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Stochastic Barbalat's Lemma and Its Applications

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***Abstract*—In the deterministic case, a significant improvement on stability analysis of nonlinear systems is caused by introducing Barbalat's lemma into control area after Lyapunov's second method and LaSalle's theorem were established. This note considers the extension of Barbalat's lemma to the stochastic case. To this end, the uniform continuity and the absolute integrability are firstly described in stochastic forms. It is nevertheless a small generalization upon the existing references since our result can be used to adapted processes which are not necessarily Itô diffusions. When it is applied to Itô diffusion processes, many classical results on stochastic stability are covered as special cases.**

***Index Terms*—Barbalat's lemma, stochastic stability, stochastic systems.**

I. Introduction

In the end of the 19th century, Lyapunov introduced the concept of stability of a dynamic system and created a very powerful tool known as Lyapunov's second method in [1], where the stability can be proven without requiring knowledge of the exact solution. The main difficulty is that the asymptotic stability can not be analyzed when the derivative of Lyapunov function is negative semidefinite. To resolve this problem,

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LaSalle's theorem was proposed by [2] for locating limit sets of time-invariant systems. Barbalat's lemma in [3] has been a well-known and powerful tool to deduce asymptotic stability of nonlinear systems (specially time-varying systems) since Popov introduced it into control area in [4]. For more details, please refer to some standard textbooks such as [5]–[9].

On the other hand, the stochastic Lyapunov's second method has been developed to deal with the stochastic stability by many authors, and here we only mention [10]–[16]. These classical results typically state that the solutions of a stochastic differential equation converge in probability to an invariant set if the initial condition approaches this invariant set. There appeared some stochastic versions of LaSalle-like theorem that locate limit sets of a system with linear growth condition (see,[17]) or without linear growth condition (see,[18]–[20]). To analyze the stochastic integral input-to-state stability, an important corollary of Barbalat's lemma was extended by [21] to the stochastic case. In theory, the gap left by stochastic Lyapunov stability theorems and stochastic LaSalle-like theorems should be closed by a general stochastic Barbalat's lemma as in the deterministic case.

The purpose of this technical note is to extend some appropriate forms of Barbalat's lemma to the stochastic case. Definitions of the absolute integrability, the (strong) boundedness in probability and the uniform continuity in probability together with their propositions are first presented as preliminaries. By the aid of many materials from [18], [20] and [21], some stochastic versions of Barbalat's lemma are proposed. The efficiency of our results can be demonstrated by covering many classical conclusions about stochastic stability as special cases. The most important advantage is that the application is not limited to the Itô diffusion process given by a SDE.

The technical note is organized as follows: Some preliminaries are given in Section II. The stochastic Barbalat's lemma is proposed in Section III, and their application are presented in Section IV. Conclusion is drawn in Section V.

Notations

The following notations are used throughout the technical note: For a vector x, $|x|$ denotes its usual Euclidean norm and x^T denotes its transpose; $|X|$ denotes the Frobenius norm of a matrix X defined by $|X| = (\mathrm{Tr}\{XX^T\})^{1/2}$, where $\mathrm{Tr}(\cdot)$ denotes the matrix trace; $\mathbb{R}^n$ denotes the real n-dimensional space; $\mathbb{R}_+$ denotes the set of all nonnegative real numbers; $\mathcal{C}^i$ denotes the set of all functions with continuous i-th partial derivative; $\mathcal{K}$ denotes the set of all functions: $\mathbb{R}_+ \to \mathbb{R}_+$, which are continuous, strictly increasing and vanish at zero; $\mathcal{K}_\infty$ denotes the set of all functions which are of class $\mathcal{K}$ and unbounded.

II. Preliminaries

A. Barbalat's Lemma in the Deterministic Case

In many textbooks about adaptive control, one can find the following standard Barbalat's lemma and one of its useful corollaries.

Lemma 1 ([7, Lemma A.6]): If function $\phi : \mathbb{R}_+ \to \mathbb{R}$ is uniformly continuous, and the limit $\lim_{t\to\infty} \int_0^t \phi(s)ds$ exists and is finite, then $\lim_{t\to\infty} \phi(t) = 0$.

Corollary 1 ([7, Corollary A.7]): If $\phi, \dot{\phi}$ are bounded and ϕ is absolutely integrable, then $\lim_{t\to\infty} \phi(t) = 0$.

It will be shown that the extension of Lemma 1 to the stochastic case is difficult. There is no means to extend Corollary 1 directly to the stochastic case, because $\dot{\phi}$ has no meaning for stochastic process ϕ. It is essential to rewrite Lemma 1 and Corollary 1 in suitable forms convenient for the extensions to the stochastic case.

Since absolute integrability of ϕ implies that $\lim_{t\to\infty} \int_0^t \phi(s)ds$ exists and is finite, a special version of Barbalat's lemma can be easily obtained.

Corollary 2: If function $\phi : \mathbb{R}_+ \to \mathbb{R}$ is uniformly continuous and absolutely integrable, then $\lim_{t\to\infty} \phi(t) = 0$.

The popularity of Barbalat's lemma in control area comes from the fact that it is often used to analyze the convergence of solutions of closed-loop systems.

Corollary 3: Consider a nonlinear system $\dot{x} = f(x,t), x(t_0) = x_0$, where f is locally Lipschitz in x and piecewise continuous in t, and $f(0,t) = 0$. If x is bounded, $\beta(x)$ is a continuous function and $\beta(x(t))$ is absolutely integrable, then $\lim_{t\to\infty} \beta(x(t)) = 0$.

Proof: Locally Lipschitz condition, $f(0,t) = 0$ and boundedness of x imply the global Lipschitz condition and linear growth condition, then system $\dot{x} = f(x,t)$ has a unique solution in $[t_0, \infty)$. It comes from the linear growth condition and boundedness of x that $f(x,t)$ is bounded, which means that the solution $x(t)$ starting from x_0 is uniformly continuous, according to mean value theorem. There exists an interval $[a,b]$ such that $\beta(x)$ is uniformly continuous on $[a,b]$ and $x(t) \in [a,b], \forall t \in [t_0, \infty)$ because of the continuity of $\beta(x)$ and boundedness of x. The uniform continuities of $\beta(x)$ and $x(t)$ results in the uniform continuity of $\beta(x(t))$ on $\mathbb{R}_+$. From Corollary 2, it is concluded that $\beta(x(t)) \to 0$ as $t \to \infty$. ■

The purpose of this technical note is to extend Corollaries 2 and 3 instead of Lemma 1 and Corollary 1 to the stochastic case.

B. Integrability and Boundedness of Stochastic Processes

In the rest of this technical note, the underlying complete probability space is taken to be the quartet $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ with a filtration $\mathcal{F}_t$ satisfying the usual conditions (i.e., it is increasing and right continuous while $\mathcal{F}_0$ contains all P-null sets).

Definition 1: The stochastic process $\phi(t) : \mathbb{R}_+ \times \Omega \to \mathbb{R}^n$ is absolutely integrable on $\mathbb{R}_+ \times \Omega$ if

$$E\int_0^\infty |\phi(s)|ds < \infty \tag{1}$$

and we denote $\phi \in \mathcal{M}(\mathbb{R}_+ \times \Omega)$.

When $\phi(t) \in \mathcal{M}(\mathbb{R}_+ \times \Omega)$ is continuous, according to Fubini's theorem [22, Theorem 2.36], (1) is equivalent to $\int_0^\infty E|\phi(s)|ds < \infty$.

According to Chebyshev's inequality, we can obtain a natural conclusion.

Proposition 1: If the stochastic process $\phi(t) \in \mathcal{M}(\mathbb{R}_+ \times \Omega)$, then for any $\epsilon \in (0,1]$ there exists a $\delta > 0$ such that

$$P\left\{\int_0^\infty |\phi(s)|ds > \delta\right\} < \epsilon.$$

Definition 2: The stochastic process $\phi(t) : \mathbb{R}_+ \times \Omega \to \mathbb{R}^n$ is called (strongly) bounded in probability if for any $\epsilon > 0$ there exists an $r > 0$ such that

$$P\left\{\sup_{t\geq 0} |\phi(t)| > r\right\} \leq \epsilon.$$

Remark 1: The boundedness in probability here is stronger than the (weak) boundedness in probability proposed in [12, P.15] considering the following facts $P\{\sup_{t\geq 0} |\phi(t)| > r\} \geq \sup_{t\geq 0} P\{|\phi(t)| > r\}$. □

C. Stochastic Uniform Continuity

The stochastic version of uniform continuity is recited.

Definition 3 ([23, Definition 3.2.4]): Stochastic process $x(t) : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}^n$ is uniformly continuous in probability (or, stochastic uniformly continuous), if for any $\varepsilon, \epsilon > 0$, there exists a $\delta(\varepsilon, \epsilon) > 0$ such that for all $s, t \in \mathbb{R}_+$ satisfying $|t - s| \leq \delta$ the following condition holds:

$$P\{|x(t) - x(s)| \leq \varepsilon\} \geq 1 - \epsilon. \quad (2)$$

This definition can be further weakened.

Definition 4: Stochastic process $x(t)$ is locally uniformly continuous in probability, if for any parameters $\varepsilon, \epsilon, r > 0$, there exists a constant $\delta(\varepsilon, \epsilon, r) > 0$ such that for each $s, t \in \mathbb{R}_+$ satisfying $|t - s| \leq \delta$, the following inequality holds:

$$P\{|x(t \wedge \sigma_r) - x(s \wedge \sigma_r)| \leq \varepsilon\} \geq 1 - \epsilon \quad (3)$$

where $\sigma_r = \inf\{t \geq 0 : |x(t)| \geq r\}$ (with special case $\inf \vee = \infty$).

By adding boundedness condition to the local definition, one can obtain the global result.

Proposition 2: A process is uniformly continuous in probability if it is locally uniformly continuous in probability and bounded in probability.

Proof: For any $\varepsilon, \epsilon > 0$, let us prove that there exists a $\delta(\varepsilon, \epsilon) > 0$ such that, for all $s, t \in \mathbb{R}_+$ with $|t - s| \leq \delta$, inequality (2) holds.

Since $x(t)$ is bounded in probability, then for any $\epsilon > 0$ there exists an $r(\epsilon) > 0$ such that

$$P\left\{\sup_{t \geq 0} |x(t)| > r\right\} \leq \frac{1}{2}\epsilon.$$

From the local uniform continuity of x, for r given above and any $\varepsilon > 0$, there exists a $\delta(\varepsilon, \epsilon) > 0$ such that, for all s, t satisfying $|t - s| \leq \delta$, we have

$$P(|x(t \wedge \sigma_r) - x(s \wedge \sigma_r)| > \varepsilon\}) \leq \frac{1}{2}\epsilon$$

where $\sigma_r = \inf\{t \geq 0 : |x(t)| \geq r\}$ (with special case $\inf \vee = \infty$). Therefore, for all s, t with $|t - s| \leq \delta$, we obtain

$$\begin{aligned} &P(|x(t) - x(s)| > \varepsilon) \\ &\leq P\left(|x(t) - x(s)| > \varepsilon \text{ and } \sup_{s \leq t \leq s+\delta} |x(t)| \leq r\right) \\ &\quad + P\left(\sup_{s \leq t \leq s+\delta} |x(t)| > r\right) \\ &\leq P\left(|x(t \wedge \sigma_r) - x(s \wedge \sigma_r)| > \varepsilon\}\right) + P\left(\sup_{t \geq 0} |x(t)| > r\right) \leq \epsilon \end{aligned}$$

that is, $x(t)$ is uniformly continuous in probability. ■

Remark 2: Under the condition of boundedness in probability, uniform continuity in probability and local uniform continuity in probability are equivalent to each other. □

Based on the above preliminaries, the conditions in Corollary 2: the uniform continuity and the absolute integrability are extended to the stochastic case.

Now, consider Kolmogorov's continuity condition [11, P.7]. For an $\mathbb{R}^n$-valued stochastic process $x(t)$, there exist constants $\alpha, \beta, K > 0$ such that

$$E|x(t) - x(s)|^\alpha \leq K|t - s|^{1+\beta} \quad (4)$$

for all $s, t \in [0, \infty)$. This is used to analyze the sample continuity of stochastic process, and can also be borrowed to analyze the uniform continuity in probability.

Proposition 3: Stochastic process satisfying Kolmogorov's continuity condition is uniformly continuous in probability.

Proof: Suppose that there exist constants $\alpha, \beta, K > 0$ such that (4) holds for all $s, t \in [0, \infty)$. For any $\varepsilon, \epsilon > 0$, by Chebyshev's inequality, it follows that:

$$P\{|x(t) - x(s)| > \varepsilon\} \leq \frac{E|x(t) - x(s)|^\alpha}{\varepsilon^\alpha} \leq \frac{K|t - s|^{1+\beta}}{\varepsilon^\alpha}.$$

There exists a $\delta = (\varepsilon^\alpha \epsilon / K)^{1/(1+\beta)}$ such that, for each s, t satisfying $|t - s| \leq \delta$, inequality

$$P\{|x(t) - x(s)| > \varepsilon\} \leq \epsilon$$

holds, which means that $x(t)$ is uniformly continuous in probability. ■

D. Stochastic Differential Equations

The preliminary on stochastic differential equation is presented for self-contained.

Consider nonlinear stochastic system

$$dx(t) = f(x, t)\, dt + g(x, t)\, dW(t),\ x(t_0) = x_0 \in \mathbb{R}^n \quad (5)$$

where $x(t) \in \mathbb{R}^n$ is the state of system, $W(t)$ is an m- dimensional independent standard Wiener process, the underlying complete probability space is taken to be the quartet $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ with a filtration $\mathcal{F}_t$ satisfying the usual conditions (i.e., it is increasing and right continuous while $\mathcal{F}_0$ contains all P-null sets), functions $f : \mathbb{R}^n \times \mathbb{R}_+ \rightarrow \mathbb{R}^n$ and $g : \mathbb{R}^n \times \mathbb{R}_+ \rightarrow \mathbb{R}^{n \times m}$ are locally Lipschitz in x and piecewise continuous in t, and $f(0, t) = 0$ and $g(0, t) = 0$ are also assumed.

For system (5), for any $r > 0$, define the first exit time as

$$\sigma_r = \inf\{t : t \geq t_0,\ |x(t)| \geq r\} \quad (6)$$

where $\inf \vee = \infty$. This means that σ_r is increasing and has its limit

$$\sigma_\infty := \lim_{r \rightarrow \infty} \sigma_r \quad (7)$$

almost surely, named as the escape (or explosion) time of the solution. It was pointed out in [16] that system (5) has a unique solution $x(t) = x(t_0, x_0; t)$ on the maximal interval $[t_0, \sigma_\infty)$. To prove the existence and boundedness of solution on $[t_0, \infty)$, we need the following result.

Lemma 2: For system (5), if there exist a nonnegative continuous function $V(x)$ and parameters c and d such that

$$EV(x(\sigma_r \wedge t)) \leq d e^{c(t - t_0)}, \quad (8)$$

$$\lim_{R \rightarrow \infty} \inf_{|x| > R} V(x) = \infty \quad (9)$$

then there exists a strong solution $x(t)$ in $[t_0, \infty)$ to system (5) for every $x_0 \in \mathbb{R}^n$. If $c \leq 0$, the unique solution is bounded in probability.

Proof: The existence and uniqueness of strong solution in $[t_0, \infty)$ come from [24, Lemma 1], then $\sigma_\infty = \infty$. From Remark 1, the boundedness here is different from the weak one. The detailed proof is given as follows. When $c \leq 0$, letting $r \rightarrow \infty$ in (8) gives

$$EV(x(t)) \leq d$$

thus, we have

$$\begin{aligned} d \geq EV(x(t)) &\geq \int_{\{\sup_{0 \leq s} |x(s)| > R\}} V(x(t)) dP \\ &\geq P\left\{\sup_{0 \leq s} |x(s)| > R\right\} \inf_{\{|x| > R\}} V(x). \end{aligned}$$

Combining this with (9) concludes the boundedness in probability of the solution. ■

III. Stochastic Barbalat's Lemma

It is expected to find some easily-checked conditions to guarantee the convergence of stochastic processes. Originated but distinguished from [18, Theorem 2.1] and [21, Lemma 3], a general form of stochastic Barbalat's lemma is presented, which is the main result in this technical note.

Theorem 1 (Stochastic Barbalat's Lemma): If a continuous and adapted process $\phi(t) : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ is uniformly continuous in probability and absolutely integrable on $\mathbb{R}_+ \times \Omega$, then $\lim_{t\to\infty} \phi(t) = 0, a.s.$

Proof: Since the uniform continuity of ϕ implies the uniform continuity of $|\phi|$ and the convergence-to-zero of $|\phi|$ means that of ϕ, then, without loss of generality, $\phi(t) \geq 0$ is assumed.

The sample space can be decomposed into the following three mutually exclusive events:

i. $A_1 = \left\{w : \limsup_{t\to\infty} \phi(t) = 0\right\}$;

ii. $A_2 = \left\{w : \liminf_{t\to\infty} \phi(t) > 0\right\}$;

iii. $A_3 = \left\{w : \liminf_{t\to\infty} \phi(t) = 0 \text{ and } \limsup_{t\to\infty} \phi(t) > 0\right\}$.

Let us prove $P\{A_1\} = 1$ by showing $P\{A_2\} = P\{A_3\} = 0$.

(1) To show $P(A_2) = 0$. Suppose $P\{A_2\} > 0$, then there exist $\varepsilon_0 > 0$ and $\epsilon_0 > 0$ such that

$$P\{D\} \geq \epsilon_0 \tag{10}$$

where $D := \{\omega : \liminf_{t\to\infty} \phi(t) > \varepsilon_0\}$. Since $\phi(t) \in \mathcal{M}(\mathbb{R}_+ \times \Omega)$, then according to Proposition 1, for ϵ_0 in (10), there exists a constant $r_0 > 0$ such that

$$P\left\{\int_0^\infty \phi(s)ds > r_0\right\} < \epsilon_0. \tag{11}$$

On the other hand, for $\omega \in D$, there exists a τ_0 such that

$$\phi(s) \geq \frac{1}{2}\varepsilon_0, \ \forall s > \tau_0 \tag{12}$$

which indicates that for r_0 in (11) and any $t' > r_0/\varepsilon_0$ the following result holds:

$$\int_0^\infty \phi(s)ds \geq \int_{\tau_0}^{\tau_0+2t'} \phi(s)ds \geq t'\varepsilon_0 > r_0, \ \forall\omega \in D \tag{13}$$

hence, from (10), we have

$$P\left\{\int_0^\infty \phi(s)ds > r_0\right\} \geq \epsilon_0$$

a contradiction to (11). Therefore, $P\{A_2\} = 0$.

(2) Next to show $P\{A_3\} = 0$ by contradiction. Suppose $P\{A_3\} > 0$, then there exist $\varepsilon_1 > 0$ and $\epsilon_1 > 0$ such that

$$P\{G\} \geq \epsilon_1 \tag{14}$$

where

$$G := \{\phi(\cdot) \text{ crosses from below } \varepsilon_1 \text{ to above } 2\varepsilon_1 \text{ and back infintely many times}\}.$$

Since $\phi(t) \in \mathcal{M}(\mathbb{R}_+ \times \Omega)$, then according to Proposition 1, for this ϵ_1, there exists a constant $r_1 > 0$ such that

$$P\left\{\int_0^\infty \phi(s)ds > r_1\right\} < \epsilon_1. \tag{15}$$

Noting ϕ being continuous and adapted to $\mathcal{F}_t$, according to [22, Theorem 2.32], we can define stopping times as follows:

$$\begin{aligned} T^1_{\varepsilon_1} &= \inf\{t \geq 0 : \phi(t) \in B_{\varepsilon_1}\}, \\ T^1_{2\varepsilon_1} &= \inf\{t > T^1_{\varepsilon_1} : \phi(t) \notin B_{2\varepsilon_1}\}, \\ T^i_{\varepsilon_1} &= \inf\left\{t > T^{i-1}_{2\varepsilon_1} : \phi(t) \in B_{\varepsilon_1}\right\}, \\ T^i_{2\varepsilon_1} &= \inf\left\{t > T^i_{\varepsilon_1} : \phi(t) \notin B_{2\varepsilon_1}\right\}, \ i = 2, 3, \ldots \end{aligned}$$

where $B_{\varepsilon_1} = \{x \in \mathbb{R}^n : \phi(t) \leq \varepsilon_1\}$, $B_{2\varepsilon_1} = \{x \in \mathbb{R}^n : \phi(t) \leq 2\varepsilon_1\}$. Obviously

$$T^1_{\varepsilon_1} < T^1_{2\varepsilon_1} < T^2_{\varepsilon_1} < T^2_{2\varepsilon_1} < \cdots < T^i_{\varepsilon_1} < T^i_{2\varepsilon_1} < \cdots.$$

On G, $T^i_{\varepsilon_1}, T^i_{2\varepsilon_1} \to \infty$ as $i \to \infty$, therefore, from definitions of G, $\tau^i_{\varepsilon_1}$ and $\tau^i_{2\varepsilon_1}$, we have

$$\begin{aligned} \int_0^\infty \phi(s)ds &\geq \sum_{i=1}^\infty \int_{T^i_{2\varepsilon_1}}^{T^{i+1}_{\varepsilon_1}} \phi(s)ds \\ &\geq \varepsilon_1 \sum_{i=1}^\infty (T^{i+1}_{\varepsilon_1} - T^i_{2\varepsilon_1}), \forall\omega \in G. \end{aligned} \tag{16}$$

From the uniform continuity in probability of $\phi(t)$, for any $\varepsilon, \epsilon > 0$, there exists a $\delta^* > 0$ such that for each i and any s satisfying $|s - T^i_{2\varepsilon_1}| \leq \delta^*$, we have

$$P\left\{|\phi(s) - \phi\left(T^i_{2\varepsilon_1}\right)| \leq \varepsilon\right\} \geq 1 - \epsilon. \tag{17}$$

If take $\varepsilon = \varepsilon_1/2$ here, the selected δ^* must satisfy

$$\delta^* < T^{i+1}_{\varepsilon_1} - T^i_{2\varepsilon_1}, \ a.s. \tag{18}$$

for each i. Otherwise, there exist an i_0 and an $s_0 \in (T^{i_0}_{2\varepsilon_1}, T^{i_0}_{2\varepsilon_1} + \delta^*]$ such that $|\phi(s_0) - \phi(T^{i_0}_{2\varepsilon_1})| \geq \varepsilon_1 > \varepsilon$ is a certain event, contradicting to (17). By submitting (18) in (16), for r_1 given in (15), we have

$$\int_0^\infty \phi(s)ds \geq \sum_{i=1}^\infty \delta^*\varepsilon_1 > r_1, \ \forall\omega \in G$$

which, together with (14), gives

$$P\left\{\int_0^\infty \phi(s)ds > r_1\right\} \geq \epsilon_1$$

a contradiction to (15). Therefore, $P\{A_3\} = 0$. Summarizing all the above results, we have $P\{A_1\} = 1$ which implies $P\{\lim_{t\to\infty} \phi(t) = 0\} = 1$. ■

To understand the necessary of integrable condition of stochastic Barbalat's lemma, an example is presented here.

Example 1: Consider a standard Wiener process $W(t)$. By Proposition 3, we can verify that it is uniformly continuous in probability from Kolmogorov's continuity condition

$$E|W(t) - W(s)|^2 \leq |t - s|.$$

It is also continuous and adapted to $\mathcal{F}_t$. We can verify $W(t) \to 0 (t \to \infty)$ does not hold and $E\int_0^\infty W(s)ds$ does not exist. □

Compared with the standard results on stochastic stability in many references such as [12], [14], [15], [19], [23] and [25] and the recent work of [21] and [26], stochastic Barbalat's lemma does not require the researched process being an Itô diffusion process generated by a SDE.

IV. Applications of Stochastic Barbalat's Lemma

A. Stochastic Barbalat's Lemma About Diffusion Processes

Stochastic version of Barbalat's lemma is presented in the above section, while how to apply it into dynamic systems is a challenging task. Consider stochastic nonlinear system (5) where locally Lipschitz condition is given. The unique strong solution $x(t)$ if there exists is an Itô diffusion process. The aim of this section is to extend Corollary 3 to the stochastic case to obtain a criterion on the convergence of the composed function—function of diffusion process.

The following proposition casts light on the uniform continuity of composed functions, which can be regarded as an extension of Proposition 2.

Proposition 4: If $x(t)$ is locally uniformly continuous in probability and bounded in probability, and $\beta(x)$ is a continuous function, then $\beta(x(t))$ is uniformly continuous in probability.

Proof: Since $x(t)$ is bounded in probability, then for any $\epsilon > 0$ there exists an $r(\epsilon) > 0$ such that

$$P\left\{\sup_{t\geq 0}|x(t)| > r\right\} \leq \frac{1}{2}\epsilon.$$

Because $\beta(x)$ is continuous, $\beta(x)$ is uniformly continuous in the ball $B_r = \{|x(t)| \leq r\}$. That is, for any $\varepsilon > 0$, there exists an $\eta > 0$ such that $|\beta(x(t)) - \beta(x(s))| < \varepsilon$ for any $|x(t) - x(s)| < \eta$ $(x(t), x(s) \in B_r)$.

From the local uniform continuity of x, for r and η given above, there exists a $\delta(\varepsilon, \epsilon) > 0$ such that, for all s, t with $|t - s| \leq \delta$, we have

$$P(|x(t\wedge\sigma_r) - x(s\wedge\sigma_r)| > \eta\}) \leq \frac{1}{2}\epsilon.$$

Therefore for all s, t with $|t - s| \leq \delta$, we obtain

$$\begin{aligned} &P\{|\beta(x(t)) - \beta(x(s))| > \varepsilon\} \\ &\leq P\left\{|\beta(x(t)) - \beta(x(s))| > \varepsilon \text{ and } \sup_{s\leq t\leq s+\delta}|x(t)| \leq r\right\} \\ &\quad + P\left\{|\beta(x(t)) - \beta(x(s))| > \varepsilon \text{ and } \sup_{s\leq t\leq s+\delta}|x(t)| > r\right\} \\ &\leq P\left\{|x(t) - x(s)| > \eta \text{ and } \sup_{s\leq t\leq s+\delta}|x(t)| \leq r\right\} \\ &\quad + P\left\{\sup_{s\leq t\leq s+\delta}|x(t)| > r\right\} \\ &\leq P\left\{|x(t\wedge\sigma_r) - x(s\wedge\sigma_r)| > \eta\right\} + P\left\{\sup_{t\geq 0}|x(t)| > r\right\} \leq \epsilon \end{aligned}$$

which means that $\beta(x(t))$ is uniformly continuous in probability. ■

For an Itô diffusion, the locally Lipschitz condition of vector fields implies the locally uniform continuity of solution.

Proposition 5: The unique solution of system (5) (if there exists) is locally uniformly continuous in probability.

Proof: Since system (5) satisfies locally Lipschitz condition with $f(0,t) = g(0,t) = 0$, there exist two functions ρ_1 and ρ_2 given by

$$\rho_1(r) \triangleq \max_{|x|\leq r}\sup_{t\geq 0}|f(x,t)|, \quad \rho_2(r) \triangleq \max_{|x|\leq r}\sup_{t\geq 0}|g(x,t)|$$

where $|g(x,t)|$ denotes Frobenius norm of $g(x,t)$. Thus, for all $s, t \in [0,\infty)$ with $0 \leq t - s \leq \delta$, we have

$$\begin{aligned} &E|x(t\wedge\sigma_r) - x(s\wedge\sigma_r)|^2 \\ &= E\left|\int_{s\wedge\sigma_r}^{t\wedge\sigma_r} f(x,t)dt + \int_{s\wedge\sigma_r}^{t\wedge\sigma_r} g(x, t\wedge\sigma_r)dw\right|^2 \\ &\leq 2E\left|\int_{s\wedge\sigma_r}^{t\wedge\sigma_r} f(x,t)\,dt\right|^2 + 2E\left|\int_{s\wedge\sigma_r}^{t\wedge\sigma_r} g(x,t)\,dw\right|^2. \end{aligned}$$

Because

$$E\left|\int_{s\wedge\sigma_r}^{t\wedge\sigma_r} f(x,t)\,dt\right|^2 \leq \rho_1^2(r)\delta^2,$$

$$\text{and}\quad E\left|\int_{s\wedge\sigma_r}^{t\wedge\sigma_r} g(x,t)\,dw\right|^2 \leq \rho_2^2(r)\delta$$

we can obtain that

$$E|x(t\wedge\sigma_r) - x(s\wedge\sigma_r)|^2 \leq 2\rho_1^2(r)\delta^2 + 2\rho_2^2(r)\delta.$$

According to Chebyshev's inequality, $x(t)$ is locally uniformly continuous in probability. ■

Indeed, the uniform continuity is a necessary condition for the boundedness of solutions to stochastic systems satisfying locally Lipschitz condition.

Proposition 6: If the unique solution of system (5) is bounded in probability, then it is uniformly continuous in probability.

Proof: From Proposition 5, it is known that $x(t)$ is locally uniformly continuous in probability. Noting x being bounded in probability, then from Proposition 2, we can conclude that $x(t)$ is uniformly continuous in probability. ■

It is ready to give a stochastic Barbalat's lemma about diffusion process, which is the natural extension of Corollary 3, a differential form of Barbalat's lemma.

Theorem 2: If a diffusion process $x(t)$ given by (5) is bounded in probability and the continuous function $\beta : \mathbb{R}^n \to \mathbb{R}_+$ satisfies $\beta(x(t)) \in \mathcal{M}([t_0,\infty)\times\Omega)$, then

$$\lim_{t\to\infty}\beta(x(t)) = 0, \ a.s.$$

Proof: Since system (5) satisfies locally Lipschitz condition and its unique solution $x(t)$ exists, then $\beta(x(t))$ is continuous and adapted to $\mathcal{F}_t$. From Proposition 5, it is known that $x(t)$ is locally uniformly continuous in probability, noting x being bounded in probability and β being continuous, then from Proposition 4, it is known that $\beta(x(t))$ is uniformly continuous in probability. By the aid of $\beta(x(t)) \in \mathcal{M}([t_0,\infty)\times\Omega)$, from Theorems 1, we can obtain $\lim_{t\to\infty}\beta(x(t)) = 0$, a.s. ■

The following interesting result was first presented in [21], which is an extension of a corollary of the deterministic Barbalat's lemma. We will verify it by the above results.

Theorem 3: For system (5), suppose there exist a nonnegative continuous function $V(x)$ and a parameter d such that (8) and (9) hold. If there exists a continuous nonnegative function $W: \mathbb{R}^n \to \mathbb{R}$ such that

$$E\left\{\int_0^\infty W(x(t))dt\right\} < \infty \tag{19}$$

then

$$P\left\{\lim_{t\to\infty} W(x(t)) = 0\right\} = 1, \quad \forall x_0 \in \mathbb{R}^n. \tag{20}$$

Furthermore, if W is continuous and positive definite then the following result holds

$$P\left\{\lim_{t\to\infty}|x(t)| = 0\right\} = 1, \quad \forall x_0 \in \mathbb{R}^n. \tag{21}$$

Proof: According to Lemma 2, from (8) and (9), one can find that there exists a solution $x(t)$ in $[t_0, \infty)$ to system (5) and the unique solution is bounded in probability. Therefore, the conditions of Theorem 2 are satisfied, which means (20). From (20) and W being continuous and positive definite, we can obtain (21). ■

It had been shown by the recent work of [21] and [26] that Theorem 3 is a powerful tool to analyze the stability of stochastic systems satisfying locally lipschitz condition. For stochastic systems with state-dependent switch, locally Lipschitz condition is often destroyed by switch behaviors, then the standard results in the existing work can not be used directly. By the aid of stochastic Barbalat's lemma (Theorem 1), a criterion on stability of this type of stochastic systems has been established in [27].

B. Lyapunov Function Method

The power and the popularity of stochastic Barbalat's lemma can be demonstrated by covering some standard results as special cases.

Consider the stochastic nonlinear system with external stochastic disturbance

$$\begin{aligned} dx(t) &= f\left(x, v(t), t\right) dt + g\left(x, v(t), t\right) dW(t), \\ x(t_0) &= x_0 \in \mathbb{R}^n \end{aligned} \tag{22}$$

where v is a piecewise continuous and adapted process, functions f and g are locally Lipschitz in x, v and piecewise continuous in t, $f(0,0,t) = 0$ and $g(0,0,t) = 0$, and the other explanations are similar with those to system (5).

A general result on stochastic stability analysis via $\mathcal{C}^2$ Lyapunov function is presented as a theorem.

Theorem 4: For system (22), if there exist a $\mathcal{C}^2$ function $V : \mathbb{R}^n \to \mathbb{R}_+$, class $\mathcal{K}_\infty$ functions α_1 and α_2 such that for any $x \in \mathbb{R}^n$ and $t \geq 0$

$$\alpha_1\left(|x|\right) \leq V\left(x\right) \leq \alpha_2\left(|x|\right), \tag{23}$$

$$\begin{aligned} \mathcal{L}V\left(x\right) &= \frac{\partial V}{\partial x} f\left(x\right) + \frac{1}{2} Tr \left\{ g^T\left(x\right) \frac{\partial^2 V}{\partial x^2} g\left(x\right) \right\} \\ &\leq -\beta(x) + \gamma(v(t)) \end{aligned} \tag{24}$$

where $\beta : \mathbb{R}^n \to \mathbb{R}$ is continuous and nonnegative and $\gamma(v(t)) \in \mathcal{M}([t_0, \infty) \times \Omega)$. Then for each $x_0 \in \mathbb{R}^n$, system (22) has a unique strong solution, which is stochastic bounded and

$$P\left\{ \lim_{t \to \infty} \beta\left(x\left(t\right)\right) = 0 \right\} = 1, \quad \forall x_0 \in \mathbb{R}^n. \tag{25}$$

Proof: The boundedness of v comes from $\gamma(v(t)) \in \mathcal{M}([t_0, \infty) \times \Omega)$. For system (22), given any $r > 0$, define the first exit time σ_r as (6). From (24), there exists a constant d such that

$$EV(x(t \wedge \sigma_r)) \leq V(x_0) + E \int_0^\infty |\gamma(v(s))| ds < d.$$

Combining this with V being radially unbounded results in the existence and uniqueness of solution in $[t_0, \infty)$ as well as its boundedness in probability, according to Lemma 2. Since $x(t)$ is the unique solution of the system, if we can obtain $\beta(x(s)) \in \mathcal{M}([t_0, \infty) \times \Omega)$, then it follows from Theorem 2 that (25) holds. The proof of $\beta(x(s)) \in \mathcal{M}([t_0, \infty) \times \Omega)$ can be referred to [19, Theorem 3.2]. ■

Remark 3: Compared with [17, Theorem 2.1] or [28, Lemma 1], the same result is obtained although linear growth condition is removed and requirement on $\gamma(v(t))$ is milder in this technical note. When $\gamma(v(t)) \equiv 0$, Theorem 4 reduces to [19, Theorem 3.2]. As corollaries of our stochastic Barbalat's lemma, these standard results were wildly applied into controller designs for stochastic nonlinear systems, and here we only mention [29]–[34] and the references therein. □

V. Conclusion

The idea to construct general stochastic versions of Barbalat's lemma should be devoted to the authors of [21], who pointed it out as an open problem. The original innovation should be devoted to the excellent work [18] and [20], from which we have found lots of materials to develop our main result. After the uniform continuity and the absolute integrability are extended to the stochastic case, the stochastic version of stochastic Barbalat's lemma is presented. The power of stochastic Barbalat's lemma is verified by covering the standard results as special cases. There are many new directions deserving considerations. The obtained results will further improve the theory of stochastic dissipative systems [26]. By the aid of stochastic Barbalat's lemma and the stochastic dissipativity, several natural definitions of input-to-state stability in the survey and expository paper [35] will be extended to the stochastic case. The relations with some tools in stochastic analysis such as Boerel-Cantelli lemma and martingale convergence theorem (see,[36]) should be casted light on, which seems to be a great challenge.

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Discrete-Time Observer Error Linearizability via Restricted Dynamic Systems

Hong-Gi Lee and Jin-Man Hong

***Abstract*—In this technical note, we define the observer error linearization problem of a discrete-time autonomous nonlinear system via a restricted dynamic system. This is the dual to restricted dynamic feedback linearization in a loose sense. Necessary and sufficient conditions for this problem are obtained in terms of the index.**

***Index Terms*—Dynamic observer error linearizable, nonlinear discrete-time systems, nonlinear observer canonical form (NOCF), restricted dynamic system.**

I. Introduction

Several methods are available to design an observer for a nonlinear system. One is to find a new state coordinate that transforms the given system into a nonlinear observer canonical form (NOCF), as first suggested in [1], [2]. A simple Luenberger-like observer design is feasible for a NOCF, as in Remark 1. We call this problem *observer error linearization* or the state equivalence of a system to a NOCF. Many researchers have studied this problem for a single output (SO) autonomous system in [3]–[5], for a multi-output (MO) autonomous system in [4], [6]–[9], for a SO non-autonomous system in [10], [11], and for a MO non-autonomous system in [12], [15]. The state equivalence to an extended observer canonical form (OCF) and the state equivalence to a generalized OCF can also be found in [5] and [13]–[15], respectively. Other methods of nonlinear observer design without using state equivalence can be found in [16]–[25].

Consider a discrete-time autonomous system of the form

$$x(t+1) = f\Big(x(t)\Big)\,;\;\; y(t) = h(x(t)) \tag{1}$$

with $f(0) = 0$, $h(0) = 0$, state $x \in \mathbb{R}^n$, output $y \in \mathbb{R}^p$, and $\operatorname{rank}\Big((\partial h/\partial x)(0)\Big) = p$.

Definition 1: System (1) is said to be state equivalent to a *nonlinear observer canonical form*, if there exists a smooth diffeomorphism $T : V_0 \to \mathbb{R}^n$, defined on some neighborhood of the origin $V_0 \subset \mathbb{R}^n$, which transforms (1), in the variable $\zeta = T(x)$, to

$$\zeta(t+1) = A\zeta(t) + \gamma(y(t))\,;\;\; y(t) = C\zeta(t) \tag{2}$$

where $\gamma : \mathbb{R}^p \to \mathbb{R}^n$ is a smooth function, $A = \operatorname{blockdiag}\{A_1, \ldots, A_p\}$, $C = \operatorname{blockdiag}\{C_1, \ldots, C_p\}$, $C_i = [\,O_{1\times(\nu_i-1)} \quad 1\,]$, and $A_i = \begin{bmatrix} O_{1\times(\nu_i-1)} & 0 \\ I_{(\nu_i-1)\times(\nu_i-1)} & O_{(\nu_i-1)\times 1}\end{bmatrix}$.

$\{\nu_1, \ldots, \nu_p\}$ are the *observability indices* of system (1). By Lemma2 in [4], the necessary and sufficient conditions for the state equivalence of system (1) to a NOCF are

$$y_i(t+\nu_i) = \sum_{\ell=1}^{\nu_i} \varphi_\ell^i\Big(y(t+\ell-1)\Big), \quad i = 1, \ldots, p \tag{3}$$

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