0 = \begin{bmatrix} 0 & 1 \end{bmatrix}^\top$. Fig. 1 shows the sample means of the Lyapunov function $\|x(k)\|_f$ and the Euclidean norm $\|x(k)\|$. While the sample mean of our Lyapunov function is almost decreasing, that of the Euclidean norm shows oscillation. Fig. 2 shows the average of the sample paths and the contour plot of our Lyapunov function and the Euclidean norm.

Remark 3.7: Since $\operatorname{supp}\mu$ is infinite in Example 3.6, we cannot use the methods to construct a Lyapunov function for positive systems proposed in the literature (see, e.g., [8]).

IV. Limiting Behavior of pth Mean Stability

This section studies the limiting behavior of pth mean stability as $p \to \infty$. We start with the following observation. Let μ be a probability distribution on $\mathbb{R}^{d\times d}$ and let $\mathscr{M} = \operatorname{supp}\mu$. Then the definitions of p-radius and joint spectral radius show $\rho_{p,\mu} \le \hat\rho(\mathscr{M})$. Since $\rho_{p,\mu}$ is non-decreasing with respect to p by Lemma 2.4 we have

$$\lim_{p\to\infty} \rho_{p,\mu} \le \hat\rho(\mathscr{M}). \tag{10}$$

It is then natural to ask when the equality holds in this inequality. We will show that the equality still holds under the following assumption, which is weaker than the one in [2].

Assumption 4.1:

a) $\mathscr{M}$ leaves a proper cone invariant;
b) The singular part μ_s of μ consists of only point measures, i.e., $\mu_s = \sum_{i=1}^N p_i \delta_{M_i}$ for some positive numbers $p_1, \dots, p_N$ and matrices $M_1, \dots, M_N$.

The next theorem is the main result of this section.

Theorem 4.2: If μ satisfies Assumption 4.1 then

$$\lim_{p\to\infty} \rho_{p,\mu} = \hat\rho(\mathscr{M}).$$

This theorem has two corollaries, both of which can be proved easily using Proposition 2.3. The first one shows a novel relationship between the stability of the deterministic switched system $\Sigma_{\mathscr{M}}$ and the stochastic switched system Σ_μ.

Corollary 4.3: Assume that μ satisfies Assumption 4.1. Then $\Sigma_{\mathscr{M}}$ is absolutely asymptotically stable if and only if there exists $\gamma < 1$ such that Σ_μ is pth mean stable with the decay rate at most γ; i.e., the expectation $E[\|x(k)\|^p]^{1/p}$ is of order $O(\gamma^k)$ for every p.

The second corollary gives an expression of joint spectral radius.

Corollary 4.4: If μ satisfies Assumption 4.1 then

$$\hat\rho(\mathscr{M}) = \lim_{p\to\infty} \rho\left(E[A^{\otimes p}]\right)^{1/p}.$$

To prove Theorem 4.2 we need the next lemma. Its proof is omitted due to limitations of space.

Lemma 4.5: Assume that μ satisfies Assumption 4.1. Then, for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\mu(\{X : X \ge^K (1-\varepsilon)M\}) \ge \delta$$

for every $M \in \mathscr{M}$.

Let us prove Theorem 4.2.

Proof of Theorem 4.2: By Proposition 2.3 and the inequality (10) it is sufficient to show $\lim_{p\to\infty}\rho_{p,\mu} \geq \hat{\rho}(\mathscr{M})$. Let $\gamma := \hat{\rho}(\mathscr{M})$. Let us take any cone linear absolute norm $\|\cdot\|$ with respect to the invariant cone. By Proposition 2.5, we can show that there exist $C > 0$ and $\{M_k\}_{k=1}^{\infty} \subset \mathscr{M}$ such that $\|M_k \cdots M_1\| > C\gamma^k$ for infinitely many k.

Take arbitrary γ_1 and γ_2 such that $\gamma > \gamma_1 > \gamma_2$. Define $\varepsilon := (\gamma - \gamma_1)/\gamma$ and take the corresponding $\delta > 0$ given by Lemma 4.5. Observe that, by Lemma 1.2, if $X_i \geq^K (1-\varepsilon)M_i = (\gamma_1/\gamma)M_i$ then $\|X_k \cdots X_1\| \geq (\gamma_1/\gamma)^k \|M_k \cdots M_1\| > C\gamma_1^k$. Therefore,

$$\begin{aligned} E[\|A_k \cdots A_1\|^p] >& \mu^k \left(\|A_k \cdots A_1\| > C\gamma_1^k\right)\left(C\gamma_1^k\right)^p \\ \geq& \left(\prod_{i=1}^{k} \mu(\{X_i : X_i \geq^K (1-\varepsilon)M_i\})\right) C^p \gamma_1^{pk} \\ \geq& \delta^k C^p \gamma_1^{pk} \end{aligned}$$

and hence we have $E[\|A_k \cdots A_1\|^p]^{1/kp} > C^{1/k}\delta^{1/p}\gamma_1$. Choose a sufficiently large p_0 such that $\delta^{1/p_0}\gamma_1 > \gamma_2$. Then $E[\|A_k \cdots A_1\|^{p_0}]^{1/kp_0} > C^{1/k}\gamma_2$ for infinitely many k. This implies $\rho_{p_0,\mu} \geq \gamma_2$ and therefore $\lim_{p\to\infty}\rho_{p,\mu} \geq \gamma_2$. This completes the proof since γ_2 can be made arbitrary close to $\gamma = \hat{\rho}(\mathscr{M})$. ■

The next example gives a simple illustration of Theorem 4.2.

Example 4.6: Let μ be the uniform distribution on $[0,\gamma]$ for some $\gamma > 0$. We can see that μ satisfies Assumption 4.1. Also clearly $\hat{\rho}(\mathscr{M}) = \gamma$. On the other hand we have $\rho_{p,\mu}^p = \rho(E[A^{\otimes p}]) = \gamma^p/(p+1)$. Therefore $\rho_{p,\mu} = \gamma(p+1)^{-p}$ and hence $\lim_{p\to\infty}\rho_{p,\mu} = \gamma = \hat{\rho}(\mathscr{M})$, as expected. We remark that, since $\operatorname{supp}\mu$ is infinite, we cannot apply the result in [2] to this example.

Finally the next theorem generalizes another characterization of joint spectral radius in [25], which does not need the existence of an invariant cone.

Theorem 4.7: If μ satisfies the condition b) of Assumption 4.1 then

$$\hat{\rho}(\mathscr{M}) = \limsup_{p\to\infty}\left(\rho(E[A^{\otimes p}])\right)^{1/p}. \tag{11}$$

Sketch of the proof: Using Theorem 4.2 and the semidefinite lifting [2] of matrices one can show $\hat{\rho}(\mathscr{M}) = \lim_{p\to\infty}\rho(E[X^{\otimes(2p)}])^{1/(2p)}$. Also it is possible to see that the sequence $\{(\rho(E[A^{\otimes p}]))^{1/p}\}_{p=1}^{\infty}$ is not decreasing. These observations yield (11). ■

V. Markovian Case

So far we have restricted our attention to the case when the random variables $\{A_k\}_{k=0}^{\infty}$ are identically and independently distributed. In this section we study a more practical case of when the process $\{A_k\}_{k=0}^{\infty}$ is a time-homogeneous Markov chain. Let $\{\sigma_k\}_{k=0}^{\infty}$ be a time-homogeneous Markov chain taking values in $\{1,\dots,N\}$ with the transition probability matrix $\mathbb{P} = [p_{ij}]$ and the constant initial state $\sigma_0 \in \{1,\dots,N\}$. Let $\mathscr{M} = \{M_1,\dots,M_N\}$ be a set of $d \times d$ square matrices. We define the stochastic process $A = \{A_k\}_{k=0}^{\infty}$ by $A_k = M_{\sigma_k}$ and the switched system Σ_A by

$$\Sigma_A : x(k+1) = A_k x(k).$$

The pth mean and moment stability of Σ_A is defined in the same way as Definition 2.1. Also we define the L^p-norm joint spectral radius $\rho_{p,A}$ of A by (7).

The next theorem summarizes the relationship between p-radius, mean stability, and moment stability for the Markovian switched system Σ_A. Though the result actually holds for a general p, for simplicity of presentation we will restrict our attention to $p = 1$ or 2.

Theorem 5.1: Assume that either $p = 2$, or $\mathscr{M}$ leaves a proper cone K invariant and $p = 1$. Then the following conditions are equivalent.

i) $\rho_{p,A} < 1$ for every σ_0;
ii) Σ_A is pth mean stable;
iii) Σ_A is pth moment stable;
iv) Define the $Nd^p \times Nd^p$ matrix T_p by
$$T_p = (\mathbb{P}^\top \otimes I_{d^p})\operatorname{diag}(M_1^{\otimes p},\dots,M_N^{\otimes p}).$$
Then $\rho(T_p) < 1$.

To prove this theorem we introduce the following definitions [5]. Let $x(\cdot;x_0,\sigma_0)$ denote the trajectory of Σ_A with the initial conditions x_0 and σ_0. For each k define $Q(k) = (Q_i(k),\dots,Q_N(k)) \in (\mathbb{R}^d)^N$ by $Q_i(k;x_0,\sigma_0) := E[x(k;x_0,\sigma_0)_{\sigma_k=i}]$, and the operator $\mathscr{L}$ on $(\mathbb{R}^d)^N$ by $\mathscr{L}(H) := (\mathscr{L}_1(H),\dots,\mathscr{L}_N(H))$ and $\mathscr{L}_j(H) := \sum_{i=1}^n p_{ij}A_iH_i$. The proof of the next lemma is omitted because it can be shown in the same way as [5].

Lemma 5.2: Let $\mathscr{L}$ and Q be as above.

1) $E[x(k;x_0,\sigma_0)] = \sum_{j=1}^N Q_j(k;x_0,\sigma_0)$.
2) $Q(k+1;x_0,\sigma_0) = \mathscr{L}(Q(k;x_0,\sigma_0))$.
3) For every $H \in (\mathbb{R}^d)^N$, $\operatorname{vec}\mathscr{L}(H) = T_1(\operatorname{vec}H)$.

With this lemma we can prove Theorem 5.1.

Proof of Theorem 5.1: The equivalence i) $\Leftrightarrow$ ii) $\Leftrightarrow$ iii) can be proved in the same way as Proposition 2.3. Also, when $p = 2$, the equivalence ii) $\Leftrightarrow$ iv) is proved in [5]. Therefore it is sufficient to show the implications iv) $\Rightarrow$ iii) and ii) $\Rightarrow$ iv) under the assumption that $p = 1$ and $\mathscr{M}$ leaves a proper cone K invariant

[iv) $\Rightarrow$ iii)]: Assume $\rho(T_1) < 1$ and let x_0 and σ_0 be arbitrary. Then, by Lemma 5.2,

$$\begin{aligned} \operatorname{vec}Q(k;x_0,\sigma_0) &= \operatorname{vec}(\mathscr{L}^k(Q(0;x_0,\sigma_0))) \\ &= T_1^k \operatorname{vec}Q(0;x_0,\sigma_0) \to 0, \end{aligned}$$

which shows $Q(k;x_0,\sigma_0) \to 0$ as $k \to \infty$ and therefore $E[x(k;x_0,\sigma_0)] \to 0$ for all x_0 and σ_0. Thus Σ_A is first moment stable.

[ii) $\Rightarrow$ iv)]: Omitted due to limitations of space. ■

The next corollary of Theorem 5.1 enables us to compute p-radius efficiently.

Corollary 5.3: If p and μ satisfies the assumption in Theorem 5.1 then it holds that

$$\rho_{p,A} = \rho(T_p)^{1/p}.$$

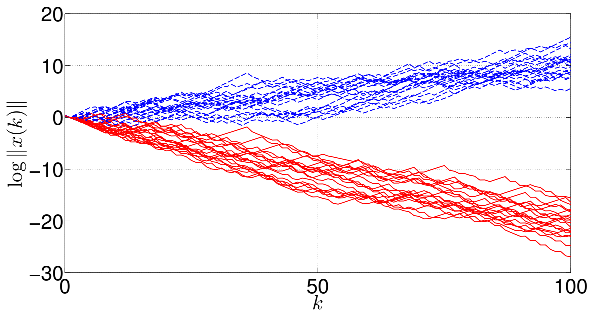

Fig. 3. Sample paths of switched systems. Dashed: Before stabilization. Solid: After stabilization

Finally let us apply the results obtained in this section to the stabilization of Markovian switched systems Let

$$N = 3,\ \mathbb{P} = \begin{bmatrix} 0.3 & 0.5 & 0.2 \\ 0.5 & 0.3 & 0.2 \\ 0.2 & 0.2 & 0.6 \end{bmatrix},$$

$$M_1 = \begin{bmatrix} .32 & .49 \\ .24 & .33 \end{bmatrix},\ M_2 = \begin{bmatrix} .53 & .65 \\ .75 & .85 \end{bmatrix},\ M_3 = \begin{bmatrix} 1.5 & .51 \\ .18 & .69 \end{bmatrix}.$$

Corollary 5.3 gives $\rho_{1,A} = 1.221$ and therefore Σ_A is not first mean stable. Let us consider the stabilization of Σ_A. Define the switched system with input by $x(k+1) = A_k x(k) + b_k u(k)$ where $b_k = n_{\sigma_k}$ with

$$n_1 = \begin{bmatrix} -0.56 \\ 0.39 \end{bmatrix},\ n_2 = \begin{bmatrix} 0.40 \\ -1.70 \end{bmatrix},\ n_3 = \begin{bmatrix} -0.37 \\ -0.49 \end{bmatrix}.$$

As an input we use the static state feedback $u(k) = f x(k)$ for some $f \in \mathbb{R}^{1\times 2}$. This yields the controlled system

$$\Sigma_{A+bf} : x(k+1) = (A_k + b_k f) x(k)$$

Let $f = \begin{bmatrix} 0.36 & 0.50 \end{bmatrix}$. Then all the matrices $M_i + n_i f$ ($i = 1,2,3$) have only positive entries. Therefore we can use Corollary 5.3 to find $\rho_{1,A+bf} = 0.9554$. Therefore the controlled system Σ_{A+bf} is first mean stable by Theorem 5.1. Fig. 3 shows the 20 sample paths of the original switched system and the stabilized switched system. Finding a systematic way to obtain a stabilizing feedback gain is left as an open problem.

VI. Conclusion

We investigated the mean stability of a class of discrete-time stochastic switched systems. First we presented the equivalence between mean stability and the existence of a homogeneous Lyapunov function. Then we showed that, in the limit of $p \to \infty$, the pth mean stability becomes equivalent to the absolute asymptotic stability of an associated deterministic switched system. Finally the characterization of the stability of a class of Markovian switched systems was given. Throughout the paper L^p-norm joint spectral radius has played a key role.

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