A note on stability of arbitrarily switched homogeneous systems

David Angeli [a]

[a] *Dip. Sistemi e Informatica,*
Università di Firenze,
Via di Santa Marta 3,
50139 Firenze, Italy

Abstract

A family of switched systems is exponentially stable if there exists positive constants M and λ such that the solution at time t satisfies an estimate of the following kind $|x(t)| \leq Me^{-\lambda t}|x(0)|$, for all possible switching sequences. Clearly exponential stability implies attractivity of the origin; we show that for homogeneous systems (and as a special case for linear systems) the converse implication is also true.

Key words: stability properties, Lyapunov methods, switched systems, robust stability

1 Introduction and basic definitions

The study of switched systems is a fastly growing area of research in control theory. Informally a switched system is a dynamical system which is able to commute between different behaviours according to some external input variable, which we will in the following refer to as switching signal. The practical relevance of this wide class of systems has been often emphasized, see for instance [4] for a recent and very interesting survey on the subject. On the other hand, many challenging theoretical questions which arise in this area are still waiting for an answer.
From a mathematical point of view a family of switched systems is a nonlinear system of the following form

$$\dot{x} = f(x, \sigma) \tag{1}$$

[1] E-mail: angeli@dsi.unifi.it .

with state x evolving in $\mathbb{R}^n$. The exogenous input σ (the *switching signal*), plays the role of a time-varying uncertain parameter of the system. In order to guarantee existence of solutions we assume $\sigma(\cdot) : \mathbb{R}_{\geq 0} \to \Sigma$ to be a measurable function taking values in some compact set Σ. For simplicity in the following we will think of Σ as a finite set although the main results carries over to more general compact sets, provided that some continuity assumption is made on f as a function of σ. The homogeneity assumption refers to the dependence of f on x; in particular f satisfies

$$\forall \lambda > 0, \forall x \in \mathbb{R}^n, \forall \sigma \in \Sigma \qquad f(\lambda x, \sigma) = \lambda f(x, \sigma). \tag{2}$$

We also assume that f satisfies some uniform Lipschitzianity assumption

$$\|f(x_1, \sigma) - f(x_2, \sigma)\| \leq M \|x_1 - x_2\| \qquad \forall x_1, x_2 \in \mathbb{R}^n, \ \forall \sigma \in \Sigma \tag{3}$$

We remark that, because of compactness of Σ and homogeneity of f this is not stronger than the usual local Lipschitzianity condition used in order to guarantee existence and unicity of classical solutions for (1).
One major instance of the previous class of systems is clearly represented by linear switched systems

$$\dot{x} = \Phi_\sigma x, \qquad \sigma \in \Sigma. \tag{4}$$

In the area of robust control, the stability of a family of linear systems is usually studied employing common quadratic Lyapunov functions whose expression can be determined solving an LMI. It is well known that the existence of a common Lyapunov function is a necessary and sufficient condition for stability of 1 under arbitrary switchings, [5], however, quadratic Lyapunov functions are not universal, not even for linear systems, meaning that there might be stable families of systems for which no common quadratic Lyapunov function exists, [3]. Nevertheless, it was shown in [6] that a Lyapunov function of the following kind always exists

$$V(x) = \max_i (v_i' x)^2 \tag{5}$$

where $v_i \in \mathbb{R}^n$ are constant vectors, but the question of how to build such Lyapunov functions in general is still open.

In this paper we investigate the stability properties of homogeneous, switched systems; in particular we will show equivalence between all reasonable definitions of asymptotic stability. In the following $x(t, \xi, \sigma)$ will denote the response at time t, to the input signal σ and initial condition ξ. It is straightforward to see from (2) that (1) is forward complete and hence solutions are unique and maximally defined over $[0,+\infty)$.

Definition 1.1 We say that system (1) is *exponentially stable* if there exists positive constants M and λ such that

$$|x(t,\xi,\sigma)| \leq Me^{-\lambda t}\,|\xi| \qquad \forall t \geq 0,\ \forall \xi \in \mathbb{R}^n, \forall \sigma \in \mathcal{M}_\Sigma. \tag{6}$$

□

Definition 1.2 We say that system (1) is *uniformly globally asymptotically stable* if there exists a $\mathcal{KL}$ function β such that the following estimate holds

$$|x(t,\xi,\sigma)| \leq \beta(|\xi|,t) \qquad \forall t \geq 0,\ \forall \xi \in \mathbb{R}^n, \forall \sigma \in \mathcal{M}_\Sigma. \tag{7}$$

□

Both stability notions are uniform with respect to σ, since the switching signal does not affect the speed of convergence of the system to 0. It is also of interest the following, apparently weaker, notion of attractivity.

Definition 1.3 We say that system (1) is *attractive* if

$$\forall \xi \in \mathbb{R}^n,\ \forall \sigma \in \mathcal{M}_\Sigma, \quad \lim_{t\to+\infty} |x(t,\xi,\sigma)| = 0 \tag{8}$$

□

2 Main result

With the definitions given in the previous section we are ready to state the main result of the paper, which was recently conjectured in [3] .

Theorem 1 *Consider the family of switched systems in equation (1); let (2) be satisfied, then the following facts are equivalent*

(1) system (1) is exponentially stable
(2) system (1) is uniformly globally asymptotically stable
(3) system (1) is attractive.

□

Remark 2.1 We remark that by virtue of (2), local stability properties are equivalent to global ones. In particular then, by the previous theorem attractivity in a neighborhood of the origin is equivalent to global exponential stability. □

The proof, which we discuss in the following section, heavily relies on a powerful technical lemma which was proved by Eduardo Sontag and Yuan Wang in a recent paper concerning ISS, [7].

3 Proof of the main result

Some of the implications, in particular $1 \Rightarrow 2 \Rightarrow 3$, are straightforward from the definition. The homogeneity assumption clearly comes in when proving the converse implications. Let us start from the easier one, $2 \Rightarrow 1$, which was already stated without proof in [6]. Let $\xi \in \mathbb{R}^n$ be such that $|\xi| = 1$ and β be as in (7). By definition of class $\mathcal{KL}$ function there exists $\bar{T}$ such that $\beta(1, \bar{T}) \leq 1/2$. Let $M = 2\beta(1, 0)$ and $\lambda = \log(2)/\bar{T}$. We claim that (6) holds with M and λ defined above. In order to see this, recall that for homogeneous systems $x(t, \lambda\xi, \sigma) = \lambda x(t, \xi, \sigma)$, for all $\lambda > 0$. Since all estimates hold independently of the particular switching signal and in order to keep the notation simple we drop the dependence of x on σ. Hence, for arbitrary $k \in \mathbb{N}$ we have,

$$\begin{aligned} |x(k\bar{T}, \xi)| &= |x(\bar{T}, x((k-1)\bar{T}, \xi))| \\ &\leq |x((k-1)\bar{T}, \xi)|\beta(1, \bar{T}) \leq |x((k-1)\bar{T}, \xi)|/2. \end{aligned} \tag{9}$$

By induction, $|x(k\bar{T}, \xi)| \leq |\xi|/2^k = e^{-\lambda k\bar{T}}|\xi|$. Let t belong to $[(k-1)\bar{T}, k\bar{T})$ for some $k \in \mathbb{N}$, then we have

$$\begin{aligned} |x(t, \xi)| &= |x(t - (k-1)\bar{T}, x((k-1)\bar{T}, \xi))| \\ &\leq |x((k-1)\bar{T}, \xi)|\beta(1, 0) \leq e^{-\lambda(k-1)\bar{T}}\beta(1, 0)|\xi| \leq 2\beta(1, 0)e^{-\lambda t}|\xi|. \end{aligned} \tag{10}$$

We now turn to the most interesting implication, $3 \Rightarrow 2$. We define the set of reachable states in time T, starting from initial conditions is some compact $K \subset \mathbb{R}^n$ as

$$\mathcal{R}^T(K) = \Big\{x \in \mathbb{R}^n : \exists \xi \in K, \exists \sigma \in \mathcal{M}_\Sigma, \exists \bar{t} \in [0, T] : x = x(\bar{t}, \xi, \sigma)\Big\}. \tag{11}$$

Further, we let $\mathcal{R}(K)$ be the set of states reachable from K for arbitrary time

$$\mathcal{R}(K) = \bigcup_{T \geq 0} \mathcal{R}^T(K). \tag{12}$$

It is again a consequence of (2) that $\mathcal{R}^T(\lambda K) = \lambda \mathcal{R}^T(K)$ for any positive λ. For a given set $\mathcal{S}$ and input u one may consider the "first crossing time",

$$\tau(\xi, \mathcal{S}, u) = \inf \Big\{ t > 0 : \; x(t, \xi, u) \in \mathcal{S} \Big\}. \tag{13}$$

Let $\varepsilon > 0$ be arbitrary; we define the set $\mathcal{C}_\varepsilon$ as

$$\mathcal{C}_\varepsilon = \Big\{ x \in \mathbb{R}^n : \varepsilon \leq |x| \leq 2\varepsilon \Big\}. \tag{14}$$

and $\mathcal{B}_\varepsilon$ the closed ball of radius ε. Clearly, if $\mathcal{R}(\mathcal{C}_\varepsilon)$ is bounded, $\mathcal{R}(\mathcal{B}_\varepsilon)$ is also bounded; in particular we have $\|\mathcal{R}(\mathcal{B}_\varepsilon)\| \leq \|\mathcal{R}(\mathcal{C}_\varepsilon)\|$. We will show that, for attractive systems, $\mathcal{R}(\mathcal{C}_\varepsilon)$ is bounded. In particular, we have

$$\forall \xi \in \mathcal{C}_\varepsilon, \; \forall \sigma \in \mathcal{M}_\Sigma \quad \exists t \geq 0 : \; x(t, \xi, \sigma) \in \mathcal{B}_\varepsilon \tag{15}$$

By Corollary III.3 of [7], we have that

$$T_\varepsilon \doteq \sup_{\xi \in \mathcal{C}_\varepsilon, \sigma \in \mathcal{M}_\Sigma} \tau(\xi, \mathcal{B}_\varepsilon, \sigma) < +\infty. \tag{16}$$

Since, for a forward complete family of systems, the set of reachable states in bounded time from bounded initial conditions is bounded,[1], we have that

$$\|\mathcal{R}(\mathcal{B}_\varepsilon)\| \leq \|\mathcal{R}(\mathcal{C}_\varepsilon)\| \leq \left\| \mathcal{R}^{T_\varepsilon}(\mathcal{C}_\varepsilon) \right\| \doteq \delta_\varepsilon < +\infty \tag{17}$$

Notice that, without using assumption (2) we proved that attractivity implies uniform Lagrange stability, viz.

$$\forall \varepsilon > 0, \; \exists \delta_\varepsilon : |\xi| \leq \varepsilon \Rightarrow |x(t, \xi, \sigma)| \leq \delta_\varepsilon, \qquad \forall t \geq 0, \; \forall \sigma(\cdot). \tag{18}$$

It turns out that for homogeneous systems, this is equivalent to uniform Lyapunov stability. In fact,

$$\forall \varepsilon > 0, \; |\xi| \leq \varepsilon \Rightarrow |x(t, \xi, \sigma)| = |\xi| |x(t, \xi/|\xi|, \sigma)| \leq \varepsilon \delta_1 \qquad \forall t \geq 0, \; \forall \sigma(\cdot) \tag{19}$$

and hence,

$$\forall \varepsilon > 0, \exists \tilde{\delta}_\varepsilon = \varepsilon / \delta_1 : |\xi| \leq \tilde{\delta}_\varepsilon \Rightarrow |x(t, \xi, \sigma)| \leq \varepsilon, \qquad \forall t \geq 0, \; \forall \sigma(\cdot). \tag{20}$$

Thus the main result follows from Theorem 2 in [7], where it is shown that uniform Lyapunov stability plus attractivity are equivalent to uniform global asymptotic stability.

4 An example of transition from stability to instability

In order to see how the result in the previous sections is not obvious even for very simple switched systems, consider the following parameterized family of linear switched systems:

$$\dot{x} = \Phi_\sigma(\theta)x, \qquad \sigma \in \{1,2\} \tag{21}$$

with

$$\Phi_1(\theta) = \begin{bmatrix} -1 & \theta \\ 0 & -1 \end{bmatrix} \qquad \Phi_2(\theta) = \begin{bmatrix} -1 & 0 \\ \theta & -1 \end{bmatrix}. \tag{22}$$

with θ, a parameter varying in $[0,2]$. Both systems are asymptotically stable, moreover, for all θ in $[0,2)$ we have:

$$\Phi_1'(\theta) + \Phi_1(\theta) = \Phi_2(\theta)' + \Phi_2(\theta) < 0. \tag{23}$$

Hence, $\Phi_1(\theta)$ and $\Phi_2(\theta)$ admit the identity as a common Lyapunov function and the resulting switched system is quadratically stable. For $\theta = 2$, however, it is not difficult to see that the system (21) fails to be exponentially stable; a necessary condition for exponential stability is in fact that all convex combination of the Φ_is be such. In this case instead

$$\frac{1}{2}\Phi_1 + \frac{1}{2}\Phi_2 = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}, \tag{24}$$

which has a zero eigenvalue, corresponding to the eigenvector $v_0 = [1,1]'$. It is not obvious instead, without making use of Theorem 1, how to show that the system is not attractive. As a matter of fact, taking as a Lyapunov function $V(x) = x'x$ one easily obtains

$$\dot{V}(x) = 2x'\Phi_i x = -2([1,-1]x)^2 \leq 0 \qquad i = 1,2. \tag{25}$$

Thus $x(t)'x(t) \leq x(0)'x(0)$ and $[1,-1]x \in L_2$. Besides, deriving the quantity $([1,-1]x)^2$ along trajectories of (21) yields

$$\frac{d}{dt}([1,-1]x)^2 = \pm 2([1,-1]x)([1,1]x) - 4([1,-1]x)^2 \tag{26}$$

where + or − is obtained according to which value σ assumes. Since $[1,1]x$ is uniformly bounded, and $[1,-1]x \in L_2$ we have that $[1,-1]x \to 0$.

One might even be brought to think, by the above considerations, that (21) be attractive; the only source of instability comes from the fact that solutions of the linear system $\dot{x} = (\Phi_1 + \Phi_2)x/2$ can be approximated, arbitrarily close, switching between Φ_1 and Φ_2, for equally long time intervals. Thus one is tempted to conjecture that exponential stability is violated because there is not a uniform estimate of the rate of convergence to zero, even though non-converging trajectories would only show up in the limit, viz. for an infinite frequency switching between Φ_1 and Φ_2, (for instance taking into account Filippov solutions). As a matter of fact, the relaxation theorem only ensures approximation of Filippov solution by classical ones on compact time intervals, which would not exclude in principle the possibility of having attractivity, (see [2]). Theorem (1) clearly indicates that this is not the case and that non convergent trajectories of (21) exist also taking into account only classical solutions.

5 Conclusions

Equivalence between attractivity and global exponential stability for homogeneous switched systems is shown. This makes, for the considered class of systems, all definitions of asymptotic stability equivalent except for the notion of *quadratic stability* which is already known to be strictly stronger than standard asymptotic stability. The result turned out to be a direct consequence of a general result proven in [7]. The transition from stability to instability of a simple switched linear system is also illustrated with an example.

References

[1] D. Angeli and E. D. Sontag, Forward completeness, unboundedness observability, and their Lyapunov characterization, Systems and Control letters, to appear

[2] J. P. Aubin and A. Cellina, Differential inclusions: set-valued maps and viability theory, Springer, Berlin, 1984

[3] W. P. Dayawansa and C. F. Martin, A converse Lyapunov theorem for a class of dynamical systems which undergo switching, IEEE Transactions on Automatic Control, 44, 1999, pp. 751-760

[4] D. Liberzon and A. S. Morse, Basic problems in stability and design of switched systems, submitted to IEEE Control Systems Magazine

[5] Y. Lin, E. D. Sontag and Y. Wang, A smooth converse Lyapunov theorem for robust stability", SIAM Journal on Control and Optimization, 34, 1996, pp. 124-160

[6] A. P. Molchanov and Y. S. Pyatnitskiy, Criteria fr absolute stability of differential and difference inclusions encountered in control theory, Systems and Control Letters, 13, 1989, pp. 59-64

[7] E. D. Sontag and Y. Wang, New characterizations of Input to State Stability, IEEE Transactions on Automatic Control, 41, 1996, pp. 1283-1294