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Brief paper

Switch observability for switched linear systems[☆]

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ABSTRACT

Mode observability of switched systems requires observability of each individual mode. We consider other concepts of observability that do not have this requirement: Switching time observability and switch observability. The latter notion is based on the assumption that at least one switch occurs. These concepts are analyzed and characterized both for homogeneous and inhomogeneous systems.

1. Introduction

Mode observability of switched systems is concerned with recovering the initial state as well as the switching signal from the output (and the input) and has been widely studied, see e.g. Vidal, Chiuso, Soatto, and Sastry (2003) for homogeneous systems, Elhamifar, Petreczky, and Vidal (2009) for inhomogeneous discrete-time systems, Babaali and Pappas (2005) for a generic observability notion of inhomogeneous systems and Lou and Si (2009) for inhomogeneous systems. For a recent overview of observability for general hybrid systems, see De Santis and Di Benedetto (2016).

Since for mode observable systems it is in particular possible to recover the state for constant switching signals, each mode necessarily has to be observable. In the context of fault-detection (or diagnosis) the different modes of a switched system describe faulty and non-faulty variants of the system and a switch represents a fault. Requiring observability of each mode, in particular of each faulty mode, might be a too strong assumption. Instead of mode observability, it would be sufficient to compute the switching signal and the state *if an error occurs*. This idea is formalized in the novel notion of switch observability, (x, σ_1)-observability for short.

Before characterizing (x, σ_1)-observability, we first have to consider the problem of detecting switches (switching time observability or t_S-observability). This has been done in Vidal et al. (2003) in the homogeneous case, but the generalization to inhomogeneous systems is not straightforward as the switch might occur in an interval where the state is zero. This difficulty has been avoided so far, e.g. in Elhamifar et al. (2009) by assuming mode observability. We are able to relax this assumption and to fully characterize t_S-observability without any additional assumptions.

Similar to the classical observability of linear systems, we derive characterizations of the observability notions based on rank-conditions on the Kalman observability matrices. Our results are summarized in Fig. 1, where $\mathcal{O}_i$ and Γ_i are the Kalman observability matrix and Hankel matrix of mode i, respectively. These notions are defined in Sections 2 and 3; $\mathrm{rk}(A)$ denotes the rank of A.

The first column in Fig. 1 gives the result for the homogeneous case: The strongest notion considered here is (x, σ)-observability, which coincides with switching signal observability (σ-observability). It implies (x, σ_1)-observability and t_S-observability. The reverse implications are false in general, we will show this by some examples. For the inhomogeneous case, we consider two different setups. First we restrict our attention to systems with analytic input and with some restriction on the input matrices (assumption (A2)). Then we drop (A2) and require only smooth input. This makes it necessary to consider equivalence classes of switching signals, but gives observability notions with the same characterizations as in the more restrictive setup

Our main contribution is the concept of (strong) (x, σ_1)-observability and its characterization. Also the characterization of strong switching time observability for inhomogeneous systems is new.

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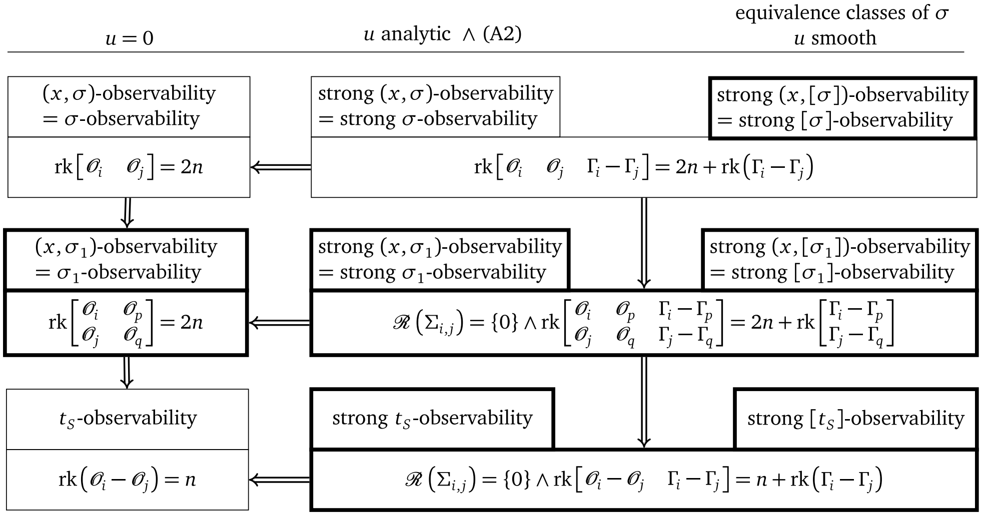

Fig. 1. Brief characterizations of the observability notions and their relations. Novel results are indicated by bold boxes.

2. Homogeneous systems

2.1. System class and preliminaries

A *switching signal* is a piecewise constant, right-continuous function $\sigma\colon \mathbb{R} \to \mathcal{P} := \{1, \ldots, N\}$, $N \in \mathbb{N}$, with locally finitely many discontinuities. The discontinuities of σ are also called *switching times*:

$$T_\sigma := \{\, t_S \in \mathbb{R} \mid t_S \text{ is a discontinuity of } \sigma \,\}.$$

We assume that all switches occur for $t > 0$, i.e. $T_\sigma \subset \mathbb{R}_{>0}$. Consider switched linear systems of the form

$$\dot{x} = A_\sigma x, \quad x(0) = x_0, \tag{1a}$$

$$y = C_\sigma x, \tag{1b}$$

with switching signal σ and $A_i \in \mathbb{R}^{n\times n}$, $C_i \in \mathbb{R}^{p\times n}$ for all $i \in \mathcal{P}$ and denote its solution and output by $x_{(x_0,\sigma)}$ and $y_{(x_0,\sigma)}$, respectively.

Furthermore, let $\mathcal{O}_i^{[\nu]}$ be the Kalman observability matrix for mode i with ν row blocks, i.e.

$$\mathcal{O}_i^{[\nu]} = \begin{bmatrix} C_i^\top & (C_iA_i)^\top & (C_iA_i^2)^\top & \cdots & (C_iA_i^{\nu-1})^\top \end{bmatrix}^\top$$

and let $\mathcal{O}_i^{[\infty]}$ be the corresponding infinite Kalman observability matrix. For observability of unswitched systems, it suffices to consider $\nu = n$. In our setting, the required size increases as we have to compare the output from different modes.

For any sufficiently smooth function $y : \mathbb{R} \to \mathbb{R}^p$ denote by $y^{[\nu]} : \mathbb{R} \to \mathbb{R}^{\nu p}$ the vector of y and its first $\nu - 1$ derivatives and by $y^{[\infty]}$ the (countably) infinite vector of y and its derivatives. The same can be done for piecewise-smooth functions, where $y(t^-)$ and $y(t^+)$ denote the left-hand side and right-hand side limit at t, respectively. Then the output $y_{(x_0,\sigma)}$ of (1) satisfies for all $t \in \mathbb{R}$:

$$y_{(x_0,\sigma)}^{[\nu]}(t^+) = \mathcal{O}_{\sigma(t^+)}^{[\nu]} x_{(x_0,\sigma)}(t), \quad \nu \in \mathbb{N} \cup \{\infty\},$$

$$y_{(x_0,\sigma)}^{[\nu]}(t^-) = \mathcal{O}_{\sigma(t^-)}^{[\nu]} x_{(x_0,\sigma)}(t), \quad \nu \in \mathbb{N} \cup \{\infty\}.$$

2.2. Known results and definitions

Definition 1. The switched system (1) is called

- (x,σ)-*observable* iff for all $(x_0, \widetilde{x}_0) \neq (0, 0)$ the following implication holds:

$$(x_0 \neq \widetilde{x}_0 \ \vee\ \sigma \not\equiv \widetilde{\sigma}) \quad \Rightarrow \quad y_{(x_0,\sigma)} \not\equiv y_{(\widetilde{x}_0,\widetilde{\sigma})}, \tag{2}$$

  i.e., iff it is possible to determine simultaneously the state and current mode from the output;
- σ-*observable* iff for all $(x_0, \widetilde{x}_0) \neq (0, 0)$

$$\sigma \not\equiv \widetilde{\sigma} \quad \Rightarrow \quad y_{(x_0,\sigma)} \not\equiv y_{(\widetilde{x}_0,\widetilde{\sigma})}, \tag{3}$$

  i.e., iff it is possible to determine the current mode from the output;
- t_S-*observable* (or *switching time observable*) iff for all $x_0 \neq 0$, σ nonconstant and all $\widetilde{x}_0, \widetilde{\sigma}$:

$$T_\sigma \neq T_{\widetilde{\sigma}} \quad \Rightarrow \quad y_{(x_0,\sigma)} \not\equiv y_{(\widetilde{x}_0,\widetilde{\sigma})},$$

  i.e., iff it is possible to determine the switching times from the output.

Clearly, (x,σ)-observability implies σ-observability which in turn implies t_S-observability. Furthermore, it seems quite obvious that it is much harder to determine both the state and the switching signal compared to just determining the current mode from the output. However, this intuition is wrong:

Lemma 2. *For the switched system (1) it holds that*

$$(x,\sigma) - \textit{observability} \quad \Leftrightarrow \quad \sigma - \textit{observability}.$$

Proof. The implication "⇒" is clear. Now let the system be σ-observable, but not (x,σ)-observable. This means that there exist $(x_0, \widetilde{x}_0) \neq (0, 0)$ and $\sigma, \widetilde{\sigma}$ with

$$(x_0 \neq \widetilde{x}_0 \ \vee\ \sigma \not\equiv \widetilde{\sigma}) \ \wedge\ y_{(x_0,\sigma)} \equiv y_{(\widetilde{x}_0,\widetilde{\sigma})}.$$

$\sigma \not\equiv \widetilde{\sigma}$ would contradict σ-observability. Hence we have $\sigma \equiv \widetilde{\sigma}$ and $x_0 \neq \widetilde{x}_0$. This means that $y_{(x_0,\sigma)} \equiv y_{(\widetilde{x}_0,\sigma)}$ and, by linearity, $y_{(x_0-\widetilde{x}_0,\sigma)} \equiv 0$. This contradicts σ-observability, as it implies $y_{(x_0-\widetilde{x}_0,\sigma)} \equiv 0 \equiv y_{(0,\hat{\sigma})}$ for all $\hat{\sigma}$. □

This relation was already implicitly stated in Elhamifar et al. (2009) for discrete-time systems. Note that observability of the (continuous) state in each mode is necessary for (x,σ)-observability (just consider the constant switching signals). However, state-observability in each mode is not sufficient for (x,σ)-observability (c.f. Babaali and Pappas, 2005). A trivial counterexample for the latter is a system for which each mode describes the same observable system.

The next example shows that t_S-observability is indeed weaker than (x,σ)-observability:

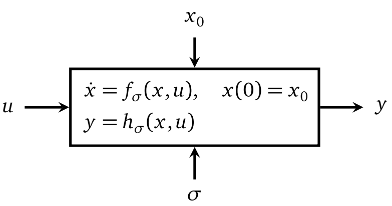

Fig. 2. General nonlinear switched system with initial state x_0, input u, switching signal σ and output y.

Table 1
Comparison of different observability notions based on the sought inverse maps.

Sought map			Name, reference footnotes
(y, u, σ)	$\mapsto$	x_0	Observability[a]
(y, x_0)	$\mapsto$	(u, σ)	Invertibility[b]
$(y, u \equiv 0)$	$\mapsto$	(x_0, σ)	(x, σ)-observability[c]
$(y, u \equiv 0)$	$\mapsto$	σ	σ-observability
(y, u)	$\mapsto$	(x_0, σ)	Strong (x, σ)-observability[d]
(y, u)	$\mapsto$	σ	Strong σ-observability

[a] Petreczky, Tanwani, and Trenn (2015).
[b] Vu and Liberzon (2008) and Tanwani and Liberzon (2010).
[c] Vidal et al. (2003) and Babaali and Pappas (2005).
[d] Babaali and Pappas (2005) and Lou and Si (2009).

Example 3. The system (1) with modes

$$(A_1, C_1) = \left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \end{bmatrix}\right), \quad (A_2, C_2) = \left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \end{bmatrix}\right)$$

is t_S-observable, but not (x, σ)-observable as the individual modes are not observable.

Remark 4 (*Observability and Invertibility*)**.** Most observability notions are concerned with the invertibility of certain maps involving the output and it is helpful to compare the different concepts side-by-side in regard of these sought inverse maps, see Table 1. For this comparison we consider general nonlinear switched systems as in Fig. 2.

Note that most results on observability of switched systems are only for the linear case (one exception is Tanwani & Liberzon, 2010).

We now recall the known characterization for t_S- and (x, σ)-observability in terms of the Kalman observability matrices:

Lemma 5 (Vidal et al., 2003)**.** *System (1) is t_S-observable if, and only if,*

$$\operatorname{rk}\left(\mathcal{O}_i^{[2n]} - \mathcal{O}_j^{[2n]}\right) = n \quad \forall i, j \in \mathscr{P} \text{ with } i \neq j.$$

It is (x, σ)-observable if, and only if,

$$\operatorname{rk}\begin{bmatrix} \mathcal{O}_i^{[2n]} & \mathcal{O}_j^{[2n]} \end{bmatrix} = 2n \quad \forall i, j \in \mathscr{P} \text{ with } i \neq j. \tag{4}$$

The characterization (4) can be nicely interpreted by considering the homogeneous *augmented system* $\Sigma_{i,j}^{\text{hom}}$, $i, j \in \mathscr{P}$:

$$\Sigma_{i,j}^{\text{hom}}: \quad \begin{aligned} \dot{\xi} &= \begin{bmatrix} A_i & 0 \\ 0 & A_j \end{bmatrix} \xi, \\ y_{\Delta_{i,j}} &= \begin{bmatrix} C_i & -C_j \end{bmatrix} \xi, \end{aligned} \tag{5}$$

because (4) is equivalent to (classical) observability of $\Sigma_{i,j}^{\text{hom}}$; indeed $\mathcal{O}_{ij}^{[\nu]} = [\mathcal{O}_i^{[\nu]}, \; -\mathcal{O}_j^{[\nu]}]$. This also justifies why it suffices to consider the order $\nu = 2n$ in (4).

2.3. σ_1-observability

As already mentioned in the introduction assuming observability of each (in particular, each faulty) mode is often too restrictive. Furthermore, the notion of (x, σ)-observability (and hence σ-observability) reduces to the ability to determine the current mode of (locally) unswitched systems. In particular, the event of the switch itself is not utilized for recovering the switching signal. We illustrate this with the following example:

Example 6. The system (1) with modes

$$(A_1, C_1) = (0, 1), \; (A_2, C_2) = (0, 2)$$

is not (x, σ)-observable, because both systems produce constant outputs for constant switching signals. However, in the presence of a switch, the output is either halved or doubled, which allows us to determine whether we switched from mode 1 to 2 or vice versa. This observability property is lost if we modify C_2 to -1, because the output then just changes its sign and we are not able to distinguish the two possible mode sequences. However it is still possible to detect the switching time, because of the sign change (which always occurs as long as $x_0 \neq 0$, which we assumed here).

This motivates us to define the following more suitable observability notion:

Definition 7. The system (1) is called (x, σ_1)-*observable* (or *switch observable*) iff (2) holds for all $x_0 \neq 0$ and all σ *with at least one switch*, i.e. σ nonconstant, and all $\widetilde{x}_0, \widetilde{\sigma}$. It is called σ_1-*observable* iff (3) holds for x_0, $\widetilde{x}_0$, σ, $\widetilde{\sigma}$ as above.

Lemma 2 holds accordingly and gives

$$(x, \sigma_1) - \text{observability} \quad \Leftrightarrow \quad \sigma_1 - \text{observability}. \tag{6}$$

We now present our first main result which characterizes (x, σ_1)-observability for homogeneous switched linear systems.

Theorem 8. *The system (1) is (x, σ_1)-observable if, and only if, for all $i, j, p, q \in \mathscr{P}$ with $i \neq j$, $p \neq q$ and $(i, j) \neq (p, q)$:*

$$\operatorname{rk}\begin{bmatrix} \mathcal{O}_i^{[2n]} & \mathcal{O}_p^{[2n]} \\ \mathcal{O}_j^{[2n]} & \mathcal{O}_q^{[2n]} \end{bmatrix} = 2n. \tag{7}$$

Proof. "$\Rightarrow$": Assume that (7) does not hold, i.e. there exist i, j, p, q as above and $(x_1, \widetilde{x}_1) \neq (0, 0)$ such that

$$\begin{bmatrix} \mathcal{O}_i^{[2n]} & \mathcal{O}_p^{[2n]} \\ \mathcal{O}_j^{[2n]} & \mathcal{O}_q^{[2n]} \end{bmatrix} \begin{bmatrix} x_1 \\ -\widetilde{x}_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{8}$$

Without loss of generality, we can assume $x_1 \neq 0$. Define $(x_0, \widetilde{x}_0) := \left(\mathrm{e}^{-A_i t_S} x_1, \mathrm{e}^{-A_p t_S} \widetilde{x}_1\right)$ and

$$\sigma(t) = \begin{cases} i, & t < t_S, \\ j, & t \geq t_S, \end{cases} \qquad \widetilde{\sigma}(t) = \begin{cases} p, & t < t_S, \\ q, & t \geq t_S. \end{cases} \tag{9}$$

Then we have $x_0 \neq 0$ and $\sigma \neq \widetilde{\sigma}$. From (8) we can conclude

$$y_{(x_0,\sigma)}^{[2n]}(t_S^-) = y_{(\widetilde{x}_0,\widetilde{\sigma})}^{[2n]}(t_S^-) \; \wedge \; y_{(x_0,\sigma)}^{[2n]}(t_S^+) = y_{(\widetilde{x}_0,\widetilde{\sigma})}^{[2n]}(t_S^+).$$

In terms of (5) with initial value $(x_1, \widetilde{x}_1)$ this is equivalent to $y_{\Delta_{i,p}}^{[2n]}(0) = 0$ and $y_{\Delta_{j,q}}^{[2n]}(0) = 0$. By the classical observability theory, this implies $y_{\Delta_{i,p}}^{[\infty]}(0) = 0$ and $y_{\Delta_{j,q}}^{[\infty]}(0) = 0$, i.e. $y_{\Delta_{i,p}} \equiv 0$ and $y_{\Delta_{j,q}} \equiv 0$. We can conclude $y_{(x_0,\sigma)} \equiv y_{(\widetilde{x}_0,\widetilde{\sigma})}$. "$\Leftarrow$": Using (6), it suffices to show σ_1-observability. (7) implies t_S-observability as for $p = j \neq i = q$ we have

$$\operatorname{rk}\begin{bmatrix} \mathcal{O}_i^{[2n]} & \mathcal{O}_j^{[2n]} \\ \mathcal{O}_j^{[2n]} & \mathcal{O}_i^{[2n]} \end{bmatrix} = 2n \;\Rightarrow\; \operatorname{rk}\begin{bmatrix} \mathcal{O}_i^{[2n]} - \mathcal{O}_j^{[2n]} \\ \mathcal{O}_j^{[2n]} - \mathcal{O}_i^{[2n]} \end{bmatrix} = n.$$

Now let $x_0, \widetilde{x}_0, \sigma$ and $\widetilde{\sigma}$ be given with $x_0 \neq 0$, σ nonconstant and $\sigma \not\equiv \widetilde{\sigma}$. It remains to show $y_{(x_0,\sigma)} \neq y_{(\widetilde{x}_0,\widetilde{\sigma})}$. For $T_\sigma \neq T_{\widetilde{\sigma}}$ this follows directly from t_S-observability, hence let $T_\sigma = T_{\widetilde{\sigma}}$. Then there exists a common switching time t_S with $\sigma(t_S^-) \neq \widetilde{\sigma}(t_S^-)$ or $\sigma(t_S^+) \neq \widetilde{\sigma}(t_S^+)$. Let i, j, p, q be as in (9). As $x_{(x_0,\sigma)}(t_S) \neq 0$, (7) implies

$$y^{[2n]}_{(x_0,\sigma)}(t_S^-) \neq y^{[2n]}_{(\widetilde{x}_0,\widetilde{\sigma})}(t_S^-) \ \vee\ y^{[2n]}_{(x_0,\sigma)}(t_S^+) \neq y^{[2n]}_{(\widetilde{x}_0,\widetilde{\sigma})}(t_S^+).$$

Thus the system is σ_1-observable. □

Condition (7) also appears in Johnson, DeCarlo, and Žefran (2014) as a characterization of what those authors call ST-observability. The main difference to our approach is that observability of the individual modes i, j, p is assumed there.

Remark 9. Vidal et al., (2003) chose a different approach for observability of systems with nonconstant switching signals. They required for all $i \neq j$:

$$\operatorname{rk}\begin{bmatrix}\mathcal{O}_i^{[2n]} & \mathcal{O}_j^{[2n]}\end{bmatrix} = \operatorname{rk}\mathcal{O}_i^{[2n]} + \operatorname{rk}\mathcal{O}_j^{[2n]}, \tag{10}$$

which guarantees that one can determine the current mode whenever the output is nonzero. Together with t_S-observability, this gives that mode and state can be determined whenever the switching signal is nonconstant and the initial state is nonzero. This means that (10) and t_S-observability imply (x, σ_1)-observability. The reverse is not true, as the first part of Example 6 shows.

Clearly, (x, σ_1)-observability works also for systems with more than one switch, but then each switching instant is treated independently of the others (analogously as for (x, σ)-observability each mode is treated independently of the others). If we restricted our attention to systems with at least two (or more generally at least k) switches and defined (x, σ_k)-observability accordingly, one would get even weaker conditions than (7). However, these conditions would then depend on the differences of the switching times, i.e. the *duration times*. It is questionable whether these weaker observability notions are really relevant in praxis and whether the technical effort to find corresponding characterizations is justified.

The results of this section for homogeneous linear switched systems are summarized in the left column of Fig. 1 and Example 6 shows that the converse implications do not hold in general.

3. Inhomogeneous systems

For unswitched systems or switched systems with known switching signal the system dynamics are known and thus the output's dependence on the input can be computed a priori; it is therefore common to restrict the analysis to homogeneous systems. For unknown switching signals this reduction to the homogeneous case is not possible, because the effect of the input on the output depends on the switching signal.

There are several ways to generalize the observability notions to inhomogeneous systems, depending on the treatment of the inhomogeneity. We consider strong observability notions, i.e. we require the system to be t_S-/σ-/(x, σ)-/(x, σ_1)-observable for *all* inputs. Other approaches are that one requires the *existence* of an input that makes the system observable (weak notion) or requires observability for *almost all* inputs. This generic notion actually coincides with the weak one, see Babaali and Pappas (2005). The literature focuses on the weak or the generic case, see e.g. De Santis and Di Benedetto (2016) and Baglietto, Battistelli, and Scardovi (2007) and we are not aware of available results for strong observability notions.

We consider the switched system

$$\dot{x} = A_\sigma x + B_\sigma u, \quad x(0) = x_0, \tag{11a}$$

$$y = C_\sigma x + D_\sigma u, \tag{11b}$$

with matrices $A_i \in \mathbb{R}^{n\times n}$, $B_i \in \mathbb{R}^{n\times q}$, $C_i \in \mathbb{R}^{p\times n}$, $D_i \in \mathbb{R}^{p\times q}$ for $i \in \mathscr{P}$. Solutions and outputs are denoted by $x_{(x_0,\sigma,u)}$ and $y_{(x_0,\sigma,u)}$, respectively. In order to define suitable observability notions we make the following two assumptions:

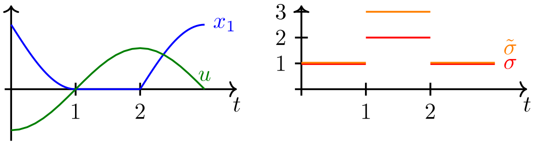

Fig. 3. For u and x_0 the solutions of Example 11 are the same for the switching signals σ and $\widetilde{\sigma}$.

$$u \text{ analytic}, \tag{A1}$$

$$\ker\begin{bmatrix} B_i \\ B_j \\ D_i - D_j \end{bmatrix} = \{0\} \quad \forall i \neq j. \tag{A2}$$

Definition 10. Consider the switched system (11) satisfying (A2). Then we define (11) to be *strongly* (x, σ)-/σ-/(x, σ_1)-/t_S-*observable* iff the analogous conditions of Definitions 1 and 7 hold for *all* inputs u satisfying (A1).

Analogously to Lemma 2 it can be shown that strong (x, σ)-observability is equivalent to strong σ-observability.

We have seen in the homogeneous case that a zero state trajectory makes it impossible to observe the switching signal because $y_{(0,\sigma)} \equiv 0$ for all σ; this problem was easily resolved by excluding the initial state zero. In the inhomogeneous case this is not sufficient as the following two examples show; in fact, these examples show that without (A1) and (A2) a zero state trajectory is possible on some interval even for nonzero initial values.

Example 11. Consider the system (11) with modes

$$(A_1, B_1, C_1, D_1) := \left(\begin{bmatrix}0&0\\0&0\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1&0\\0&1\end{bmatrix}, \begin{bmatrix}0\\0\end{bmatrix}\right),$$

$$(A_2, B_2, C_2, D_2) := \left(\begin{bmatrix}1&0\\0&1\end{bmatrix}, \begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}1&0\\0&1\end{bmatrix}, \begin{bmatrix}0\\0\end{bmatrix}\right),$$

$$(A_3, B_3, C_3, D_3) := \left(\begin{bmatrix}2&0\\0&2\end{bmatrix}, \begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}1&0\\0&1\end{bmatrix}, \begin{bmatrix}0\\0\end{bmatrix}\right).$$

This means that assumption (A2) does not hold. Define $x_0 := \begin{bmatrix}1 & 0\end{bmatrix}^\top$, $u(t) := -\frac{2}{\pi}\cos\left(\frac{\pi}{2}t\right)$ and

$$\sigma(t) := \begin{cases}1, & t < 1,\\ 2, & 1 \leq t < 2,\\ 1, & t \geq 2,\end{cases} \qquad \widetilde{\sigma}(t) := \begin{cases}1, & t < 1,\\ 3, & 1 \leq t < 2,\\ 1, & t \geq 2.\end{cases}$$

Then $x_{(x_0,\sigma,u)}(1) = x_{(x_0,\widetilde{\sigma},u)}(1) = \left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]$ and thus $x_{(x_0,\sigma,u)}(t) = x_{(x_0,\widetilde{\sigma},u)}(t) = \left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]$ for $t \in [1, 2]$. Hence the switching signals cannot be distinguished for this particular choice of input. This example is illustrated in Fig. 3.

The second example shows what can happen when assumption (A1) is not satisfied.

Example 12. Consider the system (11) with mode $(A_1, B_1, C_1, D_1) = (0, 2, 1, 0)$ and some other, not further specified mode 2. For a given x_0 and $\sigma \equiv 1$ one can choose a smooth input u with $\operatorname{supp}(u) = [0, 1] \cup [2, 3]$ such that $x_{(x_0,\sigma,u)}$ is zero on the interval $[1, 2]$. This means that $\sigma_{[1,2)}$ has no effect on the solution and hence the system cannot be t_S-observable or even (x, σ)-observable. Such a u is clearly non-analytic. In contrast to the previous example, no switch is required to achieve an interval with zero state, see Fig. 4.

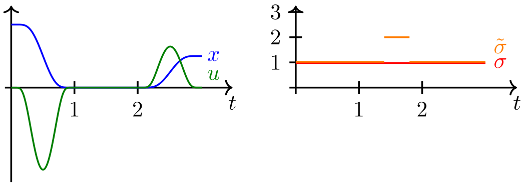

Fig. 4. In Example 12 the value of σ in the interval $[1,2]$ does not have any effect on the solution as the state is zero.

For a characterization of strong (x,σ)-observability we need to define $\Gamma^{[\nu]}$ corresponding to the unswitched inhomogeneous system

$$\Sigma: \quad \begin{aligned} \dot x &= Ax + Bu,\\ y &= Cx + Du \end{aligned}$$

by

$$\Gamma^{[\nu]} = \begin{bmatrix} D & & & \\ CB & \ddots & & \\ \vdots & \ddots & \ddots & \\ CA^{\nu-2}B & \cdots & CB & D \end{bmatrix}$$

with ν block rows and block columns. $\Gamma^{[\infty]}$ denotes the corresponding infinite matrix. Note that any solution (x,u,y) of the unswitched system Σ satisfies for any $\nu \in \mathbb{N}$:

$$y^{[\nu]} = \mathcal{O}^{[\nu]}x + \Gamma^{[\nu]}u^{[\nu]}.$$

We would like to recall the notion of unknown-input observability for unswitched systems:

Definition 13. The system Σ is *unknown-input (ui-) observable*[1] iff $y \equiv 0$ implies $x \equiv 0$ (independently of the input u).

A system Σ is ui-observable iff

$$\operatorname{rk}\left[\mathcal{O}^{[n]}\ \Gamma^{[n]}\right] = n + \operatorname{rk}\Gamma^{[n]},$$

or, equivalently,

$$\operatorname{rk}\begin{bmatrix} A - sI & B \\ C & D \end{bmatrix} = n + \operatorname{rk}\begin{bmatrix} B \\ D \end{bmatrix} \quad \forall s \in \mathbb{R},$$

see Kratz (1995) and Hautus (1983), respectively. This means that the system is ui-observable iff it has no zeros (in the sense of Hautus, 1983).

Applying this characterization on the *augmented system* $\Sigma_{i,j}$, $i,j \in \mathscr{P}$:

$$\Sigma_{i,j}: \quad \begin{aligned} \dot\xi &= \begin{bmatrix} A_i & 0 \\ 0 & A_j \end{bmatrix}\xi + \begin{bmatrix} B_i \\ B_j \end{bmatrix}u,\\ y_{\Delta_{i,j}} &= \begin{bmatrix} C_i & -C_j \end{bmatrix}\xi + \left(D_i - D_j\right)u, \end{aligned}$$

we can conclude that $\Sigma_{i,j}$ is ui-observable if and only if

$$\begin{aligned} &\operatorname{rk}\left[\mathcal{O}_i^{[2n]} \quad \mathcal{O}_j^{[2n]} \quad \Gamma_i^{[2n]} - \Gamma_j^{[2n]}\right] \\ &\quad = 2n + \operatorname{rk}\left(\Gamma_i^{[2n]} - \Gamma_j^{[2n]}\right). \end{aligned} \tag{12}$$

If (12) holds for all $i \neq j$, one can determine mode and state of the system as long as the state is nonzero. This has already been shown by Lou and Si (2009). By requiring (A1), (A2) and $x_0 \neq 0$ we can guarantee that on any interval the state is not constantly zero or the mode can be uniquely determined by the direct feedthrough. Hence we have

Lemma 14 (*cf.* Lou and Si, 2009)**.** *System* (11) *satisfying* (A1) *and* (A2) *is strongly* (x,σ)*-observable if and only if* (12) *holds for all* $i, j \in \mathscr{P}$, $i \neq j$.

For the characterization of t_S-observability, the following notion will be essential:

Definition 15 (*Trentelman, Stoorvogel, and Hautus, 2001*)**.** The set of controllable weakly unobservable states of the system Σ is

$$\mathscr{R}(\Sigma) := \left\{ x_0 \in \mathbb{R}^n \;\middle|\; \begin{array}{l} \exists\, u(\cdot) \text{ smooth, } T > 0: \\ y_{(x_0,u)} \equiv 0 \text{ and } x_{(x_0,u)}(T) = 0 \end{array} \right\}.$$

Note that one obtains the same set if we restrict the inputs to be analytic. Furthermore, $\mathscr{R}(\Sigma) = \{0\}$ if, and only if,

$$\operatorname{rk}\begin{bmatrix} A - sI & B \\ C & D \end{bmatrix} = n + \operatorname{rk}\begin{bmatrix} B \\ D \end{bmatrix}, \quad \begin{array}{r} \text{for all but finitely} \\ \text{many } s \in \mathbb{R}, \end{array}$$

see Trentelman et al. (2001).

Lemma 16. *Let* (11) *satisfy* (A1), (A2) *and*

$$\mathscr{R}\left(\Sigma_{i,j}\right) = \{0\} \quad \text{for all } i \neq j. \tag{13}$$

Let $(x_0, \widetilde{x}_0) \neq (0,0)$, *u and* σ, $\widetilde\sigma$ *be given with* $\sigma(T^+) \neq \widetilde\sigma(T^+)$ *and* $x_{(x_0,\sigma,u)}(T) = x_{(\widetilde{x}_0,\widetilde\sigma,u)}(T) = 0$ *for some* $T > 0$. *Then* $y_{(x_0,\sigma,u)} \not\equiv y_{(\widetilde{x}_0,\widetilde\sigma,u)}$.

Proof. As a nonzero state is steered to a zero state, the input u cannot be zero. Using (A1), this means that u is nonzero on any interval.

Let $\mathscr{I} := [T, T+\varepsilon]$, $\varepsilon > 0$, be an interval with σ and $\widetilde\sigma$ constant. Set $i := \sigma(T^+)$ and $j := \widetilde\sigma(T^+)$. If $B_i u \equiv B_j u \equiv 0$ on $\mathscr{I}$, (A2) implies $D_i u \not\equiv D_j u$ on $\mathscr{I}$, hence $y_{(x_0,\sigma,u)} \not\equiv y_{(\widetilde{x}_0,\widetilde\sigma,u)}$.

Thus let $B_i u \not\equiv 0$ or $B_j u \not\equiv 0$ on $\mathscr{I}$. This means that for some $\hat t \in \mathscr{I}$ we have $(x_1, \widetilde{x}_1) := \left(x_{(x_0,\sigma,u)}(\hat t), x_{(\widetilde{x}_0,\widetilde\sigma,u)}(\hat t)\right) \neq (0,0)$. $y_{(x_0,\sigma,u)} \equiv y_{(\widetilde{x}_0,\widetilde\sigma,u)}$ on $\mathscr{I}$ would imply $(x_1, \widetilde{x}_1) \in \mathscr{R}(\Sigma_{i,j})$, hence the outputs have to be different. □

Lemma 17. *Consider the switched system* (11) *satisfying* (A1) *and* (A2). *Then* (11) *is strongly* t_S*-observable if, and only if,* (13) *holds and, for all* $i \neq j$,

$$\operatorname{rk}\left[\mathcal{O}_i^{[2n]} - \mathcal{O}_j^{[2n]} \quad \Gamma_i^{[2n]} - \Gamma_j^{[2n]}\right] = n + \operatorname{rk}\left(\Gamma_i^{[2n]} - \Gamma_j^{[2n]}\right). \tag{14}$$

Proof. Necessity of (13): Assume there exists $\begin{bmatrix} x_0 \\ \widetilde{x}_0 \end{bmatrix} \in \mathscr{R}(\Sigma_{i,j}) \setminus \{0\}$. This means that there exists an analytic input u and a time $t_S > 0$ such that

$$y_{(x_0,i,u)} \equiv y_{(\widetilde{x}_0,j,u)} \;\wedge\; x_{(x_0,i,u)}(t_S) = x_{(\widetilde{x}_0,j,u)}(t_S) = 0. \tag{15}$$

Both $y_{(x_0,i,u)}$ and $y_{(\widetilde{x}_0,j,u)}$ are analytic. Define $\sigma \equiv i$ and

$$\widetilde\sigma(t) = \begin{cases} i, & t < t_S, \\ j, & t \geq t_S. \end{cases} \tag{16}$$

Then $y_{(x_0,\sigma,u)}$ and $y_{(x_0,\widetilde\sigma,u)}$ coincide on $(-\infty, t_S)$ by definition and on $[t_S, \infty)$ by (15). Hence for this specific initial value and input it is not possible to detect a switch from mode i to mode j at time t_S.

Assume that (14) does not hold for some $i \neq j$, i.e. there exist some $x_1 \neq 0$ and U with $\mathcal{O}_i^{[2n]}x_1 + \Gamma_i^{[2n]}U = \mathcal{O}_j^{[2n]}x_1 + \Gamma_j^{[2n]}U$. In particular, (12) does not hold (as the nonzero vector $\left[x_1^\top \quad -x_1^\top \quad U^\top\right]^\top$ lies in the kernel of the matrix on the left hand side). Hence by Lemma 14 there exists some input $\hat u$ with $y_{(x_1,i,\hat u)} \equiv$

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[1] Hautus (1983) uses the notion *strong observability*; however, we follow instead the naming convention from Basile and Marro (1973) in order to avoid confusing with our strong observability notion for switched systems (where we still assume that the input is known).

$y_{(x_1,j,\hat{u})}$. Now let $t_S > 0$, $u(\cdot) := \hat{u}(\cdot - t_S)$, $\sigma \equiv i$, $\widetilde{\sigma}$ as in (16) and x_0 such that $x_{(x_0,\sigma,u)}(t_S) = x_1$. By construction of σ and $\widetilde{\sigma}$, $y_{(x_0,\sigma,u)}$ and $y_{(x_0,\widetilde{\sigma},u)}$ coincide on $(-\infty, t_S)$. Due to $y_{(x_1,i,\hat{u})} \equiv y_{(x_1,j,\hat{u})}$, they also coincide on $[t_S, \infty)$. Hence the system is not strongly t_S-observable.

To show sufficiency of (13) and (14) for strong t_S-observability, consider $x_0 \neq 0$, u and σ with switching time t_S. Let $\widetilde{x}_0$ and $\widetilde{\sigma}$ be given with $t_S \notin T_{\widetilde{\sigma}}$. As we want to show that the outputs of these solutions differ in an neighborhood of t_S, it suffices to consider $T_\sigma = \{t_S\}$ and $\widetilde{\sigma}$ constant. This means that $y_{(\widetilde{x}_0,\widetilde{\sigma},u)}$ is analytic. Eq. (14) gives that for $x_{(x_0,\sigma,u)}(t_S) \neq 0$ we have $y^{[2n]}_{(x_0,\sigma,u)}(t_S^-) \neq y^{[2n]}_{(x_0,\sigma,u)}(t_S^+)$, hence $y_{(x_0,\sigma,u)} \not\equiv y_{(\widetilde{x}_0,\widetilde{\sigma},u)}$. Now let $x_{(x_0,\sigma,u)}(t_S) = 0$, then $y_{(x_0,\sigma,u)} \equiv y_{(\widetilde{x}_0,\widetilde{\sigma},u)}$ would imply that $y_{(x_0,\sigma,u)}$ is analytic, i.e. that it coincides with $y_{(x_0,\hat{\sigma},u)}$ for $\hat{\sigma}(t) = \sigma(t_S^-)$ $\forall t$. Now Lemma 16 gives a contradiction to $y_{(x_0,\sigma,u)} \equiv y_{(x_0,\hat{\sigma},u)}$. ☐

Remark 18. Regarding (13) we observe the following:

(i) In Elhamifar et al. (2009) strong t_S-observability is characterized for discrete time switched systems in terms of (14), but condition (13) does not occur. The reason is due to stronger assumption made in Elhamifar et al. (2009) which are specific to the discrete time set up; in particular, they require that each individual mode is observable.

(ii) The conditions (13) and (14) of strong t_S-observability are indeed not related. Consider for example the system given by

$$(A_1, B_1, C_1, D_1) = (0, 1, 2, 0)\,,$$
$$(A_2, B_2, C_2, D_2) = (0, 2, 1, 0)\,,$$

which satisfies (14) but not (13). On the other hand (13) holds for any system with $B_i = 0$ for all $i \in \mathscr{P}$, hence it does not imply (14) in general.

(iii) (13) does not imply $\mathscr{R}(\Sigma_i) = \{0\}$ for the individual modes. As an example, consider the system (11) with modes

$$(A_1, B_1, C_1, D_1) = (0, 1, 0, 0)\,,$$
$$(A_2, B_2, C_2, D_2) = (0, 1, 1, 0)\,.$$

It is strongly t_S-observable, in particular, $\mathscr{R}(\Sigma_{1,2}) = \{0\}$. However, for the first mode we have $\mathscr{R}(\Sigma_1) = \mathbb{R}$.

(iv) (13) and (14) are indeed weaker than (12): The example from (iii) is strongly t_S-observable, but not strongly (x, σ)-observable as $\mathscr{O}_1 = 0$.

Theorem 19. *The switched system (11) satisfying (A1) and (A2) is strongly (x, σ_1)-observable if and only if it satisfies (13) and, for all $i, j, p, q \in \mathscr{P}$ with $i \neq j$, $p \neq q$ and $(i, j) \neq (p, q)$*

$$\operatorname{rk}\begin{bmatrix} \mathscr{O}_i^{[4n]} & \mathscr{O}_p^{[4n]} & \Gamma_i^{[4n]} - \Gamma_p^{[4n]} \\ \mathscr{O}_j^{[4n]} & \mathscr{O}_q^{[4n]} & \Gamma_j^{[4n]} - \Gamma_q^{[4n]} \end{bmatrix} = 2n + \operatorname{rk}\begin{bmatrix} \Gamma_i^{[4n]} - \Gamma_p^{[4n]} \\ \Gamma_j^{[4n]} - \Gamma_q^{[4n]} \end{bmatrix}. \tag{17}$$

Here the order of the observability matrix is doubled with respect to the previous results. If we only considered $\nu = 2n$, a vector U as in the proof of Lemma 17 might be related to different inputs u and $\tilde{u}$ on the pre-switch interval and post-switch interval.

Again, the statement can be related to ui-observability of an augmented system: (17) is a necessary – but not sufficient – condition for ui-observability of the system $\Sigma_{i,j,p,q}$ defined by

$$A_{i,j,p,q} = \begin{bmatrix} A_{i,p} & 0 \\ 0 & A_{j,q} \end{bmatrix}, \quad B_{i,j,p,q} = \begin{bmatrix} B_{i,p} \\ B_{j,q} \end{bmatrix},$$
$$C_{i,j,p,q} = \begin{bmatrix} C_{i,p} & 0 \\ 0 & C_{j,q} \end{bmatrix}, \quad D_{i,j,p,q} = \begin{bmatrix} D_{i,p} \\ D_{j,q} \end{bmatrix}.$$

Proof of Theorem 19. "(13) and (17) $\Rightarrow$ strong t_S-observability": From (17) with $p = j$, $q = i$ and $i \neq j$, we can conclude (14). Then the claim follows by Lemma 17.

"Strong (x, σ_1)-observability $\Rightarrow$ (13)": Follows by Lemma 17 as strong t_S-observability is necessary for strong (x, σ_1)-observability.

"Strong (x, σ_1)-observability $\Rightarrow$ (17)": Assume that (17) does not hold for some i, j, p, q, i.e. there exist $\left(x_1, \tilde{x}_1\right) \neq (0, 0)$ and U such that

$$\begin{bmatrix} \mathscr{O}_i^{[4n]} & \mathscr{O}_p^{[4n]} & \Gamma_i^{[4n]} - \Gamma_p^{[4n]} \\ \mathscr{O}_j^{[4n]} & \mathscr{O}_q^{[4n]} & \Gamma_j^{[4n]} - \Gamma_q^{[4n]} \end{bmatrix} \begin{bmatrix} x_1 \\ -\tilde{x}_1 \\ U \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

We get that $\Sigma_{i,j,p,q}$ is not strongly observable, i.e. for the initial value $\eta_1 := \begin{bmatrix} x_1^\top & \tilde{x}_1^\top & x_1^\top & \tilde{x}_1^\top \end{bmatrix}^\top$ and some $\hat{u}$ with $\hat{u}^{[4n]}(0) = U$ we have $y_{\Delta_{i,j,p,q}} \equiv 0$, i.e. $y_{(x_1,i,\hat{u})} \equiv y_{(\widetilde{x}_1,p,\hat{u})}$ and $y_{(x_1,j,\hat{u})} \equiv y_{(\widetilde{x}_1,q,\hat{u})}$. Define σ and $\widetilde{\sigma}$ as in (9) for some $t_S > 0$ and let $u(\cdot) := \hat{u}(\cdot - t_S)$. Let x_0 and $\widetilde{x}_0$ be such that $x_{(x_0,\sigma,u)}(t_S) = x_1$ and $x_{(\widetilde{x}_0,\widetilde{\sigma},u)}(t_S) = \widetilde{x}_1$. Then we get $y_{(x_0,\sigma,u)} \equiv y_{(\widetilde{x}_0,\widetilde{\sigma},u)}$, i.e. (11) is not strongly (x, σ_1)-observable.

"(13) and (17) $\Rightarrow$ strong σ_1-observability": Let $x_0, \widetilde{x}_0, \sigma, \widetilde{\sigma}$ and u be given with $x_0 \neq 0$, σ nonconstant and $\sigma \not\equiv \widetilde{\sigma}$. We want to show that this implies $y_{(x_0,\sigma,u)} \not\equiv y_{(\widetilde{x}_0,\widetilde{\sigma},u)}$. Assume $T_\sigma = T_{\widetilde{\sigma}}$ as otherwise t_S-observability – which we have by the first step – would yield $y_{(x_0,\sigma,u)} \not\equiv y_{(\widetilde{x}_0,\widetilde{\sigma},u)}$. Then there exists a common switching time t_S with $\sigma(t_S^-) \neq \widetilde{\sigma}(t_S^-)$ or $\sigma(t_S^+) \neq \widetilde{\sigma}(t_S^+)$. Define $x_1 := x_{(x_0,\sigma,u)}(t_S)$ and $\widetilde{x}_1 := x_{(\widetilde{x}_0,\widetilde{\sigma},u)}(t_S)$. Condition (17) implies that only for $(x_1, \widetilde{x}_1) = (0, 0)$ we can have

$$y^{[4n]}_{(x_0,\sigma,u)}(t_S^-) = y^{[4n]}_{(\widetilde{x}_0,\widetilde{\sigma},u)}(t_S^-) \;\wedge\; y^{[4n]}_{(x_0,\sigma,u)}(t_S^+) = y^{[4n]}_{(\widetilde{x}_0,\widetilde{\sigma},u)}(t_S^+).$$

However, in this case Lemma 16 already implies $y_{(x_0,\sigma,u)} \not\equiv y_{(\widetilde{x}_0,\widetilde{\sigma},u)}$. As in Lemma 2, we have equivalence of strong σ_1- and strong (x, σ_1)-observability. ☐

4. Equivalent switching signals

In the previous section we have highlighted the problem that the switching signal cannot be determined when state and input are identically zero on an interval. This problem was avoided by making the assumptions (A1) and (A2). We can consider smooth instead of analytic input and can drop (A2) if we consider equivalence classes of switching signals:

Definition 20. For given $x_0 \in \mathbb{R}^n$ and $u : \mathbb{R} \to \mathbb{R}^p$ the switching signals σ and $\widetilde{\sigma}$ are equivalent for the switched system (11), denoted by $\sigma \overset{x_0,u}{\sim} \widetilde{\sigma}$, iff $x_{(x_0,\sigma,u)} \equiv x_{(x_0,\widetilde{\sigma},u)}$, $y_{(x_0,\sigma,u)} \equiv y_{(x_0,\widetilde{\sigma},u)}$ and $\sigma = \widetilde{\sigma}$, except on intervals $\mathscr{I}$ with $\left(x_{(x_0,\sigma,u)}\right)_{\mathscr{I}} = 0$. The corresponding equivalence class is denoted by

$$\left[\sigma_{(x_0,u)}\right] := \left\{ \widetilde{\sigma} \;\middle|\; \widetilde{\sigma} \overset{x_0,u}{\sim} \sigma \right\},$$

and the *essential switching times* are given by

$$T_{[\sigma_{(x_0,u)}]} := \bigcap_{\widetilde{\sigma} \overset{x_0,u}{\sim} \sigma} T_{\widetilde{\sigma}}.$$

A similar equivalence has been considered in Kaba (2014) in the context of invertibility of switched systems.

For u analytic, $(x_0, u) \neq (0, 0)$ and systems satisfying (A2) we have $[\sigma_{(x_0,u)}] = \{\sigma\}$, i.e. trivial equivalence classes.

Adaption of Definition 10 to equivalence classes of switching signals gives the following:

Definition 21. The system (11) is called

- *strongly $(x, [\sigma])$-observable* iff for all smooth u and all $x_0, \widetilde{x}_0, \sigma, \widetilde{\sigma}$ the following implication holds:

$$\left(x_0, \left[\sigma_{(x_0,u)}\right]\right) \neq \left(\widetilde{x}_0, \left[\widetilde{\sigma}_{(\widetilde{x}_0,u)}\right]\right) \Rightarrow y_{(x_0,\sigma,u)} \not\equiv y_{(\widetilde{x}_0,\widetilde{\sigma},u)}; \tag{18}$$

- *strongly* $(x,[\sigma_1])$*-observable* iff (18) holds for all smooth u and all $x_0, \widetilde{x}_0, \sigma, \widetilde{\sigma}$ with

$$1 \le \min\left\{ \, |T_{\hat{\sigma}}| \, \middle| \, \hat{\sigma} \overset{x_0,u}{\sim} \sigma \, \right\};$$

- *strongly* $[t_S]$*-observable* iff for all smooth u and all $x_0, \widetilde{x}_0, \sigma, \widetilde{\sigma}$ the following implication holds:

$$T_{[\sigma_{(x_0,u)}]} \neq T_{[\widetilde{\sigma}_{(\widetilde{x}_0,u)}]} \;\Rightarrow\; y_{(x_0,\sigma,u)} \not\equiv y_{(\widetilde{x}_0,\widetilde{\sigma},u)};$$

One can also define strong $[\sigma]$- and strong $[\sigma_1]$-observability. Lemma 2 holds accordingly. While the setup is more general, the same characterizations hold:

Theorem 22. *The system* (11) *is strongly* $[t_S]$-/$(x,[\sigma_1])$-/$(x,[\sigma])$-*observable if and only if, the conditions* (13) + (14), (13) + (17), (12) *are satisfied, respectively (c.f.* Fig. 1*).*

For the proof we need a new version of Lemma 16:

Lemma 23. *Let* $\sigma, \widetilde{\sigma}, x_0, \widetilde{x}_0$ *and* u *smooth be given such that* $t_S \in T_{[\sigma_{(x_0,u)}]} \setminus T_{[\widetilde{\sigma}_{(\widetilde{x}_0,u)}]}$ *and* $x_{(x_0,\sigma,u)}(t_S) = x_{(\widetilde{x}_0,\widetilde{\sigma},u)}(t_S) = 0$ *for the solutions of* (11)*. Then* (13) *implies* $y_{(x_0,\sigma,u)} \not\equiv y_{(\widetilde{x}_0,\widetilde{\sigma},u)}$.

Proof of Lemma 23. If the conditions for equivalent switching signals were satisfied on the interval $\mathscr{I} := (t_S - \varepsilon, t_S + \varepsilon)$ for some $\varepsilon > 0$, we had $t_S \notin T_{[\sigma_{(x_0,u)}]} \setminus T_{[\widetilde{\sigma}_{(\widetilde{x}_0,u)}]}$. Thus $y_{(x_0,\sigma,u)} \not\equiv y_{(\widetilde{x}_0,\widetilde{\sigma},u)}$ on $\mathscr{I}$ or $x_{(x_0,\sigma,u)} \not\equiv x_{(\widetilde{x}_0,\widetilde{\sigma},u)}$ on $\mathscr{I}$. Assume that $\varepsilon > 0$ is small enough such that σ and $\widetilde{\sigma}$ are constant on $(t_S - \varepsilon, t_S)$, $(t_S, t_S + \varepsilon)$. Assume that $y_{(x_0,\sigma,u)} \equiv y_{(\widetilde{x}_0,\widetilde{\sigma},u)}$ on $\mathscr{I}$. As $x_{(x_0,\sigma,u)}$ and $x_{(\widetilde{x}_0,\widetilde{\sigma},u)}$ coincide for $t = t_S$, $x_{(x_0,\sigma,u)} \not\equiv x_{(\widetilde{x}_0,\widetilde{\sigma},u)}$ on $\mathscr{I}$ implies that there exists a $T \in \mathscr{I}$ with $\sigma(T) \neq \widetilde{\sigma}(T)$ and $(x_1, \widetilde{x}_1) := \left(x_{(x_0,\sigma,u)}(T), x_{(\widetilde{x}_0,\widetilde{\sigma},u)}(T)\right) \neq (0,0)$. Then we get $(x_1, \widetilde{x}_1) \in \mathscr{R}\left(\Sigma_{\sigma(T),\widetilde{\sigma}(T)}\right)$, i.e. a contradiction to (13). □

Proof of Theorem 22. First of all, note that the arguments for necessity of (12), (13), (14) and (17) apply also in this setup. Also, Lemma 2 holds accordingly.

"Sufficiency, strong $[\sigma]$-observability": Let $[\sigma_{(x_0,u)}] \neq [\widetilde{\sigma}_{(\widetilde{x}_0,u)}]$. Then there exits a time t such that $y_{(x_0,\sigma,u)}(t) \neq y_{(\widetilde{x}_0,\widetilde{\sigma},u)}(t)$ or $\left(x_{(x_0,\sigma,u)}(t), x_{(\widetilde{x}_0,\widetilde{\sigma},u)}(t)\right) \neq (0,0)$. In the latter case, (12) gives $y_{(x_0,\sigma,u)}(t) \neq y_{(\widetilde{x}_0,\widetilde{\sigma},u)}(t)$.

"Sufficiency, strong $[t_S]$-observability": The proof is similar to the one in the previous section. For $x_{(x_0,\sigma,u)}(t_S) \neq 0$ we use (14), for $x_{(x_0,\sigma,u)}(t_S) = x_{(\widetilde{x}_0,\widetilde{\sigma},u)}(t_S) = 0$ we can use Lemma 23. Now let $x_{(x_0,\sigma,u)}(t_S) = 0$ and $x_{(\widetilde{x}_0,\widetilde{\sigma},u)}(t_S) \neq 0$. We can use (14) to obtain $y_{(x_0,\sigma,u)} \not\equiv y_{(\widetilde{x}_0,\widetilde{\sigma},u)}$ or $x_{(\widetilde{x}_0,\widetilde{\sigma},u)} \in \ker \mathscr{O}_{\widetilde{\sigma}(t_S)}$, which can be put down to the case $x_{(\widetilde{x}_0,\widetilde{\sigma},u)}(t_S) = 0$.

"Sufficiency, strong $[\sigma_1]$-observability": We can assume that σ and $\tilde{\sigma}$ have the same essential switching times, as else strong $[t_S]$-observability implies that the corresponding outputs differ. If there is a switch with $\sigma(t_S^-) \neq \widetilde{\sigma}(t_S^-)$ or $\sigma(t_S^+) \neq \widetilde{\sigma}(t_S^+)$ and nonzero state, (17) gives that the outputs differ. If all switches with $\sigma(t_S^-) \neq \widetilde{\sigma}(t_S^-)$ or $\sigma(t_S^+) \neq \widetilde{\sigma}(t_S^+)$ occur for zero states, one can show (similar to the proof of Lemma 23) that $[\sigma_{(x_0,u)}] = [\widetilde{\sigma}_{(\widetilde{x}_0,u)}]$ or $y_{(x_0,\sigma,u)} \not\equiv y_{(\widetilde{x}_0,\widetilde{\sigma},u)}$. □

5. Conclusion

Switching time observability and switch observability were introduced and characterized by rank-conditions. The relation of these notions is illustrated in Fig. 1. A possible future research topic is the extension to the case of switched differential-algebraic equations (DAEs); we already have obtained some preliminary results in Küsters, Patil, and Trenn (in press); Küsters, Trenn, and Wirsen (in press). Based on the notion of strong (x,σ_1)-observability, another future research topic is the construction of an observer; some preliminary results have been presented in Küsters, Trenn, and Wirsen (in press).

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