Nonlinear Scaling of (i)ISS-Lyapunov Functions

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***Abstract*—While nonlinear scalings of Lyapunov functions are also Lyapunov functions, we provide examples that the same statement does not necessarily hold for Input-to-State Stable (ISS) Lyapunov functions or for integral ISS (iISS) Lyapunov functions. We provide sufficient conditions under which a nonlinear scaling of an ISS or iISS Lyapunov function is also an ISS or iISS Lyapunov function. We also introduce a generalization of the iISS Lyapunov function, which we term a dissipative iISS-Lyapunov function, that allows for a larger class of nonlinear scalings so that nonlinear scalings of dissipative iISS-Lyapunov functions are again dissipative iISS-Lyapunov functions.**

***Index Terms*—Input-to-State Stability (ISS), Lyapunov methods, nonlinear scalings.**

I. Introduction

The concept of input-to-state stability (ISS) introduced in [9] has proved to be a valuable stability concept in the study of dynamical systems. In [13], it was shown that ISS of a system is equivalent to the existence of an appropriate generalization of a Lyapunov function, termed an ISS-Lyapunov function. As with the relationship between (asymptotic) stability and Lyapunov functions for systems without external inputs, ISS-Lyapunov functions are arguably the most useful tool for establishing the ISS property for a given system. Similarly, for the concept of integral ISS (iISS) introduced in [10], a Lyapunov characterization exists, termed an iISS-Lyapunov function, that is equally useful for the consideration of topics in iISS [1].

Applying a nonlinear scaling to a (ISS, iISS) Lyapunov function is a commonly used proof technique (see, e.g., [3], [8] and the references therein) whereby a (ISS, iISS) Lyapunov function is constructed and then scaled to achieve some additional property, such as an exponential decrease (cf. [8]). While it is known that any nonlinear scaling of a Lyapunov function yields another Lyapunov function, as the examples in the sequel demonstrate, this is not the case for (dissipation-form) ISS or iISS Lyapunov functions. In Section IV, we investigate an alternate form of an iISS-Lyapunov function that admits a wider class of scalings than the standard iISS-Lyapunov function. We also show that this alternate form has the nice property that quadratic functions, obtained by solving the standard Lyapunov equation for linear systems, can be used to obtain iISS-Lyapunov functions for a significant class of nonlinear systems that includes bilinear systems.

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That ISS-Lyapunov functions do not admit arbitrary nonlinear scalings is closely related to the main result of [12]. There, Sontag and Teel characterized all possible supply pairs for a dissipation-form ISS-Lyapunov function and showed that arbitrary modifications of a known supply pair are not possible. While a nonlinear scaling of a dissipation-form ISS-Lyapunov function induces a change in the original supply pair, there does not appear to be a direct relationship between the allowable modifications to supply pairs as described in [12] and the allowable nonlinear scalings of ISS-Lyapunov functions.

In what follows, we consider systems described by

$$\dot{x} = f(x, w), \quad x \in \mathbb{R}^n, w \in \mathbb{R}^m \tag{1}$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is locally Lipschitz, satisfies $f(0, 0) = 0$ and inputs are locally essentially bounded functions. Consequently, (1) gives rise to a unique solution for each initial condition and each input. We make use of the standard function classes $\mathcal{K}$ (those functions that are defined on $\mathbb{R}_{\geq 0}$, zero at zero, continuous, and strictly increasing) and $\mathcal{K}_\infty$ (functions of class-$\mathcal{K}$ that are also unbounded). See [2] or [4]. We denote the set of positive definite functions defined on $\mathbb{R}_{\geq 0}$ by $\mathcal{P}$ and any norm by $|\cdot|$.

II. ISS-Lyapunov Functions

ISS-Lyapunov functions were defined in [13] in two forms: a dissipation-form and an implication-form.

Definition 1: A dissipation-form ISS-Lyapunov function for (1) is a continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ so that there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and $\alpha, \sigma \in \mathcal{K}$ satisfying $\sup \alpha \geq \sup \sigma$ and

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|) \tag{2}$$

$$\langle \nabla V(x), f(x, w) \rangle \leq -\alpha(|x|) + \sigma(|w|) \tag{3}$$

for all $x \in \mathbb{R}^n$ and $w \in \mathbb{R}^m$.

Definition 2: An implication-form ISS-Lyapunov function is a continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ so that there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, $\chi \in \mathcal{K}$, and $\rho \in \mathcal{P}$ satisfying (2) and

$$|x| \geq \chi(|w|) \Rightarrow \langle \nabla V(x), f(x, w) \rangle \leq -\rho(|x|) \tag{4}$$

for all $x \in \mathbb{R}^n$ and $w \in \mathbb{R}^m$.

In [13], the functions α and σ of (3) were assumed to be of class-$\mathcal{K}_\infty$, in which case the condition $\sup \alpha = \sup \sigma = \infty$ is satisfied. Additionally, the function ρ of (4) was assumed to be of class-$\mathcal{K}_\infty$. Under these assumptions, a dissipation-form ISS-Lyapunov function is an implication-form ISS-Lyapunov function and vice versa (see [13, Remark 2.4]). However, while any dissipation-form ISS-Lyapunov function is an implication-form ISS-Lyapunov function, the converse does not hold in the absence of these assumptions. Nonetheless, both the dissipation-form ISS-Lyapunov function and the implication-form ISS-Lyapunov function as defined above can be shown to be equivalent to ISS, and therefore there exists a dissipation-form ISS-Lyapunov function if and only if there exists an implication-form ISS-Lyapunov function.

Definition 3: A nonlinear scaling is a function $\mu \in \mathcal{K}_\infty$, continuously differentiable on $\mathbb{R}_{>0}$, satisfying $\mu'(s) > 0$ for all $s \in \mathbb{R}_{>0}$ and $\lim_{s\to 0^+} \mu'(s) =: \mu'(0) \in \mathbb{R}_{\geq 0}$.

We observe that the nonlinear scaling, μ, defined above is a diffeomorphism on the strictly positive real numbers. We also note that, at the expense of significantly more complicated notation and statements, the above definition can be relaxed to admit locally Lipschitz scalings rather than continuously differentiable scalings. In the interest of a clear presentation of the results, we proceed with continuously differentiable scalings. Finally, the assumption on the derivative of the scaling at 0 can be removed if we do not insist on differentiability of V at the origin.

For systems without inputs, i.e.,

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n \tag{5}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is such that (5) gives rise to a unique solution for every initial condition, it is anecdotally known, and used in many proofs, that any nonlinear scaling of a Lyapunov function yields a Lyapunov function. However, we have not seen this result formally stated in the literature and so we do so here. The proof is straightforward and is omitted.

Proposition 4: Assume a Lyapunov function for (5); i.e., $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ continuously differentiable and functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and $\rho \in \mathcal{P}$ so that, for all $x \in \mathbb{R}^n$, (2) holds and

$$\langle \nabla V(x), f(x)\rangle \leq -\rho\left(|x|\right). \tag{6}$$

Then for any nonlinear scaling $\mu \in \mathcal{K}_\infty$, the function $W(x) := \mu(V(x))$ for all $x \in \mathbb{R}^n$ is a Lyapunov function for (5).

A similar result holds for the nonlinear scaling of implication-form ISS-Lyapunov functions.

Proposition 5: Suppose $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is an implication-form ISS-Lyapunov function satisfying (2) and (4) and let $\mu \in \mathcal{K}_\infty$ be a nonlinear scaling. Then the nonlinearly scaled function $W : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ defined by $W(x) := \mu(V(x))$ for all $x \in \mathbb{R}^n$ is an implication-form ISS-Lyapunov function satisfying

$$\mu \circ \alpha_1\left(|x|\right) \leq W(x) \leq \mu \circ \alpha_2\left(|x|\right) \tag{7}$$

$$|x| \geq \chi\left(|w|\right) \;\Rightarrow\; \langle \nabla W(x), f(x,w)\rangle \leq -\hat{\rho}\left(|x|\right) \tag{8}$$

where $\hat{\rho} \in \mathcal{P}$ is given by $\hat{\rho}(r) := \min_{|x|=r} \mu'(V(x))\rho(r)$.

Proof: The bounds (7) hold as the composition of two class-$\mathcal{K}_\infty$ functions yields a class-$\mathcal{K}_\infty$ function. The decrease condition (8) follows from a simple application of the chain rule. ■

In addition to the fact that any nonlinear scaling of an implication-form ISS-Lyapunov function yields another ISS-Lyapunov function, for any given implication-form ISS-Lyapunov function there exists a particular nonlinear scaling so that the scaled ISS-Lyapunov function decreases exponentially for states that are sufficiently large with respect to the input (see [8, Lemma 11]).

III. Scaling Dissipation-Form ISS-Lyapunov Functions

We now turn to nonlinearly scaled dissipation-form ISS-Lyapunov functions and first present a negative example.

Example 6: Consider the system

$$\dot{x} = -x + w \tag{9}$$

and the candidate ISS-Lyapunov function $V(x) = (1/2)x^2$ for all $x \in \mathbb{R}$. For all $s \in \mathbb{R}_{\geq 0}$, define $\alpha_1(s) = \alpha_2(s) = \alpha(s) = \rho(s) = \sigma(s) := (1/2)s^2$, and $\chi(s) := 2s$. Then, using a different completion of squares in each case, it is straightforward to see that V satisfies (2)–(4). In other words, V is both a dissipative-form and an implication-form ISS-Lyapunov function for (9).

Let the nonlinear scaling $\mu \in \mathcal{K}_\infty$ be given by $\mu(s) := \log(1+s)$ for all $s \in \mathbb{R}_{\geq 0}$ and consider the nonlinearly scaled function $W(x) := \mu(V(x))$ for all $x \in \mathbb{R}^n$. Proposition 5 states that W is an implication-form ISS-Lyapunov function for (9). This can be seen by

$$\langle \nabla W(x), f(x,w)\rangle \leq -\frac{V(x)}{1+V(x)} + \frac{1}{1+V(x)}\left(-\frac{1}{4}x^2 + w^2\right)$$

and so W satisfies (4) with $\chi(s) := 4s$ and

$$\rho(s) := \frac{s^2}{2\left(1+\frac{1}{2}s^2\right)}.$$

Note that the result of the nonlinear scaling is that the decrease rate in (4) changes from a function that is class-$\mathcal{K}_\infty$ to one that is class-$\mathcal{K}$ but not class-$\mathcal{K}_\infty$.

To see that W is not a dissipative-form ISS-Lyapunov function for (9), we argue by contradiction. Suppose there exist $\alpha, \sigma \in \mathcal{K}$ satisfying $\sup \alpha \geq \sup \sigma$ and such that

$$\langle \nabla W(x), f(x,w)\rangle = \frac{1}{1+\frac{1}{2}x^2}\left(-x^2 + xw\right) \leq -\alpha\left(|x|\right) + \sigma(|w|).$$

Fix $x = 1$ and observe that this implies

$$-\frac{2}{3} + \frac{2}{3}w \leq -\alpha(1) + \sigma\left(|w|\right).$$

This implies that $\sigma \in \mathcal{K}_\infty$ and, since $\sup \alpha \geq \sup \sigma$, consequently $\alpha \in \mathcal{K}_\infty$.

Now fix $w = 0$ and observe that

$$\frac{x^2}{1+\frac{1}{2}x^2} \leq 2$$

so that $\sup \alpha = 2$, contradicting the fact that $\alpha \in \mathcal{K}_\infty$.

In other words, consistent with Proposition 5, the nonlinear scaling μ of the implication-form ISS-Lyapunov function V yields another implication-form ISS-Lyapunov function. However, the nonlinear scaling does *not* yield another dissipative-form ISS-Lyapunov function. ■

With the above negative result for nonlinear scalings of dissipation-form ISS-Lyapunov functions, we present two sufficient conditions for a nonlinear scaling of a dissipation-form ISS-Lyapunov function to also be a dissipation-form ISS-Lyapunov function.

The Legendre-Fenchel transform of a continuously differentiable function $\kappa \in \mathcal{K}_\infty$ with $\kappa' \in \mathcal{K}_\infty$ is defined as

$$\ell\kappa(s) := \int_0^s \left(\kappa'\right)^{-1}(\tau)d\tau, \quad \forall s \in \mathbb{R}_{\geq 0}$$

where $(\kappa')^{-1}$ is the inverse function of the derivative of κ. We note that $\ell\kappa \in \mathcal{K}_\infty$ (see [4, Lemma 15], [5, Lemma A.1]). The Legendre-Fenchel transform can be used to state a general version of Young's inequality (see [4, Lemma 16], [7]); i.e., for any continuously differentiable $\kappa \in \mathcal{K}_\infty$ with $\kappa' \in \mathcal{K}_\infty$ and $a, b \in \mathbb{R}_{\geq 0}$,

$$ab \leq \kappa(a) + \ell\kappa(b). \tag{10}$$

Proposition 7: Assume a dissipation-form ISS-Lyapunov function V satisfying (2)–(3) with $\alpha, \sigma \in \mathcal{K}$ and $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$. Let $\mu \in \mathcal{K}_\infty$ be a nonlinear scaling. The function $W : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ defined by

$W(x) := \mu(V(x))$ for all $x \in \mathbb{R}^n$ is a dissipation-form ISS-Lyapunov function if any of the following conditions hold:

(i) there exist $a, b \in \mathbb{R}_{>0}$ so that $\mu'(s) \in [a, b]$ for all $s \in \mathbb{R}_{>0}$ and

$$\sup_{s\geq 0} \hat{\alpha}(s) \geq \sup_{s\geq 0} \hat{\sigma}(s) \tag{11}$$

where $\hat{\alpha}(s) := a\alpha(s)$ and $\hat{\sigma}(s) := b\sigma(s)$ for all $s \in \mathbb{R}_{\geq 0}$; or

(ii) there exist $c \in \mathbb{R}_{\geq 0}$, $\alpha_\mu \in \mathcal{K}$, and $\kappa \in \mathcal{K}_\infty$ with $\kappa' \in \mathcal{K}_\infty$ so that $\mu'(s) = \alpha_\mu(s) + c$ for all $s \in \mathbb{R}_{\geq 0}$ and
 (a) $\hat{\alpha}(s) := (\alpha_\mu \circ \alpha_1(s) + c)\alpha(s) - \kappa \circ \alpha_\mu \circ \alpha_2(s) \in \mathcal{K}$; and
 (b) $\sup \hat{\alpha} \geq \sup \hat{\sigma}$ where $\hat{\sigma}(s) := \ell\kappa(\sigma(s)) + c\sigma(s)$ for all $s \in \mathbb{R}_{\geq 0}$; or

(iii) there exist $c \in \mathbb{R}_{\geq 0}$, $\alpha_\mu \in \mathcal{K}$ so that $\mu'(s) = \alpha_\mu(s) + c$ for all $s \in \mathbb{R}_{\geq 0}$ and $\alpha \in \mathcal{K}_\infty$; or

(iv) the function $s \mapsto (\mu' \circ \mu^{-1}(s))(\alpha \circ \alpha_2^{-1} \circ \mu^{-1}(s))$, $s \in \mathbb{R}_{\geq 0}$, is of class-$\mathcal{K}_\infty$ and $\alpha \in \mathcal{K}_\infty$.

Proof: In all cases, the upper and lower bounds for W are given by $\mu \circ \alpha_2, \mu \circ \alpha_1 \in \mathcal{K}_\infty$, respectively. For Condition (i), the chain rule, the decrease condition (3), and the bounds on μ' yield

$$\begin{aligned}\langle \nabla W(x), f(x,w)\rangle &= \mu'(V(x))\langle \nabla V(x), f(x,w)\rangle \\ &\leq -\mu'(V(x))\,\alpha(|x|) + \mu'(V(x))\,\sigma(|w|) \\ &\leq -a\alpha(|x|) + b\sigma(|w|) \leq -\hat{\alpha}(|x|) + \hat{\sigma}(|w|).\end{aligned} \tag{12}$$

For Condition (ii), appealing to, in order, the chain rule, the decrease condition (3), the bounds (2), inequality (10), and the definitions of $\hat{\alpha}, \hat{\sigma} \in \mathcal{K}$ allows us to derive the bound

$$\begin{aligned}\langle \nabla W(x), f(x,w)\rangle \leq &-\alpha_\mu(V(x))\,\alpha(|x|) + \alpha_\mu(V(x))\,\sigma(|w|) \\ &- c\alpha(|x|) + c\sigma(|w|) \\ \leq &-(\alpha_\mu \circ \alpha_1(|x|))\,\alpha(|x|) \\ &+ \kappa \circ \alpha_\mu \circ \alpha_2(|x|) \\ &- c\alpha(|x|) + \ell\kappa(\sigma(|w|)) + c\sigma(|w|) \\ = &-\hat{\alpha}(|x|) + \hat{\sigma}(|w|).\end{aligned}$$

Therefore, for both Part (i) and Part (ii), W is a dissipation-form ISS-Lyapunov function.

For Conditions (iii) and (iv), recall that an implication-form ISS-Lyapunov function is a dissipation-form ISS-Lyapunov function, and vice versa, when the decrease rates satisfy $\rho, \alpha \in \mathcal{K}_\infty$. Observe that

$$\begin{aligned}&\langle \nabla W(x), f(x,w)\rangle \leq \mu'(V(x))\,(-\alpha(|x|) + \sigma(|w|)) \\ &= -\frac{1}{2}\mu'(V(x))\,\alpha(|x|) + \mu'(V(x))\Big(-\frac{1}{2}\alpha(|x|) + \sigma(|w|)\Big)\end{aligned} \tag{13}$$

so that we have the implication-form ISS-Lyapunov function

$$\begin{aligned}&|x| \geq \alpha^{-1}(2\sigma(|w|)) \implies \\ &\quad\langle \nabla W(x), f(x,w)\rangle \leq -\frac{1}{2}\mu'(V(x))\,\alpha(|x|) =: \hat{\rho}(|x|).\end{aligned} \tag{14}$$

The conditions of (iii) or (iv) then imply that $\hat{\rho} \in \mathcal{K}_\infty$ and [13, Remark 2.4] yields that W is also a dissipation-form ISS-Lyapunov function. ■

With the constants $a, b \in \mathbb{R}_{>0}$ from Proposition 7.(i), we see that condition (11) implies $\sup \alpha \geq (b/a) \sup \sigma$. Hence, when applying this sufficient condition, the gap between the largest values of the original decay rate, α, and gain, σ, can be related to the range of allowable values for the derivative of the nonlinear scaling.

Remark 8: We observe that the nonlinear scaling $\mu(s) = \log(1 + s)$ of Example 6 does not satisfy $\mu'(s) = \alpha_\mu(s) + c$ for any $\alpha_\mu \in \mathcal{K}$ and $c \in \mathbb{R}_{\geq 0}$ and so Proposition 7.(ii) is not applicable. Additionally, this nonlinear scaling fails to admit an $a > 0$ such that $\mu'(s) \geq a$ for all $s \in \mathbb{R}_{>0}$ and hence Proposition 7.(i) does not apply.

By contrast, consider the nonlinear scaling $\mu(s) = s^q$ for some $q \in \mathbb{Z}_{>1}$ and all $s \in \mathbb{R}_{\geq 0}$. Then, for Example 6, Proposition 7.(ii) yields that $W(x) = \mu((1/2)s^2)$ is a dissipation-form ISS-Lyapunov function for (9). This follows by choosing $\kappa(s) = \varepsilon s^{q/(q-1)}$ and $\varepsilon \in (0, q^{(2q-1)/q})$.

IV. Scaling Integral ISS-Lyapunov Functions

A Lyapunov characterization of iISS was introduced in [1] and has a similar form to a dissipation-form ISS-Lyapunov function.

Definition 9: An iISS-Lyapunov function is a continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ so that there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, $\sigma \in \mathcal{K}$, and $\rho \in \mathcal{P}$ satisfying (2) and

$$\langle \nabla V(x), f(x,w)\rangle \leq -\rho(|x|) + \sigma(|w|) \tag{15}$$

for all $x \in \mathbb{R}^n$ and $w \in \mathbb{R}^m$.

As before, our interest is in examining the statement that nonlinear scalings of iISS-Lyapunov functions are also iISS-Lyapunov functions. Again we start with a negative example.

Example 10: Consider the system

$$\dot{x} = -x + xw \tag{16}$$

and take as a candidate iISS-Lyapunov function $V(x) := \log(1+x^2)$ for all $x \in \mathbb{R}$. It is straightforward to see that V is an iISS-Lyapunov function for (16) with $\alpha_1(s) = \alpha_2(s) := \log(1 + s^2)$, $\sigma(s) := 2s$, and

$$\rho(s) := \frac{2s^2}{1+s^2}$$

for all $s \in \mathbb{R}_{\geq 0}$. We note that $\sigma \in \mathcal{K}_\infty$, $\rho \in \mathcal{P}$, and $\lim_{s\to\infty} \rho(s) = 2$. So V is explicitly not a dissipative-form ISS-Lyapunov function. This is consistent with the fact that (16) is not ISS.

Let the nonlinear scaling $\mu \in \mathcal{K}_\infty$ be given by $\mu(s) := e^s - 1$ and define $W(x) := \mu(V(x)) = x^2$ for all $x \in \mathbb{R}$. Consider $w = 2$ and observe that

$$\langle \nabla W(x), f(x,w)\rangle = -2x^2 + 4x^2 w = 2x^2.$$

Therefore, W is not an iISS-Lyapunov function for (16) since, for any $\sigma \in \mathcal{K}$, there is an $M > 0$ so that, for all $x > M$, $x^2 > \sigma(2)$, violating condition (15). Consequently, an arbitrary nonlinear scaling of an iISS-Lyapunov function is not necessarily an iISS-Lyapunov function. ■

Proposition 11: Assume an iISS-Lyapunov function V satisfying (2) and (15) with $\rho \in \mathcal{P}$, $\sigma \in \mathcal{K}$, and $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$. Take any nonlinear scaling $\mu \in \mathcal{K}_\infty$ such that there exists $b \in \mathbb{R}_{>0}$ so that $\mu'(s) \leq b$ for all $s \in \mathbb{R}_{>0}$. Then $W : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ defined by $W(x) := \mu(V(x))$ for all $x \in \mathbb{R}^n$ is an iISS-Lyapunov function.

Proof: The proof follows from (12) with the observation that, for $b \in \mathbb{R}_{>0}$, we have $b\sigma \in \mathcal{K}$. Furthermore, since, by definition, $\mu'(s) > 0$ for all $s \in \mathbb{R}_{>0}$, the function $r \mapsto \min_{|x|=r} \mu'(V(x))\rho(r)$, $r \in \mathbb{R}_{\geq 0}$, is positive definite. ■

Remark 12: We see that for the nonlinear scaling $\mu(s) = e^s - 1$ of Example 10, $\mu'(s) = e^s$ for all $s \in \mathbb{R}_{>0}$. Clearly, then, $\mu'(s)$ is not bounded above by any constant $b \in \mathbb{R}_{>0}$ and so the conditions of Proposition 11 do not hold.

V. Dissipative iISS-Lyapunov Functions

We now propose a more general definition of an iISS-Lyapunov function, which we call a dissipative iISS-Lyapunov function. This more general definition allows a larger class of nonlinear scalings such that scaling a dissipative iISS-Lyapunov function yields a new dissipative iISS-Lyapunov function.

Definition 13: A dissipative iISS-Lyapunov function is a continuously differentiable function $V:\mathbb{R}^n \to \mathbb{R}_{\geq 0}$ so that there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, $\gamma_1, \gamma_2 \in \mathcal{K}$, $\rho \in \mathcal{P}$, and a continuous function $\theta : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ satisfying

$$\lim_{s\to\infty} \int_0^s \frac{dr}{1+\theta(r)} = \infty \tag{17}$$

and, for all $x \in \mathbb{R}^n$ and $w \in \mathbb{R}^m$, (2) holds and

$$\langle \nabla V(x), f(x,w)\rangle \leq \gamma_1(|w|)\,\theta(V(x)) + \gamma_2(|w|) - \rho(|x|). \tag{18}$$

We observe that the iISS-Lyapunov function of Definition 9 is a special case of a dissipative iISS-Lyapunov function obtained by fixing $\theta(s) = 0$ for all $s \in \mathbb{R}_{\geq 0}$ and defining $\sigma \in \mathcal{K}$ by $\sigma(s) := \gamma_2(s)$ for all $s \in \mathbb{R}_{\geq 0}$. The form of this generalization is inspired by the iISS-Lyapunov function of [10, Theorem 2]. The nomenclature *dissipative iISS-Lyapunov function* is chosen to reflect the more general form of the so-called supply rate of dissipative systems theory [14] which does not require a strict separation of the input and state as is present in the definition of an iISS-Lyapunov function.

First note that a dissipative iISS-Lyapunov function is necessary and sufficient for iISS of (1).

Theorem 14: System (1) is iISS if and only if there exists a dissipative iISS-Lyapunov function.

Proof: That iISS implies the existence of a dissipative iISS-Lyapunov function follows from [1, Theorem 1] which, in part, shows that iISS implies the existence of an iISS-Lyapunov function; i.e., a continuously differentiable function $V:\mathbb{R}^n \to \mathbb{R}_{\geq 0}$, $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, $\gamma_2 \in \mathcal{K}$, and $\rho \in \mathcal{P}$ so that (2) is satisfied and

$$\langle \nabla V(x), f(x,w)\rangle \leq \gamma_2(|w|) - \rho(|x|).$$

Then, for any $\gamma_1 \in \mathcal{K}$ and any continuous function $\theta : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ satisfying (17) we have

$$\begin{aligned}\langle \nabla V(x), f(x,w)\rangle &\leq \gamma_2(|w|) - \rho(|x|)\\ &\leq \gamma_1(|w|)\,\theta(V(x)) + \gamma_2(|w|) - \rho(|x|)\end{aligned}$$

so that V is a dissipative iISS-Lyapunov function.

To see that a dissipative iISS-Lyapunov function implies iISS, define a scaling $\mu : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ by defining its derivative

$$\mu'(s) := \frac{1}{1+\theta(s)}, \quad \forall s \in \mathbb{R}_{>0}$$

where the fact that $\mu \in \mathcal{K}_\infty$ follows from (17). Let $W(x) := \mu(V(x))$. Therefore

$$\begin{aligned}\langle \nabla W(x), f(x,w)\rangle &= \mu'(V(x))\langle \nabla V(x), f(x,w)\rangle\\ &= \frac{1}{1+\theta(V(x))}(\gamma_1(|w|)\theta(V(x)) + \gamma_2(|w|) - \rho(|x|))\\ &\leq \gamma(|w|) - \hat{\rho}(|x|)\end{aligned}$$

where $\gamma := \max\{\gamma_1, \gamma_2\}$ and $\hat{\rho}(r) := \min_{|x|=r} \rho(r)/(1+\theta(V(x)))$. Consequently, W is an iISS-Lyapunov function and hence (1) is iISS. ■

Remark 15: The standing assumption that $f(\cdot,\cdot)$ is locally Lipschitz is only required in appealing to [1, Theorem 1] in the proof of Theorem 14. At the expense of additional technical detail, it is possible to generalize all other results in this paper to allow $f(\cdot,\cdot)$ to be continuous. Indeed, as our primary interest is in differential inequalities, uniqueness of solutions is not required.

Example 16: The system

$$\dot{x} = -x + (x^2+1)w \tag{19}$$

was presented in [11, Section 2.6] as an example of a system for which the origin is 0-input globally asymptotically stable. There, it was also observed that with the fixed input $w = 1$ the system has finite-escape time. Clearly, then, the system is not iISS.

Let $V(x) := (1/2)x^2$ for all $x \in \mathbb{R}$ and let $\gamma_1(s) := s$ and $\theta(s) := \sqrt{2s}(2s+1)$ for all $s \in \mathbb{R}_{\geq 0}$. Let $\gamma_2 \in \mathcal{K}_\infty$ be arbitrary. Then

$$\begin{aligned}\langle \nabla V(x), f(x,w)\rangle &= -x^2 + x(x^2+1)w\\ &= -2V(x) + \sqrt{2V(x)}(2V(x)+1)w\\ &\leq \theta(V(x))\,\gamma_1(|w|) + \gamma_2(|w|) - 2V(x).\end{aligned}$$

Therefore, V satisfies (2) and (18). However, we observe that θ fails to satisfy (17) since

$$\lim_{s\to\infty}\int_0^s \frac{dr}{1+\sqrt{2r}(2r+1)} \leq \lim_{s\to\infty}\int_0^s \frac{dr}{1+r^{\frac{3}{2}}} < 3.$$

Hence, we conjecture that (17) may be necessary. ■

We now present a sufficient condition for the nonlinear scaling of a dissipative iISS-Lyapunov function to again be a dissipative iISS-Lyapunov function.

Theorem 17: Suppose $V:\mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is a dissipative iISS-Lyapunov function and let $\mu \in \mathcal{K}_\infty$ be a nonlinear scaling. Suppose that, for all $s \in \mathbb{R}_{\geq 0}$

$$\hat{\theta}(s) := \mu'\left(\mu^{-1}(s)\right)\left(\theta\left(\mu^{-1}(s)\right)+1\right) \tag{20}$$

satisfies

$$\lim_{s\to\infty}\int_0^s \frac{dr}{1+\hat{\theta}(r)} = \infty. \tag{21}$$

Then the nonlinearly scaled function $W:\mathbb{R}^n \to \mathbb{R}_{\geq 0}$ defined by $W(x) := \mu(V(x))$ for all $x \in \mathbb{R}^n$ is a dissipative iISS-Lyapunov function satisfying

$$\mu\circ\alpha_1(|x|) \leq W(x) \leq \mu\circ\alpha_2(|x|) \tag{22}$$

$$\langle \nabla W(x), f(x,w)\rangle \leq \hat{\gamma}_1(|w|)\,\hat{\theta}(W(x)) + \hat{\gamma}_2(|w|) - \hat{\rho}(|x|) \tag{23}$$

where $\hat{\gamma}_2 \in \mathcal{K}_\infty$ is arbitrary, $\hat{\gamma}_1(s) := \max\{\gamma_1(s), \gamma_2(s)\}$, and $\hat{\rho}(s) := \mu'(s)\rho(s)$ for all $s \in \mathbb{R}_{\geq 0}$.

Proof: The result follows from a straightforward application of the chain rule and inequality (18) (we suppress the argument of $V(x)$ for readability)

$$\begin{aligned}\langle \nabla W(x), f(x,w)\rangle &\leq \mu'(V)\gamma_1(|w|)\,\theta(V) + \mu'(V)\gamma_2(|w|) - \mu'(V)\rho(|x|)\\ &\leq \hat{\gamma}_1(|w|)\left(\mu'(V)\theta(V) + \mu'(V)\right) - \mu'(V)\rho(|x|)\\ &= \hat{\gamma}_1(|w|)\left(\mu'\left(\mu^{-1}(W)\right)\left(\theta\left(\mu^{-1}(W)\right)+1\right)\right) - \mu'(V)\rho(|x|)\\ &= \hat{\gamma}_1(|w|)\,\hat{\theta}(W(x)) + \hat{\gamma}_2(|w|) - \hat{\rho}(|x|).\end{aligned}$$

■

Remark 18: An implication-form iISS-Lyapunov function was introduced in [6, Theorem 1] where, for $\chi \in \mathcal{K}_\infty$, the decrease condition

(18) is replaced by

$$|x| \geq \chi(|w|) \implies \langle \nabla V(x), f(x,w)\rangle \leq \gamma_1(|w|)\,\theta(V(x)) - \rho(|x|) \quad (24)$$

and all other functions are as in Definition 13. In contrast to the ISS result of Proposition 5, nonlinear scalings of implication-form iISS-Lyapunov functions are not necessarily implication-form iISS-Lyapunov functions. Note that the result of Theorem 17 holds for the function $\hat{\theta}$ of (20) defined as $\hat{\theta}(s) := \mu'(s)\theta(s)$.

Finally, we present three conditions such that (21) holds for (20).

Proposition 19: If

(i) there exists $M \in \mathbb{R}_{>0}$ so that $\theta(s) \leq M$ for all $s \in \mathbb{R}_{\geq 0}$ and if $\mu \in \mathcal{K}_\infty$ satisfies

$$\lim_{s\to\infty} \int_0^s \frac{dr}{1+\mu'(\mu^{-1}(r))} = \infty; \quad (25)$$

or

(ii) there exists $c \in \mathbb{R}_{>0}$ so that $\mu'(s) \geq c$ for all $s \in \mathbb{R}_{>0}$; or

(iii) there exist $b, c \in \mathbb{R}_{\geq 0}$ and $s^* \in \mathbb{R}_{\geq 0}$ so that, for all $s \geq s^*$

$$\mu'(s)(\theta(s)+1) \leq c\mu(s) + b \quad (26)$$

then $\hat{\theta}$ given by (20) satisfies (21).

Proof: That Condition (i) is sufficient follows from the inequality

$$\mu'\left(\mu^{-1}(s)\right)\left(\theta\left(\mu^{-1}(s)\right)+1\right) \leq \mu'\left(\mu^{-1}(s)\right)(M+1)$$

which, with (25) provides a divergent lower bound for the integral of (21).

For Condition (ii), the assumption on μ' implies that $\hat{\theta}(s) \geq c > 0$ for all $s \in \mathbb{R}_{>0}$ so that $1/\hat{\theta}(s)$ is well-defined for all $s \in \mathbb{R}_{>0}$. By a change of variables, $t = \mu^{-1}(r)$, we see that (17) and (20) imply

$$\lim_{s\to\infty} \int_0^s \frac{dt}{\hat{\theta}(t)} = \infty.$$

Furthermore, since $\hat{\theta}(s) \geq c > 0$ for all $s \in \mathbb{R}_{>0}$ we have $1 + \hat{\theta}(s) \leq ((c+1)/c)\hat{\theta}(s)$ for all $s \in \mathbb{R}_{>0}$. We thus obtain

$$\lim_{s\to\infty} \int_0^s \frac{dr}{1+\hat{\theta}(r)} \geq \lim_{s\to\infty} \int_0^s \frac{c}{c+1}\frac{dr}{\hat{\theta}(r)} = \infty$$

which shows the assertion.

Finally, for Condition (iii), with $t = \mu(s)$, (26) becomes $\hat{\theta}(t) \leq ct + b$ for all $t \geq t^* := \mu(s^*)$ which then guarantees (21). ■

Note that Proposition 11 is a special case of Proposition 19.(iii) with $\theta(s) = 0$ for all $s \in \mathbb{R}_{\geq 0}$ and $c = 0$.

Example 10 Revisited: Returning to the bilinear system in Example 10, consider $W(x) = (1/2)x^2$ for all $x \in \mathbb{R}$ so that

$$\begin{aligned}\langle \nabla W(x), -x + xw\rangle &= -x^2 + x^2 w = 2W(x)w - x^2 \\ &\leq 2W(x)|w| + |w| - x^2 \quad (27)\end{aligned}$$

and hence W satisfies (18) with $\gamma_1(s) = \gamma_2(s) = s$, $\theta(s) = 2s$, and $\rho(s) = s^2$ for all $s \in \mathbb{R}_{\geq 0}$.

In Example 10, we considered the iISS-Lyapunov function, $V(x) = \log(1+x^2)$, which can be seen to be a dissipative iISS-Lyapunov function with $\theta(s) = 0$ for all $s \in \mathbb{R}_{\geq 0}$. We see that for this θ and the nonlinear scaling $\mu(s) = e^s - 1$, and choosing $c = b = 1$, (26) is satisfied as follows:

$$\mu'(s)(\theta(s)+1) = e^s \leq ce^s - c + b.$$

Therefore, consistent with the calculation in (27) and Proposition 19.(iii), scaling the dissipative iISS-Lyapunov function V by μ yields another dissipative iISS-Lyapunov function W. Alternatively, since $\theta(s) = 0$ for all $s \in \mathbb{R}_{\geq 0}$, we see that the nonlinear scaling μ also satisfies (25) and so Proposition 19.(i) yields the same conclusion. Furthermore, $\mu'(s) > 1$ for all $s \in \mathbb{R}_{>0}$ and so Proposition 19.(ii) is also applicable.

With the notion of a dissipative iISS-Lyapunov function, we arrive at an easy way to obtain such Lyapunov functions for systems of the form

$$\dot{x} = Ax + g(x,w) \quad (28)$$

where $g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ satisfies $|g(x,w)| \leq (c|x| + d)|w|$ for some $c, d \in \mathbb{R}_{\geq 0}$ and all $x \in \mathbb{R}^n$, $w \in \mathbb{R}^m$. Such systems are iISS if and only A is Hurwitz.

Indeed, with A Hurwitz, choose symmetric, positive definite matrices $P, Q \in \mathbb{R}^{n\times n}$ such that $A^\top P + PA = -Q$ and consider the function $V(x) = x^\top Px$. Then, for (28)

$$\begin{aligned}\dot{V}(x) &= -x^\top Qx + 2x^\top Pg(x,w) \\ &\leq -x^\top Qx + (2c\,\|P\||x|^2 + 2d\,\|P\||x|)|w| \\ &\leq -x^\top Qx + \eta|w|\left(V(x) + \sqrt{V(x)}\right) \\ &= -\rho(|x|) + \eta|w|\theta(V(x))\end{aligned}$$

for $\theta(s) = s + \sqrt{s}$ for all $s \in \mathbb{R}_{\geq 0}$ and a suitably defined constant $\eta \in \mathbb{R}_{>0}$ and $\rho \in \mathcal{K}_\infty$. The conditions for a dissipative iISS-Lyapunov function are clearly satisfied and so (28) is iISS. Hence, for systems of the form (28) it is easy to derive such dissipative iISS-Lyapunov functions based on the classical theory of linear systems.

Note that a special case of (28) are bilinear systems of the form

$$\dot{x} = Ax + \sum_{j=1}^m w_j A_j x + Bw \quad (29)$$

where $x \in \mathbb{R}^n$, $w = [w_1 \ \ldots \ w_m] \in \mathbb{R}^m$, $A, A_1, \ldots, A_m \in \mathbb{R}^{n\times n}$, and $B \in \mathbb{R}^{n\times m}$. It is shown in [10, Theorem 5] that a bilinear system of the form (29) is iISS if and only if A is Hurwitz.

The result of [10, Theorem 2] is that a dissipative iISS-Lyapunov function with $\theta(s) = s$ and $\rho(s) = qs$ for some $q \in \mathbb{R}_{>0}$ and all $s \in \mathbb{R}_{\geq 0}$ implies iISS. Here, we have both generalized this form of an iISS-Lyapunov function as well as demonstrated its equivalence to iISS. The term dissipative iISS-Lyapunov function has been introduced in this context.

Example 20: Consider the system

$$\dot{x} = -\varphi(x) + \frac{1+x^2}{1+|x|}\log(1+x^2)w \quad (30)$$

where the continuous function $\varphi : \mathbb{R} \to \mathbb{R}$ satisfies $x\varphi(x) > 0$ for $x \neq 0$ and $\varphi(0) = 0$. Furthermore, assume $\lim_{|x|\to\infty} x\varphi(x) = 0$.

First, consider the candidate iISS-Lyapunov function $V_1(x) := (1/2)x^2$. Then

$$\langle \nabla V_1(x), f(x,w)\rangle = -x\varphi(x) + \frac{x(1+x^2)}{1+|x|}\log(1+x^2)w. \quad (31)$$

Since the state-dependent portion of the second term is unbounded while the first term approaches zero as the state becomes large, it is clear that there is no separation of the state and input that yields a negative definite decrease rate.

Next, consider the candidate iISS-Lyapunov function $V_2(x) = \log(1+x^2)$. Computing the time derivative along solutions yields

$$\begin{aligned}\langle \nabla V_2(x), f(x,w)\rangle &= -\frac{2x\varphi(x)}{1+x^2} + \frac{2x}{1+|x|}\log(1+x^2)w \\ &\leq -\frac{2x\varphi(x)}{1+x^2} + 2\log(1+x^2)|w| \\ &= -\frac{2x\varphi(x)}{1+x^2} + 2V(x)|w|. \end{aligned} \tag{32}$$

We see that $\theta(s) = 2s$ satisfies (17) and hence V_2 is a dissipative iISS-Lyapunov function, demonstrating that (30) is iISS. By contrast, as argued for V_1, it is clear that no separation of input and state is possible in (32) so as to obtain the classical iISS-Lyapunov function decrease (15).

Of course, the main result of [1] implies that such an iISS-Lyapunov function exists, but the difficulty remains in finding such a function. The dissipative iISS-Lyapunov function provides an alternate formulation that, in some cases, may be easier to verify. ■

VI. Conclusion

It has long been known that a nonlinear scaling of a Lyapunov function for an autonomous ordinary differential equation (5) is also a Lyapunov function for (5). However, when one considers systems subject to inputs such as (1), we have seen that this statement becomes more nuanced. In the case of implication-form ISS-Lyapunov functions, nonlinear scalings generate new implication-form ISS-Lyapunov functions. On the other hand, for dissipation-form ISS-Lyapunov functions and for iISS-Lyapunov functions we have demonstrated that this statement is not always true. Hence, we have provided sufficient conditions under which nonlinear scalings of dissipation-form ISS-Lyapunov functions or iISS-Lyapunov functions generate new ISS-Lyapunov or iISS-Lyapunov functions. We provided a generalization of an iISS-Lyapunov function, the dissipative iISS-Lyapunov function, that admits a larger class of nonlinear scalings. Additionally, for some common examples of iISS systems, the quadratic function is a dissipative iISS-Lyapunov function and has the nice property that it is not necessary to use Young's inequality in the computation of the decrease rate, potentially yielding less conservative results than those obtained via standard iISS-Lyapunov functions.

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