Systems & Control Letters 109 (2017) 24–29

Contents lists available at ScienceDirect

Systems & Control Letters

journal homepage: www.elsevier.com/locate/sysconle

On global asymptotic stability of $\dot{x} = -\phi(t)\phi^{\top}(t)x$ with ϕ not persistently exciting

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ARTICLE INFO

Article history:
Received 18 January 2017
Received in revised form 16 September 2017
Accepted 20 September 2017
Available online 18 October 2017

Keywords:
Linear time-varying systems
Stability theory
Persistency of excitation

ABSTRACT

We study global convergence to zero of the solutions of the nth order differential equation $\dot{x} = \phi(t)\phi^{\top}(t)x$. We are interested in the case when the vector ϕ is *not persistently exciting*, which is a necessary and sufficient condition for global exponential stability. In particular, we establish new necessary conditions on $\phi(t)$ for global asymptotic stability of the zero equilibrium of the "unexcited" system. A new sufficient condition, that is strictly weaker than the ones reported in the literature, is also established. Unfortunately, it is also shown that this condition is not necessary.

1. Introduction

In this paper we study the n-dimensional linear time-varying (LTV) system[1]

$$\dot{x} = -\phi(t)\phi^{\top}(t)x, \tag{1}$$

where the function $\phi\colon \mathbb{R}_+ \to \mathbb{R}^n$ is measurable and locally bounded—ensuring the forward completeness of (1). The system (1) is a particular case of that of descent algorithms of the form

$$\dot{x} = F(x,t)\frac{\partial J(x)}{\partial x}, \tag{2}$$

when the matrix $F(x,t) = F^{\top}(x,t)$ is positive semidefinite but *not* positive definite and $J(x)$ is a cost criterion to be minimised. Such equations arise when only partial information about the gradient of $J(x)$ is available at any specific time, but over time different projections of it become available, making it possible to construct an effective descent procedure. This situation arises in identification and model reference adaptive control of linear time-invariant (LTI) systems and it has widely been studied in the literature—see the textbooks [1–4] and [5,6] for recent references. In the former case (2) reduces to (1), while in the latter it takes the form

$$\dot{x} = \begin{bmatrix} I & B\phi^{\top}(x,t) \\ -\phi(x,t)B^{\top} & 0 \end{bmatrix} x, \tag{3}$$

with B the constant, input matrix of the system. See Section 5 and [7] for a discussion on extensions of our results to this case.

Taking the derivative of the function $|x(t)|^2$, along the trajectories of (1), where $|\cdot|$ is the Euclidean norm, it is clear that it is non-increasing. Via a basic Lyapunov argument, this ensures global stability of the zero equilibrium of (1).[2] However, because the quadratic form that defines its derivative is only negative semidefinite, some further hypotheses on $\phi(t)$ are needed to ensure its attractivity. To the best of the authors' knowledge necessary *and* sufficient conditions for global *asymptotic* stability (GAS) of (1) are conspicuous by their absence. On the other hand, it has been known for over 30 years that (1) is globally *exponentially* stable (GES) if and only if $\phi(t)$ is *persistently exciting* (PE). That is, when there exist positive constants T and μ such that

$$\int_t^{t+T} \phi(s)\phi^{\top}(s)ds \geq \mu I \tag{4}$$

for all $t \geq 0$. It should be underscored that if this condition is not satisfied, system (1) may still be GAS [8].

In the paper we are interested in deriving conditions for GAS when ϕ is bounded but *is not* PE.[3] Our research is motivated by the interesting results derived in [8], where an upper bound on the state transition matrix of (1) – given in terms of $\int_\tau^t \phi(s)\phi^{\top}(s)ds$ – is used to derive sufficient conditions for GAS. Besides the clear theoretical interest of this question, its study is further motivated

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1 As is customary in the systems and control literature, to avoid cluttering, whenever it is clear from the context we omit the time argument from the function $x(t)$.

2 The qualifier "zero equilibrium" is omitted in the sequel.

3 In view of the equivalence between GES and *uniform* GAS for linear systems [4], it is clear that we are aiming at a *non-uniform* GAS property. Also, since the system (1) is linear the qualifier "global" may be obviated—however, as usually done in the control literature, it is kept for clarity.

https://doi.org/10.1016/j.sysconle.2017.09.005
0167-6911/

by the fact that the PE condition is *difficult to verify* in some practical applications; see [9,10] for some discussion on this issue and two new adaptation schemes where PE is not required. Moreover, in many cases it is sufficient to ensure the – admittedly weaker – GAS property.

The main contribution of the paper is the development of a new technique for stability analysis of (1) based on a representation in *polar coordinates*, which is instrumental to obtain the following results.

- Present several new necessary conditions for GAS.
- Derive, as particular cases, the sufficient conditions for GAS of [8].
- Present examples of systems which have solutions not tending to a point at infinity.
- Derive a rather complete analysis of GAS of systems of order two.

To enhance readability the technically involved proofs are given in appendices at the end of the paper. Further we denote by $|v|$ the Euclidean norm of a vector $v \in \mathbb{R}^n$, by $\|M\|$ the induced (Euclidean) norm of a matrix $M \in \mathbb{R}^{n\times m}$ and, for symmetric matrices, by $\lambda_{\min}(M)$ its minimal eigenvalue.

2. Polar coordinates: a necessary and sufficient condition for GAS

In this section we give a representation of system (1) in *polar coordinates*. Namely, we express x as

$$x(t) = r(t)\gamma(t) \tag{5}$$

where

$$r(t) := |x(t)|, \quad \gamma(t) = \frac{x(t)}{|x(t)|}. \tag{6}$$

Then,

$$\dot{r}(t) = \frac{x(t)^\top \dot{x}(t)}{|x(t)|} = -\frac{\left(\phi^\top(t)x(t)\right)^2}{|x(t)|} = -(\phi^\top(t)\gamma(t))^2 r(t), \tag{7}$$

and

$$\begin{aligned}\dot{\gamma}(t) &= \frac{\dot{x}(t)}{|x(t)|} - \frac{x(t)}{|x(t)|^2}\frac{x^\top(t)\dot{x}(t)}{|x(t)|} = -(\phi^\top(t)\gamma(t))\phi(t)\\ &\quad + (\phi^\top(t)\gamma(t))^2\gamma(t).\end{aligned} \tag{8}$$

Notice that for every solution $\gamma(\cdot)$ of Eq. (8) if $|\gamma(0)| = 1$, then $\frac{d}{dt}|\gamma(t)| = 0$, and therefore $|\gamma(t)| = 1$ for all $t \geq 0$. In particular, it implies existence of solutions of Eqs. (7), (8) on $\mathbb{R}^+$.

As shown below, this change of coordinates, which has been used before to study stability of LTV systems, is essential for our future derivations. In particular, it is used in this section to obtain the following necessary and sufficient condition for GAS.

Proposition 1. *System (1) is GAS if and only if, for every solution $\gamma : \mathbb{R}^+ \to \mathbb{R}^n$ of the differential equation*

$$\dot{\gamma} = \gamma(\phi^\top\gamma)^2 - \phi(\phi^\top\gamma), \tag{9}$$

with initial condition $|\gamma(0)| = 1$ we have

$$\int_0^\infty (\phi^\top(s)\gamma(s))^2 ds = \infty. \tag{10}$$

Proof. Since $x(t_0) = 0$ for any t_0 implies $x(t) \equiv 0$ for all $t \geq t_0$, we assume in the sequel that $x(t) \neq 0$ for all t and the polar coordinate representation (5) is well-defined.

Eqs. (6) establish a one-to-one correspondence between non zero solutions $x(\cdot)$ of system (1) and solutions $\gamma(\cdot)$, $r(\cdot) \not\equiv 0$ of Eqs. (7), (8). Integrating (7) we get

$$r(t) = r(0)e^{-\int_0^t(\phi^\top(s)\gamma(s))^2 ds}.$$

Hence, $r(t) \to 0$ as $t \to \infty$ if and only if (10) holds. This completes the proof. □

It should be underscored that the condition of Proposition 1 is given in terms of solutions of Eq. (9) with unit initial vectors. In the next sections we use this result to present conditions for GAS in terms of the function ϕ.

3. Necessary conditions for GAS

The next simple proposition shows that $\phi(t)$ should have infinite energy to ensure GAS.

Proposition 2. *If system (1) is GAS, then*

$$\int_0^\infty |\phi(s)|^2 ds = \infty. \tag{11}$$

Proof. Since γ is a unitary vector from Cauchy–Schwarz inequality we have that

$$(\phi^\top(s)\gamma(s))^2 \leq |\phi(s)|^2$$

for all $s \geq 0$. Hence, (11) is necessary for (10) to hold, completing the proof. □

This simple claim of Proposition 2 can also be established, without the polar coordinates, analysing the time derivative of the function $|x(t)|^2$ as indicated by one of the anonymous reviewers.

The next proposition gives a less restrictive necessary condition for GAS.

Proposition 3. *Define the Gram matrix*

$$F(t) := \int_0^t \phi(s)\phi^\top(s)ds. \tag{12}$$

The condition

$$\lim_{t\to\infty} \|F(t)\| = \infty \tag{13}$$

is necessary for GAS of system (1).

Proof. Denote by e_j the jth unit vector in $\mathbb{R}^n$. Then, for every s

$$|\phi(s)|^2 = \sum_{j=1}^n (\phi^\top(s)e_j)^2,$$

and

$$n\|F(t)\| \geq \mathrm{tr}\{F(t)\} = \sum_{j=1}^n e_j^\top F(t)e_j = \int_0^t |\phi(s)|^2 ds,$$

completing the proof. □

The following statement, whose proof is given in Appendix A, is less trivial and shows that

$$\lim_{t\to\infty} \lambda_{\min}\left\{\int_0^t \phi(s)\phi^\top(s)ds\right\} = \infty \tag{14}$$

is *necessary* for GAS of system (1). Before presenting the result it is necessary to prove the existence of the limit above, which proceeds as follows. First, recall that for all $t_2 \geq t_1 \geq 0$ we have $F(t_2) \geq F(t_1)$ if and only if $y^\top F(t_2)y \geq y^\top F(t_1)y$ for all $y \in \mathbb{R}^n$. Then, the function $\lambda_{\min}\{F(\cdot)\}$ is not decreasing and, consequently, has (a finite or infinite) limit at infinity.

Proposition 4. *Assume*

$$\lim_{t\to\infty} \lambda_{\min}\left\{\int_0^t \phi(s)\phi^\top(s)ds\right\} < \infty. \tag{15}$$

Then, system (1) *is not GAS.* □

It has been shown in [11] that the condition (13) is *not sufficient* for GAS of system (1) in the general case. In particular, in [11] a two dimensional example such that condition (13) is fulfilled but the system is not GAS is given. In this example the condition number of the Gram matrix $F(t)$ tends to infinity as $t \to \infty$. Moreover, almost every solution of the system tends to a non zero point. In the proof of the next proposition – which is given in Appendix B – we also present a system for which condition (13) holds and the system is not GAS. However, in contrast with the example of [11], on one hand, the set of the condition numbers of the matrices $F(t)$ with $t \in \mathbb{R}^+$ is bounded, and, on the other hand, non-zero solutions have no limits at infinity.

Proposition 5. *There exists a bounded function $\phi(\cdot)$ such that* (14) *holds and system* (1) *is not GAS.*

Notice that the system in the proof of Proposition 5 contradicts Theorem 2 of [12]. A gap in the proof is in item (d), page 21 of [12].

4. Sufficient condition for GAS

In this section we show that the proposed approach allows to get a sufficient condition for GAS of system (1) that, to the best of our knowledge, is the strongest one and contains, as particular cases, the ones reported up to now, *e.g.*, the ones given in [8]. The proof of the proposition is given in Appendix C.

Proposition 6. *Assume there exist sequences of positive numbers $\{t_k\}$, $\{T_k\}$, $\{\mu_k\}$, with $\lim_{k\to\infty} t_k = \infty$, such that for all k we have*

$$t_{k+1} \geq t_k + T_k, \tag{16}$$

and

$$\lambda_{\min}\left\{\int_{t_k}^{t_k+T_k} \phi(t)\phi^\top(t)dt\right\} \geq \mu_k, \tag{17}$$

with

$$\sum_{k=1}^{\infty} \frac{\mu_k}{1+\left(\int_{t_k}^{t_k+T_k}|\phi(t)|^2dt\right)^2} = \infty. \tag{18}$$

Then, system (1) *is GAS.*

To gain some intuition in the connection between the condition of Proposition 6 and the standard PE condition (4) let us consider instead of (17) and (18) the more conservative conditions

$$\lambda_{\min}\left\{\int_{t_k}^{t_{k+1}} \phi(t)\phi^\top(t)dt\right\} \geq \mu_k, \tag{19}$$

and

$$\sum_{k=1}^{\infty} \frac{\mu_k}{1+\|\phi\|_\infty^4\, T_k^2} = \infty, \tag{20}$$

respectively, where $\|\cdot\|_\infty$ is the $\mathcal{L}_\infty$ norm. First, notice that the PE condition (4) implies

$$\lambda_{\min}\left\{\int_t^{t+T} \phi(t)\phi^\top(t)dt\right\} \geq \mu,$$

which compared with (19) reveals two substantial differences.

(i) The integration window is not fixed (to T) but is now time-varying $[t_k, t_{k+1}]$.

(ii) The "excitation level" μ is also time varying, but it should satisfy the non-summability condition (20).

It is interesting to check, if the criterion of Proposition 6 is necessary for GAS of system (1). The following proposition, whose proof is given in Appendix D, shows that the answer is *negative*.

Proposition 7. *There exists a (three dimensional) GAS system* (1) *such that, for all sequences of positive numbers $\{t_k\}$, $\{T_k\}$, $\{\mu_k\}$, for which* (16) *and* (17) *are satisfied for all k, but* (18) *is violated, that is,*

$$\sum_{k=1}^{\infty} \frac{\mu_k}{1+\left(\int_{t_k}^{t_k+T_k}|\phi(t)|^2dt\right)^2} < \infty. \tag{21}$$

5. The case of two-dimensional systems

In order to get further insight into the stability/instability mechanism of systems of the form (1), we consider in this section the particular case of order two, that is, $n = 2$. In this case, without loss of generality, we can represent the functions ϕ and γ in the following form:

$$\phi = p\begin{bmatrix}\sin(\alpha)\\ -\cos(\alpha)\end{bmatrix},\quad \gamma = \begin{bmatrix}\cos(\varphi)\\ \sin(\varphi)\end{bmatrix},$$

where p, α, φ are scalar, measurable functions, φ is differentiable and p is a bounded, non negative function. With this notation Eqs. (9) and (10) of Proposition 1 acquire the form

$$\begin{aligned}\dot\varphi(-\sin(\varphi)) &= p^2[\cos(\varphi)\sin^2(\alpha-\varphi) - \sin(\alpha)\sin(\alpha-\varphi)]\\ \dot\varphi\cos(\varphi) &= p^2[\sin(\varphi)\sin^2(\alpha-\varphi) + \cos(\alpha)\sin(\alpha-\varphi)],\end{aligned} \tag{22}$$

and

$$\int_0^\infty p^2(t)\sin^2(\alpha(t)-\varphi(t))dt = \infty, \tag{23}$$

respectively. After some simple manipulations, Eqs. (22) may be reduced to the single equation

$$\dot\varphi = \frac{p^2}{2}\sin[2(\alpha-\varphi)]. \tag{24}$$

Thus, Propositions 1 and 4 take the following form.

Proposition 8. *Assume $n = 2$ and fix ϕ, hence, p and α. System* (1) *is GAS if and only if for every solution $\varphi(\cdot)$ of* Eq. (24) *the equality* (23) *holds. Moreover, a necessary condition for GAS is*

$$\lim_{t\to\infty}\lambda_{\min}\left\{\int_0^t p^2(s)\begin{bmatrix}\sin^2(\alpha(s)) & -\sin(\alpha(s))\cos(\alpha(s))\\ -\sin(\alpha(s))\cos(\alpha(s)) & \cos^2(\alpha(s))\end{bmatrix}ds\right\} = \infty. \tag{25}$$

The example in the proof of Proposition 5 and the example in [11] may be easily analysed using this proposition. In [11] the function α is piecewise constant and tends to zero at infinity and such that condition (25) holds but (23) does not hold. In the example in the proof of Proposition 5 the function α is continuous, increasing to infinity, piecewise linear, with slope tending to zero, and such that, again, condition (25) holds, but (23) does not hold.

Equipped with Proposition 8 it is easy to construct other examples. Assume the function α is differentiable with bounded derivative. Denote $z := 2(\varphi - \alpha)$. Then, (24) takes the form

$$\dot z = -p^2\sin(z) - 2\dot\alpha. \tag{26}$$

Assume $p(t) \equiv 1$ and $\dot\alpha(t) \to 0$, $\alpha(t) \to \infty$ as $t \to \infty$. Then, (25) holds. If $\dot\alpha \in \mathcal{L}_2$, then $z \in \mathcal{L}_2$ for all solutions $z(\cdot)$ of system (26), and system (1) *is not* GAS. This is true, for example, for $\alpha(t) = (1+t)^\nu$ with $\nu \in (0, \frac{1}{2})$.

6. Discussion and future extensions

The following remarks concerning our problem formulation and results are in order.

R1 The assumption of bounded ϕ is done without loss of generality. Indeed, to obtain a bounded function it is possible to replace $\phi(t)$ by a normalised term. Two normalisations used in the adaptive control literature [2–4] are the static

$$\frac{\phi(t)}{\sqrt{1+|\phi(t)|^2}}$$

or the dynamic normalisation

$$\frac{\phi(t)}{\sqrt{\rho(t)}},\ \ \dot\rho(t) = -\lambda\rho(t) + |\phi(t)|^2,\ \ \lambda > 0.$$

R2 Proposition 3 can be extended to general LTV systems to recover the following result reported in Corollary 3.3.5 of [13]. The proof of the claim is given in Appendix E.

Proposition 9. *Consider the system $\dot x = A(t)x$ where $x \in \mathbb{R}^n$ and the elements of the matrix function $A\ :\ \mathbb{R}_+ \to \mathbb{R}^{n\times n}$ are measurable functions. The following implication is true*

$$\int_0^\infty \|A(t)\|dt < \infty \ \Rightarrow\ \lim_{t\to\infty} x(t) \neq 0 \text{ for any } x(\cdot)\neq 0. \tag{27}$$

R3 In [8] it is proven that the state transition matrix of (1), denoted $\mathfrak{T}(t,\tau)$, satisfies

$$\mathfrak{T}^\top(t,\tau)\mathfrak{T}(t,\tau)\ \le\ I - \frac{\int_\tau^t \phi(s)\phi^\top(s)ds}{1+\left(\int_\tau^t |\phi(s)|^2 ds\right)^2}. \tag{28}$$

This result is then used to derive a sufficient condition for GAS similar to the one given in Proposition 6.[4]

R4 Regarding least-squares estimators it is interesting to note that condition (14) is necessary and sufficient for parameter convergence—see, for instance, the proof of Proposition 4.3.4 in [2]. Under these conditions the covariance matrix tends to zero, with the unfortunate consequence that the least squares estimator looses adaptation *alertness* hampering its ability to track parameter variations, which is the main motivation in recursive estimation and adaptive systems.

R5 It is shown in [8] that the derivations to obtain (28) apply *verbatim* to the difference equation

$$x_{t+1} = \phi_t\phi_t^\top x_t \tag{29}$$

yielding conditions for GAS of (29) that are strictly weaker than the usual PE condition for discrete-time systems.

R6 Some of the results reported in this paper have been recently extended in [7] to the case of system (3). As indicated in the introduction this situation arises in model reference adaptive control of LTI systems.

R7 The results reported in the paper do not answer the question of necessary *and* sufficient conditions for GAS—a fundamental open question that we are now investigating. It would be particularly interesting to find conditions that, in contrast to the ones given here, are "robust" in the identification (and adaptive control) context.

4 In the first version of [8] the convergence condition is given with a sliding window of bounded width, *i.e.*, $[t_k, t_k+1] \subset [t_k, t_k+T]$ in the Gramian and a constant "excitation level μ. Inspired by the present paper this condition was later sharpened to match the one given in Proposition 6.

Acknowledgements

The authors would like to thank Laurent Praly for an early version of [8], which motivated our research, and for many useful discussions. They are also grateful to Rodolphe Sepulchre that brought the important reference [11] to their attention and to the anonymous reviewers for their highly thorough and professional revision of our paper that significantly helped us to improve the clarity of our contribution. This article is supported by the Ministry of Education and Science of Russian Federation (project 14.Z50.31.0031) and Government of Russian Federation (grant 074-U01, GOSZADANIE 2014/190 (project 2118)).

Appendix A. Proof of Proposition 4

As it was pointed out in Section 3, the function $\lambda_{\min}\{F(\cdot)\}$ is non-decreasing and has a limit as $t\to\infty$. Denote by ℓ the limit in (15). Then for every $t \ge 0$ the set

$$S(t) = \{y\in\mathbb{R}^n\ :\ |y| = 1,\ y^\top F(t)y \le \ell+1\}$$

contains a unit eigenvector of $F(t)$ corresponding to the minimal eigenvalue of $F(t)$. The sets $S(t)$ are closed, bounded, and non-empty. Moreover, the sets are non increasing, in the sense that, for all $t_2 > t_1$, we have $S(t_2)\subseteq S(t_1)$, because $F(t_2)\ge F(t_1)$. Hence, the intersection $S = \cap_{t\ge 0} S(t)$ is non-empty, *i.e.*, there exists a vector $b\in S(t)$. Since $b\in S(t)$ for all $t\ge 0$, we have

$$\int_0^\infty (b^\top\phi(t))^2 dt < \infty.$$

Now, fix a positive number T such that

$$\int_T^\infty (b^\top\phi(\tau))^2 d\tau < 1. \tag{A.1}$$

Integrate (9) over $[T,\tau]$ to get

$$\begin{aligned}\gamma(\tau) = {} & e^{\int_T^\tau(\phi^\top(s)\gamma(s))^2 ds}\gamma(T)\\ & - \int_T^\tau \phi(s)(\phi^\top(s)\gamma(s))e^{\int_s^\tau(\phi^\top(u)\gamma(u))^2 du} ds.\end{aligned} \tag{A.2}$$

Multiply the second right hand term of (A.2) by the row vector $b^\top$ from the left, and use the Cauchy–Schwarz inequality to get the upper bound

$$\begin{aligned}&[\int_T^\tau (b^\top\phi(s))(\phi^\top(s)\gamma(s))e^{\int_s^\tau(\phi^\top(u)\gamma(u))^2du}ds]^2\\ &\le \int_T^\tau (b^\top\phi(s))^2ds\ \int_T^\tau (\phi(s)\gamma(s))^2 e^{2\int_s^\tau(\phi^\top(u)\gamma(u))^2du}ds\\ &= \int_T^\tau (b^\top\phi(s))^2ds\left[-\frac{1}{2}e^{2\int_s^\tau(\phi^\top(u)\gamma(u))^2du}\right]_{s=T}^{s=\tau}\\ &= \frac{1}{2}\int_T^\tau (b^\top\phi(s))^2ds[e^{2\int_T^\tau(\phi^\top(s)\gamma(s))^2ds}-1].\end{aligned}$$

Therefore,

$$\begin{aligned}&|\int_T^\tau (b^\top\phi(s))(\phi^\top(s)\gamma(s))\ e^{\int_s^\tau(\phi^\top(u)\gamma(u))^2du}ds|\\ &\le \sqrt{\frac{1}{2}\int_T^\tau (b^\top\phi(s))^2ds}\ \ e^{\int_T^\tau(\phi^\top(u)\gamma(u))^2du}\\ &< \frac{1}{\sqrt{2}}e^{\int_T^\tau(\phi^\top(s)\gamma(s))^2ds},\end{aligned}$$

where have invoked (A.1) to get the second inequality.

Now, system (1) is linear. Therefore, for every vector $z\in\mathbb{R}^n$ there exists a vector $x_0\in\mathbb{R}^n$ such that for a solution of Eq. (1) with initial condition $x(0) = x_0$ we have $x(T) = z$. We take z such that $z/|z| = b$. Denote $\gamma(t) = x(t)/|x(t)|$. For this solution equality

(A.2) implies

$$1 \geq |b^\top \gamma(\tau)| \geq e^{\int_T^\tau (\phi^\top(s)\gamma(s))^2 ds}(1 - \frac{1}{\sqrt{2}})$$

for all $\tau \geq T$. Hence,

$$\int_T^\infty (\phi^\top(s)\gamma(s))^2 ds \leq \ln\left(\frac{\sqrt{2}}{\sqrt{2}-1}\right).$$

From the bound above we conclude that

$$\int_0^\infty (\phi^\top(s)\gamma(s))^2 ds < \infty.$$

As shown in Proposition 1, under this condition, (1) is not GAS, which completes the proof.

Appendix B. Proof of Proposition 5

Consider the two-dimensional system (1) with

$$\phi(t) = \begin{pmatrix} \sin \epsilon t \\ \cos \epsilon t \end{pmatrix}, \tag{B.1}$$

where ϵ is a number, $\epsilon \in (0, 1/2)$, $x(0) = (1, 0)^\top$ and $t \geq 0$. To solve system (1) introduce new variables:

$$y := \begin{pmatrix} \sin \epsilon t & \cos \epsilon t \\ -\cos \epsilon t & \sin \epsilon t \end{pmatrix} x.$$

In the new variables y system (1) has the form

$$\dot{y} = \begin{pmatrix} -1 & -\epsilon \\ \epsilon & 0 \end{pmatrix} y. \tag{B.2}$$

The eigenvalues of the system matrix of (B.2) are

$$\lambda_+ = \frac{-1 + \sqrt{1-4\epsilon^2}}{2}, \quad \lambda_- = \frac{-1 - \sqrt{1-4\epsilon^2}}{2}$$

and the initial condition, associated to $x(0)$, is $y(0) = (0, -1)^\top$.

Some tedious calculations show that

$$y(t) = \frac{\epsilon}{\beta}\begin{pmatrix} e^{\lambda_+ t} - e^{\lambda_- t} \\ -e^{\lambda_+ t}\dfrac{\lambda_+ + 1}{\epsilon} + e^{\lambda_- t}\dfrac{\lambda_- + 1}{\epsilon} \end{pmatrix},$$

where we defined the constant

$$\beta := \lambda_+ - \lambda_- = \sqrt{1 - 4\epsilon^2}.$$

From the definition of y we get

$$\begin{aligned} x(t) &= \begin{pmatrix} \sin \epsilon t & -\cos \epsilon t \\ \cos \epsilon t & \sin \epsilon t \end{pmatrix} y(t) \\ &= \frac{\epsilon}{\beta}\begin{pmatrix} e^{\lambda_+ t}(\sin \epsilon t + \dfrac{\lambda_+ + 1}{\epsilon}\cos \epsilon t) - e^{\lambda_- t}(\sin \epsilon t + \dfrac{\lambda_- + 1}{\epsilon}\cos \epsilon t) \\ e^{\lambda_+ t}(\cos \epsilon t - \dfrac{\lambda_+ + 1}{\epsilon}\sin \epsilon t) - e^{\lambda_- t}(\cos \epsilon t - \dfrac{\lambda_- + 1}{\epsilon}\sin \epsilon t) \end{pmatrix}. \end{aligned}$$

Notice that the components $x_1(t)$ and $x_2(t)$ of the vector $x(t)$ are products of sinusoidal functions and decreasing exponents. Denote by $\bar{t}$ the second positive zero of the function $x_2(t)$. It may be shown that $\bar{t} \in [\frac{2\pi}{\epsilon}, \frac{2\pi}{\epsilon} + \frac{\pi}{2\epsilon}]$, and $x_1(\bar{t}) \in (0, 1)$. Some simple calculations show that

$$\begin{aligned} 1 &> |x(\bar{t})|^2 = |x_1(\bar{t})|^2 \\ &= \frac{\epsilon^2}{\beta^2}[e^{2\lambda_+\bar{t}}(1 + (\frac{\lambda_+ + 1}{\epsilon})^2) + e^{2\lambda_-\bar{t}}(1 + (\frac{\lambda_- + 1}{\epsilon})^2) \\ &\quad + 2e^{(\lambda_+ + \lambda_-)\bar{t}}(-1 - \frac{(\lambda_+ + 1)(\lambda_- + 1)}{\epsilon^2})] \\ &= \frac{\epsilon^2}{\beta^2}[e^{2\lambda_+\bar{t}} - 10e^{-\bar{t}} + e^{2\lambda_-\bar{t}}] + e^{2\lambda_+\bar{t}}(\frac{1+\beta}{2\beta})^2 + e^{2\lambda_-\bar{t}}(\frac{1-\beta}{2\beta})^2 \\ &\geq e^{2\lambda_+\bar{t}} = e^{-\frac{4\epsilon^2\bar{t}}{1+\beta}} \geq e^{-8\pi\epsilon} \end{aligned}$$

where we have used the fact that $x_2(\bar{t}) = 0$ in the first identity.

Define $\bar{t}_1 := \bar{t}$, $\epsilon_1 := \epsilon$ and $\epsilon_2 := \frac{1}{2}\epsilon_1$. Denote by t_2 the second positive zero of the function $x_2(\bar{t}_1 + t)$. Then, $t_2 \in [\frac{2\pi}{\epsilon_2}, \frac{2\pi}{\epsilon_2} + \frac{\pi}{2\epsilon_2}]$. Now, we have a similar situation as at the beginning, but the initial value is $(x(\bar{t}_1), 0)^\top$ rather than $(1, 0)^\top$, and they are given at point $t = \bar{t}_1$. Consequently, invoking the lower bound obtained above, we get

$$x_1(\bar{t}_1 + t_2) > e^{-8\pi\epsilon_2}x_1(\bar{t}_1).$$

Repeating this construction for $k = 1, \ldots, \infty$ we see that, in each interval $(\bar{t}_k, \bar{t}_{k+1})$, we have Eq. (1) with function ϕ given by Eq. (B.1) where ϵ is replaced by $\epsilon_k = \frac{\epsilon}{2^k}$ and with initial conditions

$$x(\bar{t}_k) = \begin{pmatrix} x_1(\bar{t}_k) \\ 0 \end{pmatrix},$$

where $x_1(\bar{t}_k) \in (0, e^{-8\pi\sum_{j=1}^k \epsilon_j}]$. From here we conclude that

$$x_1(\bar{t}_k) \geq e^{-8\pi\sum_{j=0}^k 2^{-k}\epsilon} \geq e^{-16\pi\epsilon} > 0,$$

and the function $x(t)$ *does not* tend to zero at infinity.

On the other hand, setting $\bar{t}_0 = 0$, we have that

$$\begin{aligned} &\lambda_{\min}\left\{\int_0^{\bar{t}_{k+1}} \phi(\tau)\phi^\top(\tau)d\tau\right\} \\ &= \lambda_{\min}\left\{\sum_{j=0}^k \int_{\bar{t}_j}^{\bar{t}_{j+1}} \phi(\tau)\phi^\top(\tau)d\tau\right\} \\ &\geq \lambda_{\min}\left\{\sum_{j=0}^k \int_0^{\frac{2\pi}{\epsilon_j}} \begin{pmatrix} \sin^2\epsilon_j\tau & \sin\epsilon_j\tau\cos\epsilon_j\tau \\ \sin\epsilon_j\tau\cos\epsilon_j\tau & \cos^2\epsilon_j\tau \end{pmatrix} dt\right\} \\ &= \lambda_{\min}\left\{\sum_{j=0}^k \begin{pmatrix} \dfrac{\pi}{\epsilon_j} & 0 \\ 0 & \dfrac{\pi}{\epsilon_j} \end{pmatrix}\right\}, \end{aligned}$$

which tends to infinity as $k \to \infty$. Notice, that the condition number of the Gram matrix $F(T)$ is not bigger than three for $T \geq \frac{2\pi}{\epsilon}$. The proof is complete.

Appendix C. Proof of Proposition 6

Our goal is to show that there exists a positive number c such that for every solution $\gamma(\cdot)$ of Eq. (9) with $|\gamma(t)| \equiv 1$ and for all k we have

$$\int_{t_k}^{t_k+T_k} [\phi^\top(\tau)\gamma(\tau)]^2 d\tau \geq c\frac{\mu_k}{1 + (\int_{t_k}^{t_k+T_k} |\phi(s)|^2 ds)^2}. \tag{C.1}$$

This condition – together with (18) – implies (10). Then, according to Proposition 1, system (1) is GAS.

It is sufficient to show that there exists a positive number c such that for all positive numbers T, μ satisfying

$$\int_0^T \phi(s)\phi(s)^\top ds \geq \mu I,$$

and all solutions $\gamma(\cdot)$ of Eq. (9) with $|\gamma(0)| = 1$ we have

$$\int_0^T [\phi^\top(s)\gamma(s)]^2 ds \geq c\frac{\mu}{1 + (\int_0^T |\phi(s)|^2 ds)^2}. \tag{C.2}$$

Towards this end, we integrate (9) over $[0, \tau]$ yielding

$$\begin{aligned} \gamma(\tau) &= e^{\int_0^\tau (\phi^\top(s)\gamma(s))^2 ds}\gamma(0) \\ &\quad - \int_0^\tau \phi(s)\phi^\top(s)\gamma(s)e^{\int_s^\tau (\phi^\top(u)\gamma(u))^2 du} ds, \end{aligned} \tag{C.3}$$

for all $\tau \in [0, T]$. Now, multiply (C.3) by $e^{-\int_0^\tau (\phi^\top(s)\gamma(s))^2 ds}\phi^\top(\tau)$ from the left to get

$$e^{-\int_0^\tau (\phi^\top(s)\gamma(s))^2 ds}(\phi^\top(\tau)\gamma(\tau)) = \phi^\top(\tau)\gamma(0) - \int_0^\tau \phi^\top(\tau)\phi(s)\phi^\top(s)\gamma(s)e^{-\int_0^s (\phi^\top(u)\gamma(u))^2 du}ds. \tag{C.4}$$

Applying the triangle inequality to (C.4) we get

$$|\phi^\top(\tau)\gamma(0)| \le |\phi^\top(\tau)\gamma(\tau)| + |\int_0^\tau \phi^\top(\tau)\phi(s)\phi^\top(s)\gamma(s) \times e^{-\int_0^s (\phi^\top(u)\gamma(u))^2 du}ds|.$$

Therefore,

$$(\phi^\top(\tau)\gamma(0))^2 \le 2(\phi^\top(\tau)\gamma(\tau))^2 + 2\left[\int_0^\tau \phi^\top(\tau)\phi(s)\phi^\top(s)\gamma(s)e^{-\int_0^s (\phi^\top(u)\gamma(u))^2 du}ds\right]^2. \tag{C.5}$$

Integrate this inequality over the interval $[0, T]$. Notice that

$$\int_0^T (\phi^\top(\tau)\gamma(0))^2 d\tau \ge \mu,$$

since $|\gamma(0)| = 1$, and

$$\int_0^T \left[\int_0^\tau \phi^\top(\tau)\phi(s)\phi^\top(s)\gamma(s)e^{-\int_0^s (\phi^\top(u)\gamma(u))^2 du}ds\right]^2 d\tau \le \int_0^T \left[\int_0^\tau |\phi^\top(\tau)\phi(s)\phi^\top(s)\gamma(s)|ds\right]^2 d\tau \le \int_0^T \int_0^\tau (\phi^\top(\tau)\phi(s))^2 ds \int_0^\tau (\phi^\top(s)\gamma(s))^2 ds d\tau \le \int_0^T \int_0^T (|\phi(\tau)||\phi(s)|)^2 ds \int_0^T (\phi^\top(s)\gamma(s))^2 ds d\tau = \left[\int_0^T |\phi(s)|^2 ds\right]^2 \int_0^T (\phi^\top(s)\gamma(s))^2 ds.$$

Hence (C.5) implies

$$\mu \le 2\int_0^T (\phi^\top(s)\gamma(s))^2 ds \left(1 + \left[\int_0^T |\phi(s)|^2 ds\right]^2\right),$$

which is equivalent to (C.2) with $c = \frac{1}{2}$. The proposition is proved.

Appendix D. Proof of Proposition 7

Consider the sequence $\{\bar{t}_m\}_{m=1}^\infty$ such that

$$\bar{t}_{3k+1} - \bar{t}_{3k} = \bar{t}_{3k+2} - \bar{t}_{3k+1} = \bar{t}_{3(k+1)} - \bar{t}_{3k+2} = 3^k$$

for all $k = 0, 1, 2, \ldots$. Denote by $\{e_1, e_2, e_3\}$ the standard orthogonal basis in $\mathbb{R}^3$. Consider system (1) with $\phi(t) = e_1$ if $t \in [\bar{t}_{3k}, \bar{t}_{3k+1})$, $\phi(t) = e_2$ if $t \in [\bar{t}_{3k+1}, \bar{t}_{3k+2})$, and $\phi(t) = e_3$ if $t \in [\bar{t}_{3k+2}, \bar{t}_{3k+3})$ for $k = 0, 1, 2, \ldots$. Then $|\phi(t)| \equiv 1$, and for every sequence $\{t_m\}$, $\{\tilde{T}_m\}$, $\{\mu_m\}$ mentioned in Proposition 6 we have $\mu_k = 0$ if $t_m \ge \bar{t}_{3k}$ and $T_m \le 2\ 3^k$. Moreover, if $t_m \in [\bar{t}_{3k}, \bar{t}_{3(k+1)})$, then

$$\frac{\mu_m}{1 + T_m^2} \le 3^{-k}.$$

Therefore inequality (18) is not satisfied. But for every solution of system (1) the jth component is exponentially decreasing on $[\bar{t}_{3k+j-1}, t_{3k+j}]$ for all $j = 1, 2, 3$ and $k = 0, 1, 2, \ldots$. Hence, the system is GAS.

Appendix E. Proof of Proposition 9

The goal is to show that the solutions do not tend to zero at infinity. Assume, on the contrary, that $x(t) \to 0$ as $t \to \infty$. Let $x(t)$ be a solution with $x(0) \ne 0$. In view of assumption (27) it is possible to find $T > 0$ such that

$$\int_T^\infty \|A(s)\| ds < \frac{1}{2}$$

and the vector $x(T)$ is not zero. Since $|x(t)| \to 0$ there exists a point $\hat{T} \in [T, \infty)$ such that $|x(\hat{T})| = |x(T)|$, and $|x(t)| < |x(T)|$ for all $t > \hat{T}$.

Now, integrate the equation $\dot{x} = A(t)x$ over the interval $[\hat{T}, \infty)$. Since $x(t) \to 0$, we have

$$-x(\hat{T}) = \int_{\hat{T}}^\infty A(t)x(t)dt.$$

But

$$|x(\hat{T})| = |\int_{\hat{T}}^\infty A(s)x(s)ds| \le \int_{\hat{T}}^\infty \|A(s)\|\ |x(s)|ds \le |x(\hat{T})| \int_{\hat{T}}^\infty \|A(s)\| ds < \frac{1}{2}|x(\hat{T})|.$$

The contradiction proves that $x(t) \not\to 0$ at infinity. The proposition is proved.

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