Brief paper

Switch observability for switched linear systems✩

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Mode observability of switched systems requires observability of each individual mode. We consider other concepts of observability that do not have this requirement: Switching time observability and switch observability. The latter notion is based on the assumption that at least one switch occurs. These concepts are analyzed and characterized both for homogeneous and inhomogeneous systems.

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1. Introduction

Mode observability of switched systems is concerned with recovering the initial state as well as the switching signal from the output (and the input) and has been widely studied, see e.g. Vidal, Chiuso, Soatto, and Sastry (2003) for homogeneous systems, Elhamifar, Petreczky, and Vidal (2009) for inhomogeneous discrete-time systems, Babaali and Pappas (2005) for a generic observability notion of inhomogeneous systems and Lou and Si (2009) for inhomogeneous systems. For a recent overview of observability for general hybrid systems, see De Santis and Di Benedetto (2016).

Since for mode observable systems it is in particular possible to recover the state for constant switching signals, each mode necessarily has to be observable. In the context of fault-detection (or diagnosis) the different modes of a switched system describe faulty and non-faulty variants of the system and a switch represents a fault. Requiring observability of each mode, in particular of each faulty mode, might be a too strong assumption. Instead of mode observability, it would be sufficient to compute the switching signal and the state if an error occurs. This idea is formalized in the novel notion of switch observability, \((x, \sigma_1)\)-observability for short.

Before characterizing \((x, \sigma_1)\)-observability, we first have to consider the problem of detecting switches (switching time observability or \(t_s\)-observability). This has been done in Vidal et al. (2003) in the homogeneous case, but the generalization to inhomogeneous systems is not straightforward as the switch might occur in an interval where the state is zero. This difficulty has been avoided so far, e.g. in Elhamifar et al. (2009) by assuming mode observability. We are able to relax this assumption and to fully characterize \(t_s\)-observability without any additional assumptions.

Similar to the classical observability of linear systems, we derive characterizations of the observability notions based on rank-conditions on the Kalman observability matrices. Our results are summarized in Fig. 1, where \(\omega_i\) and \(I_i\) are the Kalman observability matrix and Hankel matrix of mode \(i\), respectively. These notions are defined in Sections 2 and 3; \(\text{rk}(A)\) denotes the rank of \(A\).

The first column in Fig. 1 gives the result for the homogeneous case: The strongest notion considered here is \((x, \sigma)\)-observability, which coincides with switching signal observability \((\sigma)\)-observability. It implies \((x, \sigma_1)\)-observability and \(t_s\)-observability. The reverse implications are false in general, we will show this by some examples. For the inhomogeneous case, we consider two different setups. First we restrict our attention to systems with analytic input and with some restriction on the input matrices (assumption (A2)). Then we drop (A2) and require only smooth input. This makes it necessary to consider equivalence classes of switching signals, but gives observability notions with the same characterizations as in the more restrictive setup.

Our main contribution is the concept of (strong) \((x, \sigma_1)\)-observability and its characterization. Also the characterization of strong switching time observability for inhomogeneous systems is new.

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2. Homogeneous systems

2.1. System class and preliminaries

A switching signal is a piecewise constant, right-continuous function \( \sigma : \mathbb{R} \rightarrow \mathcal{P} := \{1, \ldots, N\} \), \( N \in \mathbb{N} \), with locally finitely many discontinuities. The discontinuities of \( \sigma \) are also called switching times:

\[ T_\sigma := \{ t_\sigma \in \mathbb{R} \mid t_\sigma \text{ is a discontinuity of } \sigma \} \]

We assume that all switches occur for \( t > 0 \), i.e., \( T_\sigma \subset \mathbb{R}_{\geq 0} \).

Consider switched linear systems of the form

\[ \dot{x}(t) = A_i x(t), \quad x(0) = x_0, \quad y = C_i x \]

with switching signal \( \sigma \) and \( A_i \in \mathbb{R}^{n \times n}, C_i \in \mathbb{R}^{p \times n} \) for all \( i \in \mathcal{P} \) and denote its solution and output by \( x(t; x_0, \sigma) \) and \( y(t; x_0, \sigma) \), respectively.

Furthermore, let \( \phi_i^{[n]} \) be the Kalman observability matrix for mode \( i \) with \( n \) row blocks, i.e.,

\[ \phi_i^{[n]} = \left[ \begin{array}{c} C_i^T \quad (CA_i)^T \quad (CA_i^2)^T \quad \cdots \quad (CA_i^{n-1})^T \end{array} \right]^T \]

and let \( \phi_i^{[\infty]} \) be the corresponding infinite Kalman observability matrix. For observability of unswitched systems, it suffices to consider \( \nu = n \). In our setting, the required size increases as we have to compare the output from different modes.

For any sufficiently smooth function \( y : \mathbb{R} \rightarrow \mathbb{R}^p \) denote by \( y^{[v]} : \mathbb{R} \rightarrow \mathbb{R}^{vP} \) the vector of its first \( v \) derivatives and by \( y^{[\infty]} \) the (countably) infinite vector of its derivatives. The same can be done for piecewise-smooth functions, where \( y(t^-) \) and \( y(t^+) \) denote the left-hand side and right-hand side limit at \( t \), respectively. Then the output \( y(x_0, \sigma) \) of (1) satisfies for all \( t \in \mathbb{R} \):

\[ y^{[v]}(x_0, \sigma)(t) = \phi_i^{[v]} x(x_0, \sigma)(t), \quad v \in \mathbb{N} \cup \{\infty\}, \]

\[ y^{[\infty]}(x_0, \sigma)(t^-) = \phi_i^{[\infty]} x(x_0, \sigma)(t), \quad v \in \mathbb{N} \cup \{\infty\} \]

2.2. Known results and definitions

**Definition 1.** The switched system (1) is called

- \((x, \sigma)\)-observable iff for all \((x_0, \tilde{x}_0) \neq (0, 0)\) the following implication holds:

\[ (x_0 \neq \tilde{x}_0 \lor \sigma \neq \tilde{\sigma}) \Rightarrow y(x_0, \sigma) \neq y(\tilde{x}_0, \tilde{\sigma}) \]

\[ (x_0 \neq \tilde{x}_0 \lor \sigma \neq \tilde{\sigma}) \Rightarrow y(x_0, \sigma) \neq y(\tilde{x}_0, \tilde{\sigma}) \]

**Lemma 2.** For the switched system (1) it holds that

\[ (x, \sigma) - \text{observability} \Leftrightarrow \sigma - \text{observability} \]

**Proof.** The implication “\( \Rightarrow \)” is clear. Now let the system be \( \sigma \)-observable, but not \((x, \sigma)\)-observable. This means that there exist \((x_0, \tilde{x}_0) \neq (0, 0)\) and \( \sigma, \tilde{\sigma} \) with

\[ (x_0 \neq \tilde{x}_0 \lor \sigma \neq \tilde{\sigma}) \land y(x_0, \sigma) \equiv y(\tilde{x}_0, \tilde{\sigma}) \]

\[ \sigma \neq \tilde{\sigma} \] would contradict \( \sigma \)-observability. Hence we have \( \sigma \equiv \tilde{\sigma} \) and \( x_0 \neq \tilde{x}_0 \). This means that \( y(x_0, \sigma) \equiv y(\tilde{x}_0, \tilde{\sigma}) \), and, by linearity, \( y(x_0 - \tilde{x}_0, \sigma) \equiv 0 \). This contradicts \( \sigma \)-observability, as it implies \( y(x_0 - \tilde{x}_0, \sigma) \equiv 0 \equiv y(0, \tilde{\sigma}) \) for all \( \tilde{\sigma} \). \( \square \)

This relation was already implicitly stated in Elhamifar et al. (2009) for discrete-time systems. Note that observability of the (continuous) state in each mode is necessary for \((x, \sigma)\)-observability (just consider the constant switching signals). However, state-observability in each mode is not sufficient for \((x, \sigma)\)-observability (c.f. Babaali and Pappas, 2005). A trivial counterexample for the latter is a system for which each mode describes the same observable system. The next example shows that \( t_\sigma \)-observability is indeed weaker than \((x, \sigma)\)-observability:
if, because (4) is equivalent to (classical) observability of the Kalman observability matrices:

$$\begin{align*}
\hat{\xi} &= \begin{bmatrix} A_i & 0 \\ 0 & A_j \end{bmatrix} \xi_i \\
y_{\lambda_i j} &= \begin{bmatrix} C_i - C_j \end{bmatrix} \xi_j,
\end{align*}$$

because (4) is equivalent to (classical) observability of $\Sigma_{i,j}^\hom$; indeed $\delta_{ij}^{[2]} = [\delta_{ij}^{[1]}, -\delta_{ij}^{[1]}]$. This also justifies why it suffices to consider the order $\nu = 2n$ in (4).

2.3. $\sigma_1$-observability

As already mentioned in the introduction assuming observability of each (in particular, each faulty) mode is often too restrictive. Furthermore, the notion of $(\chi, \sigma)$-observability (and hence $\sigma$-observability) reduces to the ability to determine the current mode of (locally) switched systems. In particular, the event of the switch itself is not utilized for recovering the switching signal. We illustrate this with the following example:

Example 6. The system (1) with modes

$$(A_1, C_1) = (\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \end{bmatrix}) \quad \text{and} \quad (A_2, C_2) = (\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \end{bmatrix})$$

is not $(\chi, \sigma)$-observable, because both systems produce constant outputs for constant switching signals. However, in the presence of a switch, the output is either halved or doubled, which allows us to determine whether we switched from mode 1 to 2 or vice versa. This observability property is lost if we modify $C_2$ to $-1$, because the output then just changes its sign and we are not able to distinguish the two possible mode sequences. However it is still possible to detect the switching time, because of the sign change (which always occurs as long as $x_0 \neq 0$, which we assumed here).

This motivates us to define the following more suitable observability notion:

Definition 7. The system (1) is called $(\chi, \sigma_1)$-observable (or switch observable) iff (2) holds for all $x_0 \neq 0$ and all $\sigma$ with at least one switch, i.e. $\sigma$ nonconstant, and all $\tilde{x}_0, \tilde{\sigma}$. It is called $\sigma_1$-observable iff (3) holds for $x_0, \tilde{x}_0, \sigma, \tilde{\sigma}$ as above.

Lemma 2 holds accordingly and gives

$$(\chi, \sigma_1) - \text{observability} \iff \sigma_1 - \text{observability}.$$ (6)

We now present our first main result which characterizes $(\chi, \sigma_1)$-observability for homogeneous switched linear systems.

Theorem 8. The system (1) is $(\chi, \sigma_1)$-observable if, and only if, for all $i, j, p, q \in \mathcal{P}$ with $i \neq j, p \neq q$ and $(i, j) \neq (p, q)$:

$$\text{rk} \begin{bmatrix} e_i^{[2n]} & e_j^{[2n]} \\ e_i^{[2n]} & e_j^{[2n]} \end{bmatrix} = 2n.$$ (7)

Proof. $"\Rightarrow"$ Assume that (7) does not hold, i.e. there exist $i, j, p, q$ as above and $(x_1, \tilde{x}_1) \neq (0, 0)$ such that

$$\begin{bmatrix} e_i^{[2n]} \\ e_j^{[2n]} \end{bmatrix} \begin{bmatrix} x_1 \\ \tilde{x}_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$ (8)

Without loss of generality, we can assume $x_1 \neq 0$. Define $(x_0, \tilde{x}_0) := (e^{-L_{\lambda_i} x_1}, e^{-L_{\lambda_j} \tilde{x}_1})$ and

$$\sigma(t) = \begin{cases} i, & t < \tau_i, \\
q, & t \geq \tau_i \end{cases} \quad \tilde{\sigma}(t) = \begin{cases} p, & t < \tau_p, \\
q, & t \geq \tau_p \end{cases}.$$ (9)

Then we have $x_0 \neq 0$ and $\sigma \neq \tilde{\sigma}$. From (8) we can conclude

$$\begin{align*}
y_{(x_0, \sigma)}(t_0^+)^{(i)} &= y_{(\tilde{x}_0, \tilde{\sigma})}(t_0^+)^{(j)} \quad \text{and} \quad y_{(x_0, \sigma)}(t_0^+)^{(i)} = y_{(\tilde{x}_0, \tilde{\sigma})}(t_0^+)^{(j)}.\end{align*}$$

In terms of (5) with initial value $(x_1, \tilde{x}_1)$ this is equivalent to $y_{(x_0, \sigma)}(0) = 0$ and $y_{(\tilde{x}_0, \tilde{\sigma})}(0) = 0$. By the classical observability theory, this implies $y_{(x_0, \sigma)}(0) = 0$ and $y_{(\tilde{x}_0, \tilde{\sigma})}(0) = 0$ i.e. $y_{(\lambda_i, \sigma)} = 0$ and $y_{(\lambda_i, \tilde{\sigma})} = 0$. We can conclude $y_{(x_0, \sigma)} = y_{(\tilde{x}_0, \tilde{\sigma})}$. "\Leftarrow": Using (6), it suffices to show $\sigma_1$-observability. (7) implies $t_\sigma$-observability as for $p = j = i = q$ we have

$$\text{rk} \begin{bmatrix} e_i^{[2n]} & e_j^{[2n]} \\ e_j^{[2n]} & e_i^{[2n]} \end{bmatrix} = 2n \Rightarrow \text{rk} \begin{bmatrix} e_i^{[2n]} - e_j^{[2n]} \\ e_j^{[2n]} - e_i^{[2n]} \end{bmatrix} = n.$$
Now let \( x_0, \tilde{x}_0, \sigma \) and \( \tilde{\sigma} \) be given with \( x_0 \neq 0, \sigma \) nonconstant and \( \sigma \neq \tilde{\sigma} \). It remains to show \( y(x_0, \sigma) \neq y(\tilde{x}_0, \tilde{\sigma}) \). For \( T_0 \neq T_2 \) this follows directly from \( t_2 \)-observability, hence let \( T_0 = T_2 \). Then there exists a common switching time \( t_2 \) with \( \sigma(t_2) = \tilde{\sigma}(t_2) \) or \( \sigma(t_2) \neq \tilde{\sigma}(t_2) \). Let \( i, j, p, q \) be as in (9). As \( x_{x_0,\sigma}(t_2) \neq 0, (7) \) implies
\[
y_{x_0,\sigma}(t_2) \neq y_{\tilde{x}_0,\tilde{\sigma}}(t_2) \quad \lor \quad y_{x_0,\sigma}(t_2) \neq y_{\tilde{x}_0,\tilde{\sigma}}(t_2).
\]
Thus the system is \( \sigma_1 \)-observable. \( \square \)

Condition (7) also appears in Johnson, DeCarlo, and Žefran (2014) as a characterization of what those authors call ST-observability. The main difference to our approach is that observability of the individual modes \( i, j, p \) is assumed there.

**Remark 9.** Vidal et al., (2003) chose a different approach for observability of systems with nonconstant switching signals. They required for all \( i \neq j \):
\[
\operatorname{rk} \begin{pmatrix} \phi_i^{(2n)} \\ \phi_j^{(2n)} \end{pmatrix} = \text{rk} \phi_i^{(2n)} + \text{rk} \phi_j^{(2n)},
\]
which guarantees that one can determine the current mode whenever the output is non-zero. Together with \( t_2 \)-observability, this gives that mode and state can be determined whenever the switching signal is nonconstant and the initial state is non-zero. This means that (10) and \( t_2 \)-observability imply \((x, \sigma_1)\)-observability. The reverse is not true, as the first part of Example 6 shows.

Clearly, \((x, \sigma_1)\)-observability works also for systems with more than one switch, but then each switching instant is treated independently of the others (analogously as for \((x, \sigma)\)-observability each mode is treated independently of the others). If we restricted our attention to systems with at least two (or more generally at least \( k \)) switches and defined \((x, \sigma_k)\)-observability accordingly, one would get even weaker conditions than (7). However, these conditions would then depend on the differences of the switching times, i.e., the *duration times*. It is questionable whether these weaker observability notions are really relevant in practice and whether the technical effort to find corresponding characterizations is justified.

The results of this section for homogeneous linear switched systems are summarized in the left column of Fig. 1 and Example 6 shows that the converse implications do not hold in general.

### 3. Inhomogeneous systems

For unswitched systems or switched systems with known switching signal the system dynamics are known and thus the output’s dependence on the input can be computed a priori; it is therefore common to restrict the analysis to homogeneous systems. For unknown switching signals this reduction to the homogeneous case is not possible, because the effect of the input on the output depends on the switching signal. There are several ways to generalize the observability notions to inhomogeneous systems, depending on the treatment of the inhomogeneity. We consider strong observability notions, i.e. we require the system to be \( t_2 \)-\( \sigma \)-/\((x, \sigma)\)-/\((x, \sigma_1)\)-observable for all inputs. Other approaches are that one requires the existence of an input that makes the system observable (weak notion) or requires observability for almost all inputs. This generic notion actually coincides with the weak one, see Babaali and Pappas (2005). The literature focuses on the weak or the generic case, see e.g. De Santis and Di Benedetto (2016) and Baglietto, Battistelli, and Scardovi (2007) and we are not aware of available results for strong observability notions.

We consider the switched system
\[
\begin{align}
\dot{x} &= A_i x + B_i u, & x(0) &= x_0, \quad \text{(11a)} \\
y &= C_j x + D_j u, & \quad \text{(11b)}
\end{align}
\]
with matrices \( A_i \in \mathbb{R}^{n \times n}, B_i \in \mathbb{R}^{n \times q}, C_i \in \mathbb{R}^{p \times n}, D_i \in \mathbb{R}^{p \times q} \) for \( i \in \mathcal{I} \). Solutions and outputs are denoted by \( x_{x_0, \sigma, u}(t) \) and \( y_{x_0, \sigma, u}(t) \), respectively. In order to define suitable observability notions we make the following two assumptions:
\[
u \text{ analytic},
\]
\[
\ker \begin{pmatrix} B_1 \\ B_i \end{pmatrix} = \{0\} \quad \forall i \neq j.
\]

**Definition 10.** Consider the switched system (11) satisfying (A2). Then we define (11) to be strongly \((x, \sigma)\)-/\(\sigma\)-/\((x, \sigma_1)\)-observable iff the analogous conditions of Definitions 1 and 7 hold for all inputs \( u \) satisfying (A1).

Analogously to Lemma 2 it can be shown that strong \((x, \sigma)\)-observability is equivalent to strong \( \sigma \)-observability.

We have seen in the homogeneous case that a zero state trajectory makes it impossible to observe the switching signal because \( y(x_0, \sigma) = 0 \) for all \( \sigma \); this problem was easily resolved by excluding the initial state zero. In the inhomogeneous case this is not sufficient as the following two examples show: in fact, these examples show that without (A1) and (A2) a zero state trajectory is possible on some interval even for nonzero initial values.

**Example 11.** Consider the system (11) with modes
\[
\begin{align}
(A_1, B_1, C_1, D_1) &:= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
(A_2, B_2, C_2, D_2) &:= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \\
(A_3, B_3, C_3, D_3) &:= \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
\end{align}
\]
This means that assumption (A2) does not hold. Define \( x_0 := \begin{pmatrix} 1 \\ 0 \end{pmatrix} \), \( u(t) := -\frac{1}{\zeta} \cos \left( \frac{\pi}{2} t \right) \) and
\[
\sigma(t) := \begin{cases}
1, & t < 1, \\
1 \leq t < 2, & \tilde{\sigma}(t) := 3, & t < 1, \\
1, & t \geq 2, & 1 \leq t < 2, & 3, & t \geq 2.
\end{cases}
\]
Then \( x_{x_0, \sigma, u}(1) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \) and thus \( x_{x_0, \sigma, u}(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \) for \( t \in [1, 2] \). Hence the switching signals cannot be distinguished for this particular choice of input. This example is illustrated in Fig. 3.

The second example shows what can happen when assumption (A1) is not satisfied.

**Example 12.** Consider the system (11) with mode \((A_1, B_1, C_1, D_1) = (0, 2, 1, 0)\) and some other, not further specified mode 2. For a given \( x_0 \) and \( \sigma \equiv 1 \) one can choose a smooth input \( u \) with \( \text{supp}(u) = \{0, 1\} \cup [2, 3] \) such that \( x_{x_0, \sigma, u}(t) \) is zero on the interval \([1, 2]\). This means that \( \sigma(t, x) \) has no effect on the solution and hence the system cannot be \( t_2 \)-observable or even \((x, \sigma)\)-observable. Such a \( u \) is clearly non-analytic. In contrast to the previous example, no switch is required to achieve an interval with zero state, see Fig. 4.
For a characterization of strong \((x, \sigma)\)-observability we need to define \(\Gamma^{(v)}\) corresponding to the unswitched inhomogeneous system

\[
\begin{align*}
\dot{\xi} &= A_0 \xi + B_0 u, \\
y &= C_0 \xi + D_0 u
\end{align*}
\]

by

\[
\Gamma^{(v)} = \begin{bmatrix} D & CB & \cdots & \cdots \\ CA^{-2}B & \cdots & CB & D \end{bmatrix}
\]

with \(v\) block rows and block columns. \(\Gamma^{(\infty)}\) denotes the corresponding infinite matrix. Note that any solution \((x, u, y)\) of the unswitched system satisfies for any \(v \in \mathbb{N}\):

\[
y^{(v)} = \phi^{(v)} x + \Gamma^{(v)} u^{(v)}.
\]

We would like to recall the notion of unknown-input observability for unswitched systems:

**Definition 13.** The system \(\Sigma\) is unknown-input (ui-) observable\(^1\) iff \(y \equiv 0\) implies \(x \equiv 0\) (independently of the input \(u\)).

A system \(\Sigma\) is ui-observable iff

\[
\text{rk}\left[\phi^{(n)} \Gamma^{(n)}\right] = n + \text{rk} \Gamma^{(n)},
\]

or, equivalently,

\[
\text{rk}\left[\begin{bmatrix} A - sI & B \\ C & D \end{bmatrix}\right] = n + \text{rk} \begin{bmatrix} B \\ D \end{bmatrix} \quad \forall s \in \mathbb{R}.
\]

see Kratz (1995) and Hautus (1983), respectively. This means that the system is ui-observable if it has no zeros (in the sense of Hautus, 1983).

Applying this characterization on the augmented system \(\Sigma_{i,j}\), \(i, j \in \mathbb{P}\):

\[
\begin{align*}
\dot{\xi}_i &= [A_i, 0] \xi + [B_i, 0] u, \\
y_{\lambda_{ij}} &= [C_i - C_j] \xi + (D_i - D_j) u,
\end{align*}
\]

we can conclude that \(\Sigma_{i,j}\) is ui-observable if and only if

\[
\text{rk}\left[\phi_i^{(2n)} \phi_j^{(2n)} \Gamma_i^{(2n)} - \Gamma_j^{(2n)}\right] = 2n + \text{rk} \left(\Gamma_i^{(2n)} - \Gamma_j^{(2n)}\right).
\]

**Lemma 16.** Let \((11)\) satisfy \((A1), (A2)\) and

\[
\mathcal{R}(\Sigma_{i,j}) = \{0\} \quad \text{for all} \quad i \neq j.
\]

Then \((11)\) is strongly \(t_s\)-observable if, and only if, \((13)\) holds and, for all \(i \neq j\),

\[
\text{rk}\left[\phi_i^{(2n)} - \phi_j^{(2n)} \Gamma_i^{(2n)} - \Gamma_j^{(2n)}\right] = n + \text{rk} \left(\Gamma_i^{(2n)} - \Gamma_j^{(2n)}\right).
\]

See Hautus (1983) uses the notion strong observability; however, we follow instead the naming convention from Basile and Marro (1973) in order to avoid confusing with our strong observability notion for switched systems (where we still assume that the input is known).

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\(^1\) Hautus (1983) uses the notion strong observability; however, we follow instead the naming convention from Basile and Marro (1973) in order to avoid confusing with our strong observability notion for switched systems (where we still assume that the input is known).

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Remark 18. Regarding (13) we observe the following:

(i) In Elhamifar et al. (2009) strong $t_3$-observability is characterized for discrete time switched systems in terms of (14), but condition (13) does not occur. The reason is due to stronger assumption made in Elhamifar et al. (2009) which are specific to the discrete time set up; in particular, they require that each individual mode is observable.

(ii) The conditions (13) and (14) of strong $t_3$-observability are indeed not related. Consider for the system example given by

\[
(A_1, B_1, C_1, D_1) = (0, 1, 2, 0),
\]
\[
(A_2, B_2, C_2, D_2) = (2, 0, 1, 0),
\]

which satisfies (14) but not (13). On the other hand (13) holds for any system with $B_0 = 0$ for all $i \in \mathcal{S}$, hence it does not imply (14) in general.

(iii) (13) does not imply $\mathcal{A}(\Sigma_i \equiv 0)$ for the individual modes. As an example, consider the system (11) with modes

\[
(A_1, B_1, C_1, D_1) = (0, 1, 0, 0),
\]
\[
(A_2, B_2, C_2, D_2) = (0, 1, 1, 0).
\]

It is strongly $t_3$-observable, in particular, $\mathcal{A}(\Sigma_{1,2} \equiv 0)$. However, for the first mode we have $\mathcal{A}(\Sigma_1 \equiv 0) = \emptyset$.

(iv) (13) and (14) are indeed weaker than (12): The example from (iii) is strongly $t_3$-observable, but not strongly $(x, \sigma)$-observable as $\mathcal{A}_i \equiv 0$.

Theorem 19. The switched system (11) satisfying (A1) and (A2) is strongly $(x, \sigma)$-observable if and only if it satisfies (13) and, for all $i, j, p, q \in \mathcal{S}$ with $i \neq j, p \neq q$ and $(i, j) \neq (p, q)$,

\[
\begin{aligned}
& \big[ e^{[4n]}_{4i} e^{[4n]}_{4p} \big] \\
& \begin{bmatrix}
      e^{[4n]}_{4i} e^{[4n]}_{4p} \\
      \ell_{p - q}^{[4n]} - \ell_{q - p}^{[4n]}
    \end{bmatrix}
\]
\[
= 2n + r k
\]
\[
\begin{bmatrix}
      e^{[4n]}_{4i} e^{[4n]}_{4p} \\
      \ell_{p - q}^{[4n]} - \ell_{q - p}^{[4n]}
    \end{bmatrix}
\]
\[
\begin{bmatrix}
      e^{[4n]}_{4j} e^{[4n]}_{4q} \\
      \ell_{j - q}^{[4n]} - \ell_{q - j}^{[4n]}
    \end{bmatrix}
\]
\[
= 2n + r k
\]
\[
\begin{bmatrix}
      e^{[4n]}_{4j} e^{[4n]}_{4q} \\
      \ell_{j - q}^{[4n]} - \ell_{q - j}^{[4n]}
    \end{bmatrix}
\]
\[
\text{where}
\]
\[
\mathcal{O}(\Sigma_0, u) \equiv \left\{ \sigma : \sigma \equiv \overline{\sigma} = 0 \right\}
\]
\[
\text{and the essential switching times are given by}
\]
\[
T(\Sigma_0, u) := \bigcup_{\sigma \equiv \overline{\sigma} = 0} T_{\sigma}.
\]

A similar equivalence has been considered in Kaba (2014) in the context of invertibility of switched systems.

For $u$ analytic, $(x_0, u) \neq (0, 0)$ and systems satisfying (A2) we have $[\sigma(\Sigma_0, u)] = [\sigma]$, i.e. trivial equivalence classes.

Adaption of Definition 10 to equivalence classes of switching signals gives the following:

Definition 21. The system (11) is called

\[
\begin{aligned}
& \text{strongly $(x, \sigma)$-observable iff for all smooth u and all $x_0$, $\overline{x}_0$,} \\
& \text{the following implication holds:}
\end{aligned}
\]
\[
(x_0, \overline{x}_0) \neq (x_0, \overline{x}_0) \Rightarrow y(\Sigma_0, u) \neq y(\Sigma_0, u).
\]

Proof of Theorem 19. “(13) and (17) ⇒ strong $t_3$-observability”:

\[
\begin{aligned}
& \text{From (17) with } p = j, \text{ we can conclude (14). Then the claim follows by Lemma 17.}
\end{aligned}
\]

“Strong $(x, \sigma)$-observability ⇒ (13)”:

Follows by Lemma 17 as strong $t_3$-observability is necessary for strong $(x, \sigma)$-observability.

“Strong $(x, \sigma)$-observability ⇒ (17)”:

Assume that (17) does not hold for some $i, j, p, q$ i.e. there exist $(x_1, x_1) \neq (0, 0)$ and $U$ such that

\[
\begin{aligned}
& \left[ e^{[4n]}_{4i} e^{[4n]}_{4p} \right] \\
& \begin{bmatrix}
      e^{[4n]}_{4i} e^{[4n]}_{4p} \\
      \ell_{p - q}^{[4n]} - \ell_{q - p}^{[4n]}
    \end{bmatrix}
\]
\[
\begin{bmatrix}
      e^{[4n]}_{4j} e^{[4n]}_{4q} \\
      \ell_{j - q}^{[4n]} - \ell_{q - j}^{[4n]}
    \end{bmatrix}
\]
\[
\begin{bmatrix}
      x_1 \\
      -x_1
    \end{bmatrix}
\]
\[
\begin{bmatrix}
      U
    \end{bmatrix}
\]
\[
= 0.
\]

We get that $\Sigma_{i,p,q} \neq \emptyset$ and $\mathcal{A}(\Sigma_i \equiv 0)$ for the initial value $\eta_1 := (x_1, x_1)$ and some $u \equiv \overline{u}(0) = U$ we have $y(\Sigma_{i,p,q} \equiv 0) = y(\Sigma_{i,p,q} \equiv 0)$ and $y(\Sigma_{i,p,q} \equiv 0)$. Define $\sigma$ and $\overline{\sigma}$ as in (9) for some $t_0 > 0$ and let such that $x_0$ be such that $x(\Sigma_{i,p,q} \equiv 0) = x_1$ and $y(\Sigma_{i,p,q} \equiv 0) = x_1$. Then we get $y(\Sigma_{i,p,q} \equiv 0)$ i.e. (11) is not strongly $(x, \sigma)$-observability.

Remark 18. Regarding (13) we observe the following:

(i) In Elhamifar et al. (2009) strong $t_3$-observability is characterized for discrete time switched systems in terms of (14), but condition (13) does not occur. The reason is due to stronger assumption made in Elhamifar et al. (2009) which are specific to the discrete time set up; in particular, they require that each individual mode is observable.

(ii) The conditions (13) and (14) of strong $t_3$-observability are indeed not related. Consider for the system example given by

\[
(A_1, B_1, C_1, D_1) = (0, 1, 2, 0),
\]
\[
(A_2, B_2, C_2, D_2) = (2, 0, 1, 0),
\]

which satisfies (14) but not (13). On the other hand (13) holds for any system with $B_0 = 0$ for all $i \in \mathcal{S}$, hence it does not imply (14) in general.

(iii) (13) does not imply $\mathcal{A}(\Sigma_i \equiv 0)$ for the individual modes. As an example, consider the system (11) with modes

\[
(A_1, B_1, C_1, D_1) = (0, 1, 0, 0),
\]
\[
(A_2, B_2, C_2, D_2) = (0, 1, 1, 0).
\]

It is strongly $t_3$-observable, in particular, $\mathcal{A}(\Sigma_{1,2} \equiv 0) = \emptyset$. However, for the first mode we have $\mathcal{A}(\Sigma_1 \equiv 0) = \emptyset$.

(iv) (13) and (14) are indeed weaker than (12): The example from (iii) is strongly $t_3$-observable, but not strongly $(x, \sigma)$-observable as $\mathcal{A}_i \equiv 0$.
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References


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