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Stabilisation of state-and-input constrained nonlinear systems via diffeomorphisms: A Sontag's formula approach with an actual application

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Summary

In this work, we provide a new and constructive outlook for the control of state-and-input constrained nonlinear systems. Previously, explicit solutions have been mainly focused on the finding of a barrier-like Lyapunov function, whereas we propose the construction of a diffeomorphism to map all the trajectories of the constrained dynamics into an unconstrained one. Careful analysis has revealed that only some foundations of differential geometry and a technical assumption are necessary to construct the proposed methodology based on the well-established theories of control Lyapunov functions and Sontag's universal formulae. Altogether, it allows us to obtain an explicit solution that even includes bounded constraints in the control action, giving the designer a way to decide (to some extent) the trade-off between control saturations and robustness. Moreover, this approach does not rely on the own structure of the system dynamics, therefore covering a broad class of nonlinear systems. The main advantage of this approach is that the use of a diffeomorphism allows the splitting of the mathematical treatment of the constraint and the Lyapunov controller design. The result has been successfully applied to solve the dynamic positioning of an actual ship, where the nonlinear state constraints describe a strait. This approach enabled us to design a control Lyapunov function and thereby use Sontag's formula to solve the stabilisation problem. Realistic simulations have been executed in a real scenario on the simulator owned by an international shipbuilding company.

KEYWORDS

control applications, control design, nonlinear control, stability

1 | INTRODUCTION

This work deals with the stabilisation of nonlinear systems with both linear and nonlinear state and bounded input constraints. The underlying idea is the construction of a diffeomorphism that transforms the dynamics living in a constrained "world" into a dynamics living in an unconstrained one. Therefore, the control designer relies only on Lyapunov-based techniques in the transformed dynamics and somehow "forgets" about the constraints. Motivated by the works with barrier Lyapunov functions for strict-feedback cascade systems of related works,[1-4] the different outlook considered here gives a constructive result for a broader class of nonlinear systems. Our work is also partially related to the work of Bechlioulis and Rovithakis,[5] where an output error transformation was proposed to fulfil some prescribed performance for feedback linearisable nonlinear systems. Either way, all the aforementioned approaches rely on the structure of the system, either strict-feedback or feedback linearisable nonlinear systems, which are instrumental in those proposed solutions. The main advantages of our approach are to separate the mathematical treatment of the constraint and the Lyapunov design and cover a broader class of nonlinear systems.

Although out of the context of this work, it is also fair to mention some optimisation-based works. Thus, in the work of Wills and Heath,[6] a weighted barrier function was included in the objective function of the model predictive control approach so that, in each step, an unconstrained optimisation problem has to be solved to obtain the computational control input. In the work of Panagou et al,[7] a barrier function for first-order kinematical model of nonholonomic agents is proposed to encode collision avoidance in decision making of multiagent systems. In the work of Esterhuizen and Lévine,[8] the notion of barrier stopping points is analysed. In the work of Angeli et al,[9] a command governor is added to the primary controller acting only when necessary to avoid a violation of the constraints. In the work of Jin and Xu,[10] an iterative learning control algorithm is designed with *tan*-like barrier functions to control a robot manipulator with position constrained.

Finally, the line of research focused on "stabilisation with safety" is closely related to our work. Thus, in the work of Prajna et al,[11] the authors introduced the notion of barrier certificates (BCs) to certify that all trajectories of a system starting from a given initial set do not enter in unsafe regions. The latter approach becomes computationally tractable with sum-of-squares optimisation approach for polynomial vector fields, covering hybrid and stochastic systems. The main difference between that approach and ours is that the aforementioned work[11] could be seen as a procedure to check that all trajectories converge safely for a given closed-loop system. In the work of Wieland and Allgöwer[12] and using BC, the authors introduced the control barrier functions to construct a controller as a convex-like combination of a repulsion and stabiliser parts for steering the states from the set of initial conditions to the set of terminal conditions safely. Recently, in the work of Romdlonya and Jayawardhana,[13] the authors studied the controller design by merging control Lyapunov functions (CLFs) and control barrier function to construct a suitable function to be used through a universal formula controller.

Here, in the same line of research, we propose an alternative procedure to design a stabiliser so that, a priori, we know that all the trajectories will converge safely to the equilibrium and provide an explicit solution, ie, it does not rely on optimisation-based numerical computations. The idea relies on a different outlook of the control for state-and-input constrained nonlinear systems. All the aforementioned works have been focused on finding barrier-like Lyapunov functions, unlike here where we propose the construction of a diffeomorphism to map all the trajectories of the constrained dynamics into an unconstrained one. Once the dynamics are mapped, we made use of the classical and well-established theories of CLFs and Sontag's universal formulae to derive the controller, obtaining an explicit solution even with bounded constraints in the control action. Moreover, the use of diffeomorphisms to solve the control problem makes this approach significantly different because, it allows us to split the mathematical treatment of the constraint and the Lyapunov controller design. It is also shown that the diffeomorphisms can be constructed from geometrical and physical insights of the problem.

Thus, this work continues, refines, and completes our preliminary work,[14] where we stated part of the results but just for the state-constrained case. Thus, here, we refine the theory, removing some unnecessary assumptions and providing new results that also include the case of input constraints. In addition, we show that, once the equivalence is established, the combination of the proposed mapping between the constrained and unconstrained dynamics with Sontag's formulae, which is based on CLFs[15,16] and closely related with Artstein's work,[17] provides a constructive approach to obtain a solution that stabilises state and input (bounded) constrained nonlinear systems. As a result, the mapping approach provides a breakthrough to deal with input constraints (input saturations), and hence easing its use in practice, as it is demonstrated in the application section. In fact, with the new developments provided here, we redesign the controller in the work of Dòria-Cerezo et al,[18] complementing the numerical results on a realistic simulator of an actual ship, succinctly described in the aforementioned work,[18] owned by an internationally renowned shipbuilding company,[19] being, to the best of the authors' knowledge, the first implementation of Sontag's formula in a real problem at this level.

This paper is organised as follows. In Section 2, we formulate the problem and provide all the necessary previous results and definitions. The state-constrained case is treated in Section 3 and, in Section 4, we extend the result to use Sontag's universal formulae to deal with the state and input constraints. In Section 5, we provide the Sontag-based solution for the dynamic positioning of a ship along a strait. This paper closes with a conclusion section.

Notation. Unless otherwise indicated, all vectors are defined as column vectors including the gradient. The Jacobian of a vector function $\Phi(x)$ is denoted by $\partial_x \Phi$, $x \in \mathbb{R}^n$. The Euclidean norm of a vector $x \in \mathbb{R}^n$ is denoted by $|x| = \sqrt{x^\top x}$. Acronyms: GAS (GES) means globally asymptotically stable (globally exponentially stable).

2 | BACKGROUND

In this section, we introduce and formalise the problem and recall some foundations. Thus, consider an autonomous and affine-in-control system given by

$$\Sigma_x^u : \quad \dot{x} = f(x) + g(x)\, u, \tag{1}$$

where f and g are smooth vector fields, with $f(0) = 0$ and $g(0) \neq 0$; and $x \in \mathcal{X} \subseteq \mathbb{R}^n$ and $u \in \mathcal{U} \subseteq \mathbb{R}^m$ are the state and input vector, respectively. We consider here the stabilisation problem at the origin of (1), where the states are constrained to belong to an open and connected subset $\mathcal{X}$ of $\mathbb{R}^n$, actually a simply connected set including the origin. In this approach, we consider separately the following two cases: first a control design with $\mathcal{U} \equiv \mathbb{R}^m$ and then, second, with the input bounded and taken values in an open ball. These kinds of constraints fit with limitations in position, velocity, and acceleration in mechanical systems, also in currents and voltages in electrical circuits, and so on. For instance, $\mathcal{X}$ may be the interior of a square in $\mathbb{R}^2$ in the case of a second-order dynamics, where positions and velocities are both constrained, or a strip in $\mathbb{R}^2$ for the same system without constraints in the velocities.

In short, the proposed approach relies on mapping all the trajectories of (1) through a diffeomorphism such that the constrained state is mapped into $\mathbb{R}^n$. Recall that a diffeomorphism $\Phi : \mathcal{X} \mapsto \mathcal{Z}$ with $\mathcal{X}, \mathcal{Z} \subseteq \mathbb{R}^n$ is a one-to-one continuous and differentiable map whose inverse Φ^{-1} is also continuous and differentiable.[20] Thus, for a given diffeomorphism, say $z = \Phi(x)$, the dynamics (1) is transformed in the new coordinates z as

$$\Sigma_z^{\bar{u}} : \quad \dot{z} \; = F(z) + G(z)\, \bar{u}, \quad z \in \mathcal{Z} \subseteq \mathbb{R}^n, \tag{2}$$

with $F(z) := \partial_x \Phi\, f(x)|_{x=\Phi^{-1}(z)}$; $G(z) := \partial_x \Phi\, g(x)|_{x=\Phi^{-1}(z)}$; and where, for $u = u(x)$, then $\bar{u} := u \circ \Phi^{-1}(z) \in \mathcal{U}$. It is an exercise of calculus to show that, if $\{\mathbf{x}_k\}$ is a sequence in $\mathcal{X}$ that converges to a point in the boundary of $\mathcal{X} \subseteq \mathbb{R}^n$, then $\Phi(\{\mathbf{x}_k\})$ converges to infinity in $\mathbb{R}^n$. This means that the boundary of $\mathcal{X}$ as a (nonempty) subset of $\mathbb{R}^n$ is mapped to infinity through Φ. From now on, we can presume the existence of a diffeomorphism $\Phi : \mathcal{X} \mapsto \mathbb{R}^n$, with $\Phi(0) = 0$. Roughly speaking, the main idea is the design of a controller for a state-constrained system taking the benefit of a diffeomorphism to somehow avoid the state constraints. Notice that the change of coordinates might result in a more involved dynamics, and hence, a trade-off between simplifying the state-domain or the dynamics vector field should be considered.

The following assumption formalises the required map and provides an easy reference.

Assumption 1. There exists a diffeomorphism $\Phi : \mathcal{X} \mapsto \mathbb{R}^n$, $\Phi(0) = 0$.

Remark 1. We emphasise that the underlying idea is the construction of the diffeomorphism motivated from applications. The way we propose to solve the constrained control problem is the following: (i) to define the constraints required from the application; (ii) to construct the diffeomorphism from these constraints; and (iii) to derive the controller with the methodology provided here. Thus, we propose to construct the diffeomorphism from physical and/or geometrical insights. From our perspective of the problem, to find a general expression for diffeomorphisms is nonsense because we do not focus on any particular class of applications rather systems of the form (1). In that regard, we have included the application section with the following twofold objective: first, to show the construction of the diffeomorphism with the physical and geometrical insights of a real scenario; and second, to show the application of the theoretical results.

3 | LYAPUNOV-BASED CONTROL DESIGN BY DIFFEOMORPHISM EQUIVALENCE WITH BARRIER FUNCTIONS: STATE-CONSTRAINED CASE

In this section, we establish the equivalence between some known Lyapunov stability results and barrier functions. First, we recall from the work of Tee et al[1] the definition of a barrier Lyapunov function (BLF). We believe that it is essential to leave the definition given in the aforementioned work[1] untouched for the sake of comparison thereafter, even knowing that some extra machinery is needed for our approach. To make this work self contained, we reproduce as follows the definition given in the work of Tee et al.[1]

Definition 1. A BLF is a scalar function $W(x)$, defined w.r.t. the system $\dot{x} = \mathcal{F}(x) := f(x, u(x))$ on an open region $\mathcal{X}$ containing the origin, that is continuous, positive definite, has continuous first-order partial derivatives at every point of $\mathcal{X}$, has the property $W(x) \to \infty$ as x approaches the boundary of $\mathcal{X}$, and satisfies $W(x(t)) \leq c$, for all $t \geq 0$, along any solution of $\dot{x} = \mathcal{F}(x)$ with $x(0) \in \mathcal{X}$ and some positive constant c depending on the initial conditions.

In the work of Tee et al,[1] the controller design relies on a BLF proposed ad hoc for systems in strict feedback form, whereas, here, we propose to transform the dynamics (1) through the diffeomorphism such that the controller design for the constrained dynamics becomes a standard unconstrained design.

Remark 2. The following observations are in order.

i. A BLF $W(x)$ of Definition 1 corresponds, through the diffeomorphism $z = \Phi(x)$, to a positive real function $V(z)$, which is radially unbounded and fulfills,

$$\forall z(0) \quad \exists c(z(0)) : \qquad V(z(t)) \leq c(z(0)), \quad \forall t \geq 0,$$

ie, trajectories remain in the V-level set defined by $c(z(0)) = V(z(0)) = W(x(0))$.

ii. If $\mathcal{X}$ were unbounded, the radially unboundedness condition is needed for W to be a BLF.

iii. The existence of BLF does not allow us to conclude anything about either stability or convergence in the Lyapunov sense. This fact stems from the own Definition 1 that ensures only the boundedness and confinement of trajectories.

Let us highlight the aforementioned removal of conditions made in our previous work.[14] In particular, the definition of Φ and the points above imply that Proposition 1, Assumption **S**, and Proposition 2 are unnecessary.

Now, we are in position to state a preliminary stability result in the following theorem, which states the equivalence between the Lyapunov theory and BLFs.

Theorem 1. *Let $z = \Phi(x)$ be a diffeomorphism as in Assumption 1 and $W : \mathcal{X} \mapsto \mathbb{R}_+$ a BLF for (1), for some $u = u_l(x)$ and $x \in \mathcal{X}$. Then, W is a Lyapunov function for (1) if and only if $V(z) := W \circ \Phi^{-1}(z)$ is a Lyapunov function for (2), with $\bar{u} = u_l \circ \Phi^{-1}(z)$ and $z \in \mathbb{R}^n$. If $W(x)$ or $V(z)$ are Lyapunov functions, then the origin of (1) and (2) are Lyapunov stable.*

Proof. The proof is straightforward using the inherited smoothness properties of the diffeomorphism. □

The result of Theorem 1 allows control designers to use any Lyapunov-based technique for the stabilisation of the state-constrained problem. The idea consists in constructing a new unconstrained state z and designing a control algorithm $\bar{u}(z)$ with any available design method. The only requirement is that, for this control law $\bar{u}(z)$, there should be a Lyapunov function V radially unbounded. Finally, undoing $\bar{u}$ via Φ the control law is given by $u(x) := \bar{u} \circ \Phi(x)$ and, in the original coordinates, the BLF can be obtained from $W(x) := V \circ \Phi(x)$. Remind that there is no restriction in the structure of the smooth nonlinear system (1), for example, unlike in the work of Tee et al,[1] a cascade structure is no longer needed.

Remark 3. In Figure 1, we depict the relationship between the dynamics, maps, and functions. The diffeomorphism and its inverse are in the centre relating the open subsets $\mathcal{X}$ (left) and $\mathbb{R}^n$ (right), where the constrained dynamics (1) and the unconstrained one (2) are defined. Additionally, we define, at the top, the functions W and V and their domain $\mathcal{X}$ and $\mathbb{R}^n$ and codomain $\mathbb{R}_+$; and, at the bottom, the control inputs with their corresponding domain $\mathcal{X}$ and codomain

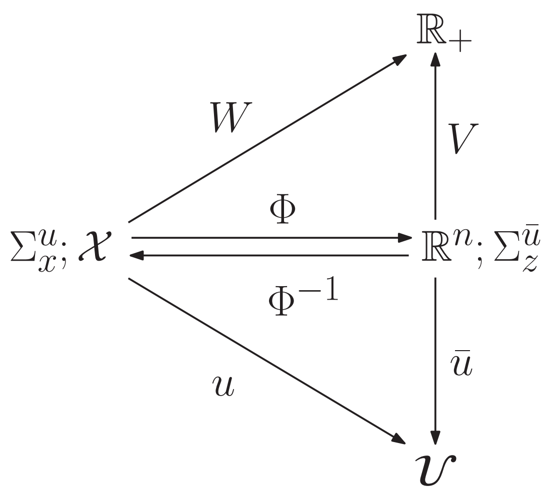

FIGURE 1 Relational diagram

$\mathcal{U}$. In summary, it sketches all the relations of the maps through the diffeomorphism Φ, as stated in Theorem 1. Finally, notice that the controller design side is on the right side, ie, V and $\bar{u}$.

The illustrative simple nonlinear example in our related work[14] is also used here to show the new developments further, and so, we summarise it as follows to show the application of Theorem 1.

3.1 | Benchmark example

Consider the stabilisation at the origin $x = 0$ of the nonlinear system given by

$$\Sigma_x^u: \quad f(x) := \begin{bmatrix} x_2 - x_1^2 x_2 \\ 0 \end{bmatrix}, \quad g(x) := \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

with $u \in \mathcal{U} \equiv \mathbb{R}$ and the following two cases for the constraints: first, a partially constrained state with $\mathcal{X} = \{|x_1| < 1, x_2 \in \mathbb{R}\}$; and, secondly, a fully constrained state with $\mathcal{X} = \{|x_1| < 1, |x_2| < 1\}$.
Partially constrained state: $\mathcal{X} = \{|x_1| < 1;\ x_2 \in \mathbb{R}\}$. The following diffeomorphism transforms the constrained state to an unconstrained one

$$z = \Phi(x) := \begin{bmatrix} \tanh^{-1} x_1 \\ x_2 \end{bmatrix}, \ \Phi^{-1}(z) = \begin{bmatrix} \tanh z_1 \\ z_2 \end{bmatrix},$$

with $\partial_x \Phi(x) > 0$, for all $x \in \mathcal{X}$. The diffeomorphism is not unique* and another possible choice was provided in our other work[14] so that a trade-off between the robustness and the simplicity of the dynamics should make the designer to decide the "best". From (2), the dynamics in the new coordinates become

$$\Sigma_z^{\bar{u}}: \quad F(z) = \begin{bmatrix} z_2 \\ 0 \end{bmatrix}, \ G(z) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Invoking Theorem 1, we design the controller for the transformed (linear) system that, in this case, it is straightforward. Then, the stabiliser $\bar{u} = -k_1 z_1 - k_2 z_2$, $k_1, k_2 > 0$, makes the origin $z = 0$ GAS, and GES, or equivalently via Theorem 1, $x = 0$ GAS, and GES, with $u(x) = -k_1 \tanh^{-1} x_1 - k_2 x_2$. The Lyapunov functions V and W to conclude stability invoking Theorem 1 are in our related work.[14]
Fully constrained state: $\mathcal{X} = \{|x_1| < 1,\ |x_2| < 1\}$. Let define the diffeomorphism Φ as

$$\Phi(x) := \begin{bmatrix} \tanh^{-1} x_1 \\ \tanh^{-1} x_2 \end{bmatrix}, \ \Phi^{-1}(z) = \begin{bmatrix} \tanh z_1 \\ \tanh z_2 \end{bmatrix},$$

where $\partial_x \Phi(x) > 0$, for all $|x_i| < 1$, $i = 1, 2$. The dynamics in the new coordinates become

$$\Sigma_z^{\bar{u}}: \quad F(z) = \begin{bmatrix} \tanh z_2 \\ 0 \end{bmatrix}, \ G(z) = \begin{bmatrix} 0 \\ \frac{1}{1-(\tanh z_2)^2} \end{bmatrix},$$

and a stabiliser that makes $z = 0$ GAS becomes $\bar{u} := (-z_1 - 2z_2 - 2\tanh z_2)(1 - (\tanh z_2)^2)$ with Lyapunov function $V = z_1^2 + z_1 z_2 + z_2^2$. The corresponding function W and $u(x)$ can be obtained by a direct application of Theorem 1 (see our previous work[14]).

4 | CLF-BASED CONTROL DESIGN: STATE- AND-INPUT CONSTRAINED CASE

In this section, we go forward in the design stage making use of CLFs with a twofold objective, ie, being constructive and including control constraints (bounded). A weak point of the aforementioned proposed approach and also in all previous referenced works is that the barrier for the state is made through a (nonlinear) "high-gain" feedback. The closer the state is to the barrier the higher (nonlinear) gain is provided so that, in practice, this might cause undesired saturations and, in turn, instabilities.† The point of view given here allows us to force additionally to the input to be constrained in some bounded set, unlike for example in the works of Tee et al.[1,2] At first view, it looks rather unintuitive, but the underlying idea is the trade-off between robustness and input boundedness. Toward this end, we make use of the CLFs approach and constructive Sontag's formulae.

*In most of technical problems, state variables are bounded, possibly to have physical meaning. Let ξ be a state that has to remain in the interval (a, b). Then, $\Phi(\xi) = \frac{1}{a-\xi} + \frac{1}{b-\xi}$ is a diffeomorphism mapping (a, b) to the real line $(-\infty, \infty)$.
†Fact elegantly described by Stein.[21]

Let us brief the CLF approach from the unconstrained transformed dynamics side (2). This approach relies on the equivalence between the *stabilisability* and the existence of Lyapunov functions with some special properties introduced in the work of Artstein[17] and later on in the work of Sontag.[15] In fact, in the work of Artstein,[17] he proved that there exists a stabiliser $\bar{u}(z)$, $z \in \mathbb{R}^n$ that makes the origin of (2) *asymptotically stable* if and only if there exists a C^1 Lyapunov function $V(z) > 0$, with $V(0) = 0$, satisfying the inequality

$$\inf_{\bar{u}\in\mathcal{U}} \left\{\partial_z V^\top (F + G\bar{u})\right\} < 0, \tag{3}$$

for at least a $\bar{u}$ if $z \neq 0$. Moreover, the origin is GAS if $V(z) \to \infty$ as $|z| \to \infty$, ie, radially unbounded. In a related work,[15] Sontag called such Lyapunov function as *control Lyapunov function*. Artstein also proved that, although smooth elsewhere, such $\bar{u}$ does not have to be continuous at the origin. However, he also provided a *necessary and sufficient* condition on V to make $\bar{u}$ continuous, which is, *for every* $\epsilon > 0$ *there is a* $\delta > 0$ *such that, whenever* $|z| < \delta$, $z \neq 0$, *there is some* $\bar{u}$ *with* $|\bar{u}| < \epsilon$ *such that the inequality* $\partial_z V^\top (F + G\bar{u}) < 0$ *holds*. Sontag[15] called such condition as the *small control property*.

Back to our approach, it becomes apparent that a BLF might or might not be a CLF, according with its aforementioned definition (3). For the sake of formality and comparison, let us define a BLF that additionally satisfies the conditions to be a CLF.

Definition 2. A barrier CLF (henceforth BCLF) is a BLF that additionally is a CLF endowed with the *small control property*.

With Definition 2, the next theorem establishes the equivalence between both constrained (1) and unconstrained (2) worlds via BCLFs and diffeomorphisms that, in turn, simplifies drastically the controller design stage, upon having a map Φ satisfying Assumption 1.

Theorem 2. *Let* $z = \Phi(x)$, $x \in \mathcal{X}$, *and* $z \in \mathbb{R}^n$ *be a map verifying Assumption 1. Then,* $W(x)$ *is a BCLF w.r.t.* Σ_x^u *if and only if* $V(z)$ *is a BCLF w.r.t.* $\Sigma_z^{\bar{u}}$, *with* $W(x) := V \circ \Phi(x)$ *and* $u(x) := \bar{u} \circ \Phi(x)$.

Proof. [W(x) ∈ BCLF ⇐ V(z) ∈ BCLF]. Since $V(z)$ is a BCLF for (2), $z \in \mathbb{R}^n$, there is a smooth stabiliser $\bar{u}$ such that the following inequality holds:

$$\begin{aligned} \dot{V}(z) &= \partial_z V^\top \dot{z} \\ &= \partial_z V^\top (F(z) + G(z)\bar{u}(z)) < 0, \quad z \neq 0. \end{aligned} \tag{4}$$

On the one hand, by Assumption 1, the function $W(x) := V \circ \Phi(x)$ is positive definite, smooth, and C^1, and the stabiliser $u = \bar{u} \circ \Phi(x)$ is smooth as well, for all $x \in \mathcal{X}$, and hence W is a BLF. On the other hand, the derivative of W along the trajectories of the system (1), for all $x \in \mathcal{X}$, becomes

$$\begin{aligned} \dot{W}(x) &= \dot{V}(\Phi(x)) \\ &= \partial_z V^\top \Big|_{z=\Phi(x)} \partial_x \Phi(x)\ \dot{x} \\ &= \partial_z V^\top \Big|_{z=\Phi(x)} (\partial_x \Phi\ f(x) + \partial_x \Phi\ g(x)u(x)) \\ &= \partial_z V^\top (F(z) + G(z)\bar{u}(z))\Big|_{z=\Phi(x)} < 0, \end{aligned}$$

where the last inequality holds from (4) and so the BLF W is a CLF as well, ie, a BCLF.
[$W(x) \in BCLF \Rightarrow V(z) \in BCLF$]. Since Φ is a diffeomorphism, replacing Φ by Φ^{-1} and flipping V and W in the previous proof one obtain the reciprocal implication. □

Remark 4. As it was aforementioned, the use of a diffeomorphism eases the control-design stage at the level of Lyapunov functions, for example, avoiding the constructions of ad-hoc cross terms in BLFs. In fact, in an unstructured nonlinear system (neither feedback nor feedforward structure), the design of a cross term might become a daunting task. To show this, in our other work,[14] we made a comparison with the numerical "feasibility check" needed in the work of Tee et al[1] and related references therein.

4.1 | Sontag formulae–based control design

The equivalence established in Theorem 2 allows the use of Sontag's universal formulae for stabilisation. The design methodology follows these two steps: first, as before, describe the constraints in x with Φ and map the dynamics to z; and second, find a BCLF in z and make use of Sontag's formulae. Additionally, here, we also propose the use of the formula for bounded control, and hence, solving constrained state problems with bounded control. Notice that, altogether, it provides a methodology to solve the hard problem of stabilisation of nonlinear systems with state and control constraints. Thus, let us brief those formulae as follows w.r.t. the control-design side $\Sigma_z^{\bar{u}}$ of (2). For that, recall from the work of Sontag[15] the definitions of the (vector) functions $\mathbf{a} := \partial_z V^\top F$ and $\mathbf{b} := \partial_z V^\top G$ and define $\vec{\mathbf{b}} := \mathbf{b}^\top / |\mathbf{b}|^2$.

Unbounded control formula $\Leftrightarrow \mathcal{U} \equiv \mathcal{R}^m$. In the aforementioned work,[15] a universal formula for stabilisation was provided upon knowing a CLF. Thus, our control design reduces to find a CLF w.r.t. $\Sigma_z^{\bar{u}}$ that globally stabilises its origin, say $V(z)$, and then the stabiliser formula reads

$$\bar{u}(z) := \begin{cases} -\left(\mathbf{a} + \sqrt{\mathbf{a}^2 + |\mathbf{b}|^4}\right)\vec{\mathbf{b}} & , \quad \mathbf{b} \neq 0; \\ 0 & , \quad \mathbf{b} = 0. \end{cases} \tag{5}$$

Recall that (5) is continuous at $z = 0$ if and only if the CLF satisfies the aforementioned *small control property*. In fact, if $V(z)$ is a BCLF of Definition 2, that property holds.

Bounded control formula $\Leftrightarrow \mathcal{U} \equiv \mathcal{B}_r^m$. The aforementioned formula (5) is essentially a (nonlinear) "high-gain" feedback that might cause saturation in practice. Thus, the trade-off between robustness and actuator saturation can be achieved with the approach proposed here and the formula provided in the work of Lin and Sontag[16] for bounded control, which is

$$\bar{u}(z) := \begin{cases} -\dfrac{\mathbf{a} + \sqrt{\mathbf{a}^2 + |\mathbf{b}|^4}}{1 + \sqrt{1 + |\mathbf{b}|^2}} r\,\vec{\mathbf{b}} & , \quad \mathbf{b} \neq 0; \\ 0 & , \quad \mathbf{b} = 0, \end{cases} \tag{6}$$

and, as before, (6) has also been defined on the unconstrained dynamics. The formula (6) was defined to confine the control in the unit ball, ie, $r = 1$. For practical issues, here, we have replaced the unit ball ($\mathcal{B}_1^m$) for one of radius r defined as

$$\mathcal{B}_r^m := \{u \in \mathcal{R}^m : |u|^2 < r\}, r > 0. \tag{7}$$

All the technical properties of (6), such as K-continuity,‡ analyticity, and boundedness, were provided in the work of Lin and Sontag.[16] However, those properties were developed for a controller taking values in the unit ball, and all those properties hold for the ball of radius r given by (7), although the "scaling" factor r has to be inserted accordingly. In this way, the following theorem establishes the way to proceed in this framework to design controllers with BCLFs as a direct consequence of the joining of a BCLF and a diffeomorphism of Assumption 1. Additionally, its proof needs a (nonlinear) rescaled version of lemma 2.3 in the work of Lin and Sontag[16] that, although straightforward, we streamline along it for clarity.

Theorem 3. *Let $z = \Phi(x)$, $x \in \mathcal{X}$, and $z \in \mathbb{R}^n$ be a map verifying Assumption 1 and V be a BCLF for the system (2) with $\mathcal{U} \equiv \mathbb{R}^m$ ($\mathcal{U} \equiv \mathcal{B}_r^m$). Then, the origin of (2) is GAS with Sontag's formula (5) (formula (6)) and so is the origin of (1) on $\mathcal{X}$ with $x(t) \in \mathcal{X}$, $t \geq 0$.*

Proof. [$\mathcal{U} \equiv \mathbb{R}^m$]. For the unbounded control case a BCLF is just a CLF with *small control property* and hence, by the result of the work of Sontag,[15] the formula (5) makes the origin of (2) GAS, which, in turn, makes the origin of (1) GAS on $\mathcal{X}$ by the equivalence established in Theorem 2.

[$\mathcal{U} \equiv \mathcal{B}_r^m$]. As it has been aforementioned, in the case of bounded control $\mathcal{U} \equiv \mathcal{B}_r^m$, for clarity, we streamline a "rescaled" version of lemma 2.3 in the work Lin and Sontag.[16] Thus, following the same line of arguments as in the aforementioned work,[16] the fact that V is a BCLF with $\bar{u} \in \mathcal{U} \equiv \mathcal{B}_r^m$ is equivalent to $\mathbf{a} + r|\mathbf{b}| < 0$. It is straightforward to see that lemma 2.3 in the work of Lin and Sontag[16] holds with a rescaled definition of the open set $\mathcal{D}$ from the aforementioned work[16], as $\mathcal{D}_r := \{(\mathbf{a}, \mathbf{b}) : \mathbf{a} < r|\mathbf{b}|, \mathbf{a}, \mathbf{b} \in \mathbb{R}\}$, and also the function $\alpha_r := r\alpha(\mathbf{a}, \mathbf{b})$. In particular, properties (a) and (b) remain unchanged and (c) becomes $|\alpha_r| < r$ and (d) $\mathbf{a} + \mathbf{b}\alpha_r < 0$, for all $(\mathbf{a}, \mathbf{b}) \in \mathcal{D}_r$. Thus, the *small control property* assumption of V implies that $(\mathbf{a}, |\mathbf{b}|) \in \mathcal{D}_r$, and hence, (d) implies that V is a BCLF and (c) from (6) implies that $\mathcal{U} \equiv \mathcal{B}_r^m$. Hence, theorem 1 in the work of Lin and Sontag[16] with $\mathcal{U} \equiv \mathcal{B}_r^m$ guarantees the origin of (2) is GAS. Finally, the stabilisability of (1) is also guaranteed recalling the equivalence established by Theorem 2, with $u(x) := \bar{u} \circ \Phi(x) = \bar{u}(\Phi(x)) \in \mathcal{U} \subseteq \mathcal{B}_r^m$. □

‡*Definition.*[16] Let $\mathcal{D} \subseteq \mathbb{R}^2$ denote the open set $\mathcal{D} := \{(\mathbf{a}, \mathbf{b}) : \mathbf{a} < |\mathbf{b}|, \mathbf{a}, \mathbf{b} \in \mathbb{R}\}$. A function $\sigma : \mathcal{D} \mapsto \mathbb{R}$ is K-continuous if, for each $\epsilon > 0$, there is a $\delta > 0$ such that $|\mathbf{b}| < \delta$ and $\mathbf{a} < \delta|b| \Rightarrow |\sigma(\mathbf{a}, \mathbf{b})| < \epsilon$.

4.2 | Benchmark example continued

We redesign the controller for the fully constrained case of the example in Section 3.1 by using the results of Theorems 2 and 3 together with the universal formulae. On the one hand, it is straightforward to verify that $V(z) = z_1^2 + z_1 z_2 + z_2^2$ is a BCLF. To see this, notice that

$$\mathbf{a} = (2z_1 + z_2)\tanh z_2,$$
$$\mathbf{b} = \frac{z_1 + 2z_2}{1 - (\tanh z_2)^2},$$

and after some straightforward manipulations, the derivative reads

$$\dot{V}(z) = -3z_2 \tanh z_2 + (z_1 + 2z_2)\left(2\tanh z_2 + \frac{\bar{u}}{1 - (\tanh z_2)^2}\right).$$

Thus, V is a BCLF and, moreover, is also a BCLF with $\mathcal{U} \subseteq \mathcal{B}_1^1$. Hence, both formulae (5) and (6) guarantee the GAS property of the origin of (2) and with Theorem 2 of the origin of (1) with $u = \bar{u} \circ \Phi(x)$. Comparison of Sontag's formulae in a batch of 100 simulations are shown in Figure 2 in the original coordinates x. Figure 2 at the top left (right) shows the simulations for unbounded control (5) (bounded control (6)). In the middle left (right) is the minimum (maximum) of the input in each simulation with unbounded control (5) (bounded control (6)). Notice that, in the bounded control case, $|u| < 1$, as expected by design. At the bottom left, it shows the cloud of initial conditions, and finally, at the bottom right is the trade-off of unbounded and bounded control performances for $x(0) = (0.9, 0.9)$, noting that the stabilisation takes about 30% longer with the bounded control.

Remark 5. Even though the constructive Sontag's result is powerful providing analytic formulae, for the sake of generality, we underscore that Artstein's result goes further because it considers nonsmoothness at the origin, even for nonaffine nonlinear systems. Moreover, Artstein's result guarantees the existence of a feedback, ie, stabilisability. In that regard, let us redesign the stabiliser in the benchmark example without the use of Sontag's formula (6) and ensuring $|\bar{u}| < 1$. Certainly, a stabiliser ensuring GAS with that bound for the input is given by the following state feedback:

$$\bar{u} = -2(k\tanh(z_1 + 2z_2) + \tanh z_2)\left(1 - (\tanh z_2)^2\right),$$

where k is a control gain such that $0 < k < 1/2 - 2/(3\sqrt{3})$ to fulfil the constraint $|\bar{u}| < 1$.

5 | APPLICATION: DYNAMIC POSITIONING OF MARINE CRAFT

This application entails a more complicated scenario because of the nonlinearities introduced by the use of two different reference frames. It consists in the dynamic positioning of a ship. The Lyapunov-based design mentioned in Section 3 is used in the work of Dòria-Cerezo et al[18] to solve the position-state–constrained problem. Here, we make use of a BCLF control design, making use of the result of Theorems 2 and 3. Thus, consider the widely accepted low-speed ship dynamics (see the works of Fossen et al[22,23]) as

$$\dot{x}_c = J(x_c)x_u, \tag{8}$$

$$M\dot{x}_u = -Dx_u + u + J^\top(x_c)d, \tag{9}$$

where $(x_c, x_u) \in \mathbb{R}^3 \times \mathbb{R}^3$ are the position coordinate vector in the Earth-fixed reference frame and the relative vessel-frame velocity coordinate vector, respectively; $u \in \mathbb{R}^3$ is the vector forces and torque applied to the vessel in the fixed reference frame; and $d \in \mathbb{R}^3$ represents the environmental disturbances because of the sea currents, waves, and wind (which is assumed known). The rotation matrix relating the Earth-fixed frame to the relative frame of the reference is $J(x_c)$, and matrices $M = M^\top > 0$ and $D + D^\top > 0$ represent constant regime inertia and damping/drag, respectively.

Remark 6. It is important to highlight that the model (8)-(9) is for control design purposes and all the simulations have been made in the aforementioned realistic shipbuilding-company simulator, keeping all the marine-craft nonlinearities, succinctly described in the work of Dòria-Cerezo et al.[18]

The control objective is twofold: (i) the stabilisation of the ship at a desired position x_c^d, ie, $\lim_{t\to\infty}(x_c(t) - x_c^d) = 0$ and $\lim_{t\to\infty} x_u(t) = 0$, $t \geq 0$; and (ii) reach the desired position with the state confined in the predefined region on the sea, defined as $\mathcal{X} := \left\{(x_{c1}, x_{c2}) \in \mathbb{R}^2 : C(x_c) < 0\right\} \times \mathbb{R}^4$ and $(x_c(0), x_u(0)) \in \mathcal{X}$. Notice that this control problem is partially constrained because only two of the states are constrained. In summary, it is a regulation control problem with nonlinear state constraints. In fact, it is an interesting problem for dynamic positioning of ships because, among others, it alleviates the computational burden of motion planning.

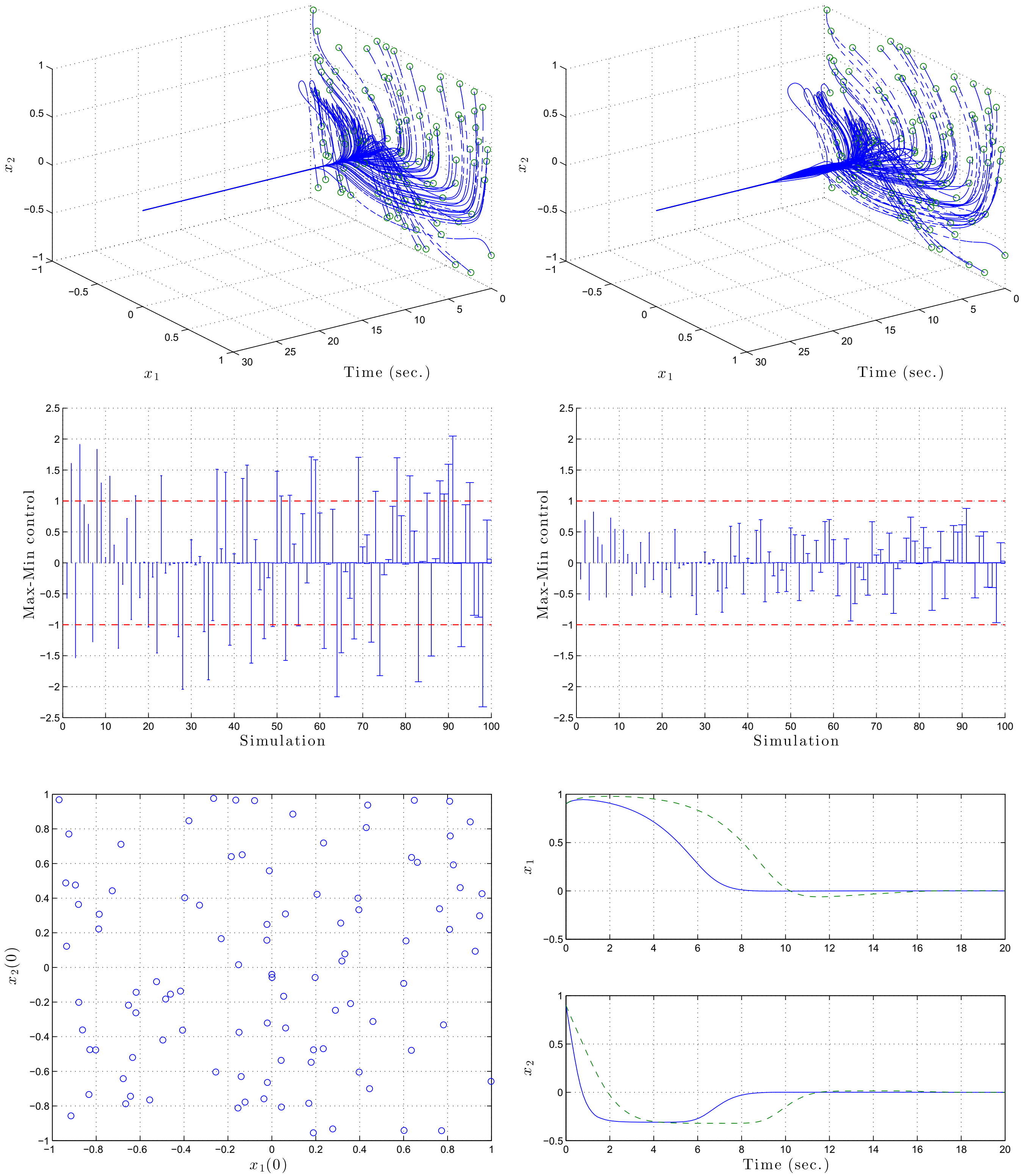

FIGURE 2 Batch of 100 simulations with Sontag's formulae. Initial conditions $x(0) \in \mathcal{X} = (-1, 1) \times (-1, 1)$ [Colour figure can be viewed at wileyonlinelibrary.com]

On the other hand, to shift the desired equilibrium to the origin and ease the calculations, we define $z_c := \phi(x_c)$, $z_u := x_u$ and the error coordinate $\tilde{z}_c := z_c - z_c^d$, with $z_c^d := \phi(x_c^d)$ as the desired position, so that the corresponding *diffeomorphism* according to the notation becomes $\Phi := [\phi(x_c)^\top, x_u^\top]^\top$. Thus, the transformed unconstrained error dynamics from (8)-(9) yield

$$\dot{\tilde{z}}_c = \partial\bar{\Psi} \cdot \bar{J} z_u, \tag{10}$$

$$M\dot{z}_u = -Dz_u + \bar{u} + \bar{J}^\top d, \tag{11}$$

where, for compactness, we have defined $\partial\bar{\Psi}(\tilde{z}_c) := \partial_{x_c}\phi \circ \phi^{-1}(\tilde{z}_c)$ and $\bar{J} = J \circ \phi^{-1}(\tilde{z}_c)$. For the sake of simplicity, we consider the case $d = 0$ because, in practice, their good estimation allows to compensate them with a feedforward action.

Unlike in the Lyapunov-based design of Dòria-Cerezo et al,[18] the design here is based on a CLF and so we need to find a suitable CLF. Unfortunately, the Lyapunov function used in the aforementioned work[18] is not a CLF. In the following proposition, we state a CLF for the dynamic positioning of ships described by (10)-(11).

Proposition 1. *Consider a map $\Phi := [\phi(x_c)^\top, x_u^\top]^\top$ satisfying Assumption 1 and the dynamics given by (10)-(11). Then, the positive definite and radially unbounded function defined as*

$$V := \frac{1}{2}\begin{bmatrix}\tilde{z}_c \\ z_u\end{bmatrix}^\top \begin{bmatrix}2I_3 & \bar{J} \\ \bar{J}^\top & I_3\end{bmatrix}\begin{bmatrix}\tilde{z}_c \\ z_u\end{bmatrix} \tag{12}$$

is a BCLF for any ϕ such that the positivity condition $\partial\bar{\Psi} + \partial\bar{\Psi}^\top > 0$ holds. Therefore, Sontag's formula (5) makes the origin of (10)-(11) GAS, or equivalently the equilibrium of (8)-(9).

Proof. Recalling the fact that $J^\top J = I_3$, the derivative of (12) along the trajectories of (10)-(11) reads

$$\begin{aligned}\dot{V} &= \left(2\tilde{z}_c^\top + z_u^\top \bar{J}^\top\right)\partial\bar{\Psi}\bar{J}z_u + \tilde{z}_c^\top \dot{\bar{J}} z_u + \left(z_c^\top \bar{J} + z_u^\top\right) M^{-1}(-Dz_u + \bar{u}) \\ &= -\frac{1}{2}\tilde{z}_c^\top\left(\partial\bar{\Psi} + \partial\bar{\Psi}^\top\right)\tilde{z}_c + \left(z_c^\top \bar{J} + z_u^\top\right)\left[M^{-1}(-Dz_u + \bar{u}) + \partial\bar{\Psi}z_u + \partial\bar{\Psi}^\top \bar{J}^\top z_c + Sz_u\right],\end{aligned}$$

where we have made use of the property on $\dot{\bar{J}}$ of S being its corresponding skew-symmetric matrix. Recall that the function (12) is radially unbounded on $\mathbb{R}$ and positive definite and define the set $\Omega := \{(\tilde{z}_c, z_u) \in \mathbb{R}^6 : z_u = -\bar{J}^\top \tilde{z}_c\}$. Then, by definition of CLF, we have

$$\dot{V}\big|_\Omega = -\frac{1}{2}\tilde{z}_c^\top\left(\partial\bar{\Psi} + \partial\bar{\Psi}^\top\right)\tilde{z}_c,$$

which, under the positivity condition, becomes negative definite restricted to the set Ω, and hence V is a CLF. Additionally, since (12) satisfies the *small control property*, it is a BCLF. Therefore, Sontag's formula (5) is a global *smooth* stabiliser for the origin of (10)-(11), and Theorem 3 establishes the equivalence needed for (8)-(9). □

Remark 7. We underscore that it is not difficult to satisfy the positivity condition $\partial\bar{\Psi} + \partial\bar{\Psi}^\top > 0$. The own construction of the diffeomorphism makes in many applications its Jacobian sign definite. In the following controller design for a typical scenario, this fact becomes apparent.

Construction of Φ. Following Remark 1, let us show the construction of the diffeomorphism for a specific application with geometrical insights. To this end, we consider a strait scenario, as shown in Figure 3 left. In this figure, three regions have been differentiated, ie, the green is the ground and the dotted line on the sea is splitting the ground and the continental sea area of relatively shallow water. The position constraints are defined by the safe margin with dotted line and become

$$\mathcal{X}_c := \begin{cases} x_{c1} \in \mathbb{R} : & x_{c1}^{\min} < x_{c1} < x_{c1}^{\max}, \\ x_{c2} \in \mathbb{R} : & x_{c2}^{\min}(x_{c1}) < x_{c2} < x_{c2}^{\max}(x_{c1}), \\ x_{c3} \in \mathbb{R}, & \end{cases}$$

with functions $x_{c2}^{\min}(x_{c1}) := a_m x_{c1}^2 + b_m$ and $x_{c2}^{\max}(x_{c1}) := a_M x_{c1}^2 + b_M$, where constants $x_{c1}^{\min}, a_m, b_m < 0$ and $x_{c1}^{\max}, a_M, b_M > 0$, becoming apparent the quadratic boundaries for the x_{c2} coordinate. Thus, a suitable diffeomorphism fulfilling all the required properties becomes

$$z_c = \phi(x_c) := \begin{bmatrix} \frac{k_1 x_{c1}}{(x_{c1}^{\min} - x_{c1})(x_{c1}^{\max} - x_{c1})} \\ \frac{k_2 x_{c2}}{(x_{c2}^{\min}(x_{c1}) - x_{c2})(x_{c2}^{\max}(x_{c1}) - x_{c2})} \\ x_{c3} \end{bmatrix}.$$

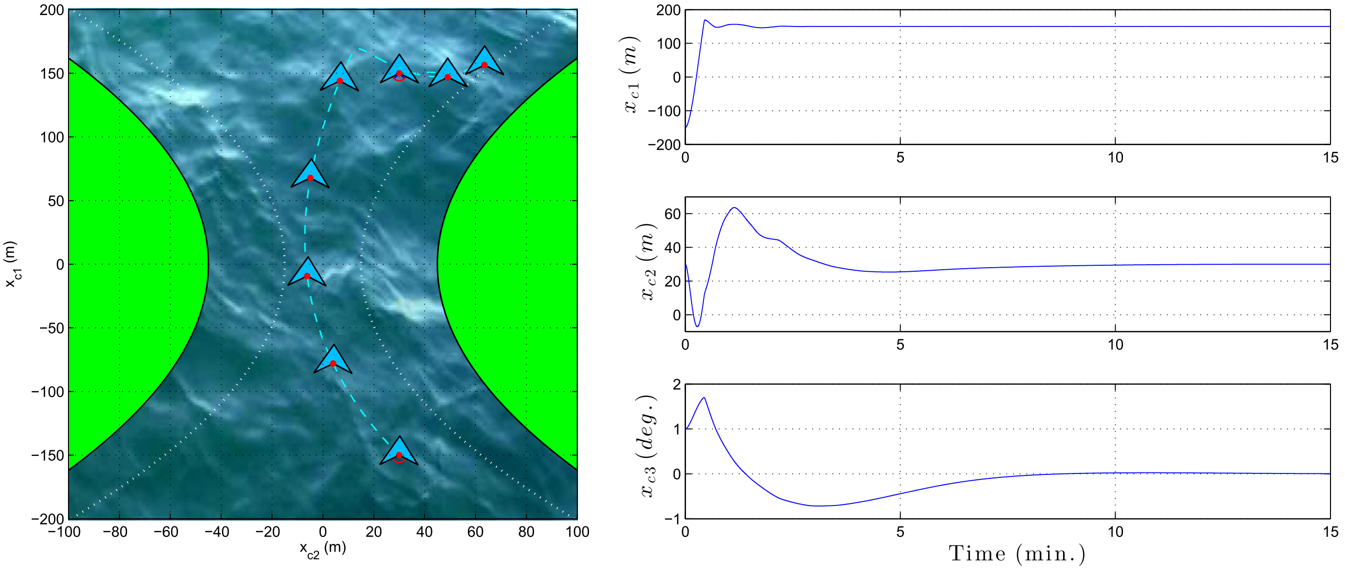

FIGURE 3 Position and orientation of the ship: (left) along the path where the dotted line is the safe constraint programmed, the green patch is the earth and the dashed line is the trajectory described by the center of gravity; (right) time histories [Colour figure can be viewed at wileyonlinelibrary.com]

It is straightforward to check that the positivity requirement is satisfied noting that

$$\partial_{x_c}\phi = \begin{bmatrix} \alpha_{11} & 0 & 0 \\ \alpha_{21} & \alpha_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

with

$$\begin{aligned} \alpha_{11} &= k_1 \frac{{x_{c1}}^2 - x_{c1}^{\min} x_{c1}^{\max}}{\left(x_{c1} - x_{c1}^{\min}\right)^2 \left(x_{c1} - x_{c1}^{\max}\right)^2}, \\ \alpha_{21} &= 2k_2 x_{c1} x_{c2} \frac{2a_m a_M {x_{c1}}^2 + a_m b_M - a_m x_{c2} + a_M b_m - a_M x_{c2}}{\left(x_{c2}^{\min}(x_{c1}) - x_{c2}\right)\left(x_{c2}^{\max}(x_{c1}) - x_{c2}\right)}, \\ \alpha_{22} &= k_2 \frac{{x_{c2}}^2 - a_m {x_{c1}}^4 a_M - a_m {x_{c1}}^2 b_M - b_m a_M {x_{c1}}^2 - b_m b_M}{\left(x_{c2}^{\min}(x_{c1}) - x_{c2}\right)\left(x_{c2}^{\max}(x_{c1}) - x_{c2}\right)}, \end{aligned}$$

and where k_1, k_2 are positive control gains, $\partial_{x_c}\phi$ is positive definite and consequently $\partial\bar{\Psi}$ in $\mathcal{X}_c$. In Figure 3, we show a representative simulation result. To explore the performance and capabilities of the controller, all the simulations have been made without wind, current, and wave disturbances, but we keep all the nonlinearities of the model so that we can test the robustness to parametric uncertainties. The simulation was made starting from the initial condition $x_c(0) = (-150, 30, 1°)$ with a destination at $x_c^d = (150, 30, 0°)$, where the arrow points to the course in an obvious way. Notice that the initial condition is forced to be in the direction of the prohibited area and the controller confines the trajectory fulfilling the objective. However, due to the high nonlinearity, the tuning of the controller gains is not straightforward. For readers not familiar with ship thrusters' configurations, it is also worth to mention that this ship is overactuated, ie, it can be driven in any direction with its thrusters. In this way, since there is not restriction on the heading angle, the controller forces the confinement of the ship inside the constraints and keeps tight to the heading reference. This can be seen in the Figure 3 right, where the ship goes at cruise speed of 19 knots[§] from $x_{c1} = -150$ to $x_{c1} = 150$ and then slowly to the final destination. Current research is under way to analyse the relationship between the controller gains and the performance.

Remark 8. After analysing a massive amount of data from simulations, we realised that the mathematical model (8)-(9) considered for the design is not precise enough in the sense that, by physical considerations, the ship dynamics are obviously bounded. The latter means that we were able to compute the necessary bounds to identify the minimum r that qualifies the function (12) as a BCLF as well in the case of bounded inputs with formula (6), although r depends

[§] Maximum speed of 21-29 knots.

on the actual ship configuration and disturbances considered. A complete simulation analysis including all those real data and disturbances is pending for the approval and certification of the company.

Remark 9. Recall that the realistic simulator succinctly described in the work of Dòria-Cerezo et al[18] includes environmental disturbances and uncertainties giving insight of the robustness of the controller. However, we had to reshape the norms in the formulae (5) and (6) to achieve good performances. To the best of authors' knowledge, it is the first implementation of Sontag's formulae in a real problem at this level.

6 | CONCLUSIONS

A methodology to find explicit solutions for the stabilisation of constrained nonlinear systems has been provided. The approach relies on the construction of a diffeomorphism so that standard Lyapunov-based techniques can be applied. Moreover, the proposed methodology makes use of Sontag's universal formulae as a constructive alternative to the controller design, and therefore being able to deal with bounded input constraints. To wrap up this paper, we provide the solution of the dynamic positioning of a ship along a strait, making use of Sontag's formula and reporting representative realistic simulations on the simulator owned by a renowned shipbuilding company.

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