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Brief paper

Stability of Markov regenerative switched linear systems☆

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ABSTRACT

In this paper, we give a necessary and sufficient condition for mean stability of switched linear systems having a Markov regenerative process as its switching signal. This class of switched linear systems, which we call Markov regenerative switched linear systems, contains Markov jump linear systems and semi-Markov jump linear systems as special cases. We show that a Markov regenerative switched linear system is mth mean stable if and only if a particular matrix is Schur stable, under the assumption that either m is even or the system is positive.

1. Introduction

Among switched linear systems, those having a time-homogeneous Markov process as its switching signal, called Markov jump linear systems (Costa, Fragoso, & Todorov, 2013), are of particular importance. One of the reasons for their importance is that time-homogeneous Markov processes are well suited to model stochastic phenomena presenting a constant rate of occurrence. Another reason is that the analysis and synthesis of Markov jump linear systems can be performed in a rather similar way to those for linear time-invariant systems by the introduction of auxiliary variables (Costa et al., 2013).

However, it is often restricting to assume that the switching signals present constant transition rates, or even the Markovian property. To overcome this restriction, we find a wide variety of alternative switching signals in the literature (Antunes, Hespanha, & Silvestre, 2013; Hou, Luo, & Shi, 2005; Ogura & Martin, 2014, 2015). A natural extension to time-homogeneous Markov processes are time-homogeneous semi-Markov processes, which are Markovian-like processes with time-varying transition rates (see Çinlar, 1975). The stability analysis of the corresponding switched linear systems can be found in Antunes et al. (2013) and Ogura and Martin (2014). Another extension are regenerative processes (Smith, 1955), which are, roughly speaking, stochastic processes that can be obtained by concatenating independent and identically distributed random functions (see Sigman and Wolff (1993) for the details). The mean stability analysis of linear systems subject to regenerative switchings is performed in Ogura and Martin (2015).

In this paper, we extend the works in Antunes et al. (2013), Ogura and Martin (2015), and Ogura and Martin (2014) to analyze the stability of *Markov regenerative switched linear systems*, which are switched linear systems whose switching signal is a *Markov regenerative process* (also called a *semi-regenerative* process) (Choi, Kulkarni, & Trivedi, 1994; Çinlar, 1975). Markov regenerative processes form a large class of stochastic processes which contains as special cases all the Markov, semi-Markov, and regenerative processes. We show that exponential mth mean stability of a Markov regenerative switched linear system is characterized by the spectral radius of a matrix, under the assumption that either m is even or the system is positive. Extending various results in the literature (Antunes et al., 2013; Fang & Loparo, 2002; Ogura & Martin, 2014, 2015), the obtained result enables us to analyze the stability of, for example, state-feedbacked Markov jump linear systems with periodically observed mode signals (see a discrete-time setting in Cetinkaya and Hayakawa (2015)), as well as controlled system with failure-prone controllers with only one repairing facility (Distefano & Trivedi, 2013, Section 3.2).

The paper is organized as follows. After introducing necessary notations, in Section 2 we introduce the class of Markov regenerative switched linear systems under consideration and then state a necessary and sufficient condition for their exponential mean stability. Then, in Section 3, we present various applications of the main result. The notation used in this paper is standard. When $x \in \mathbb{R}^n$ is nonnegative entrywise we write $x \geq 0$. The standard

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Euclidean norm on $\mathbb{R}^n$ is denoted by $\|\cdot\|$. Let I and O denote the identity and zero matrices, respectively. The block diagonal matrix with block diagonals $A_1, \ldots, A_N$ is denoted by $\bigoplus_{i=1}^N A_i$. The Kronecker product of two matrices A and B is denoted by $A \otimes B$. We say that $A \in \mathbb{R}^{n\times n}$ is Schur stable if A has the spectral radius less than one. Also we say that A is Hurwitz stable if all the eigenvalues of A have negative real parts. For an integrable random variable X, its expected value is denoted by $E[X]$ and its conditional expectations by $E[X \mid \cdot]$.

2. Stability characterization

This section introduces the class of Markov regenerative switched linear systems and then presents a necessary and sufficient condition for their stability. We need to first recall the definition of Markov renewal processes (Çinlar, 1975). Let N be a positive integer. A stochastic process $(\theta, \tau) = \{(\theta_k, \tau_k)\}_{k\geq 0}$ taking values in $\{1, \ldots, N\} \times [0, \infty)$ and satisfying $0 = \tau_0 \leq \tau_1 \leq \cdots$ is called a Markov renewal process (Çinlar, 1975) if

$$P(\theta_{k+1} = j, \tau_{k+1} - \tau_k \leq t \mid \theta_k, \ldots, \theta_0, \tau_k, \ldots, \tau_0) = P(\theta_{k+1} = j, \tau_{k+1} - \tau_k \leq t \mid \theta_k)$$

holds for every k and $t \geq 0$. We assume that (θ, τ) is time-homogeneous, that is, for all i, j and $t \geq 0$, the probability $P(\theta_{k+1} = j, \tau_{k+1} - \tau_k \leq t \mid \theta_k = i)$ is independent of k. We note that, in this case, θ is a time-homogeneous Markov chain and therefore has the constant transition probabilities $P(\theta_{k+1} = j \mid \theta_k = i)$. In this paper, we furthermore assume that

$$\tau_{k+1} - \tau_k > 0 \tag{1}$$

with probability one for every k. Then, we can state the definition of Markov regenerative processes as follows.

Definition 1 (*Çinlar, 1975, Choi et al., 1994*)**.** Let $\sigma = \{\sigma_t\}_{t\geq 0}$ be a stochastic process taking values in a finite set Λ. We say that σ is *Markov regenerative* if there exists a Markov renewal process (θ, τ) taking values in $\{1, \ldots, N\} \times [0, \infty)$ and satisfying the following conditions for every k:

(P1) τ_k is a stopping time for σ;
(P2) There exists a function $\pi : \Lambda \to \{1, \ldots, N\}$ such that $\pi(\sigma_{\tau_k}) = \theta_k$;
(P3) For $j \in \{1, \ldots, N\}$, $\ell \geq 1$, $0 \leq t_1 < t_2 < \cdots < t_\ell$, and a function $f : \Lambda^\ell \to [0, \infty)$,

$$E[f(\sigma_{\tau_k+t_1}, \ldots, \sigma_{\tau_k+t_\ell}) \mid \theta_k = j, \sigma_s; s \leq \tau_k] = E[f(\sigma_{t_1}, \ldots, \sigma_{t_\ell}) \mid \theta_0 = j].$$

We call τ the *regeneration times* of σ, and (θ, τ) the *embedded Markov renewal process* of σ.

Among the three conditions in the definition, (P3) is the most important. It implies that, as far as prediction is concerned, at the regeneration time τ_k, the past information of the process $\{\sigma_s\}_{s\leq\tau_k}$ is irrelevant and only the value of θ_k is needed. Actually, (P3) implies that

$$P(\sigma_{\tau_k+t} = \lambda \mid \theta_k = j, \sigma_s; s \leq \tau_k) = P(\sigma_t = \lambda \mid \theta_0 = j) \tag{2}$$

for all $\lambda \in \Lambda$, $i, j \in \{1, \ldots, N\}$, $t \geq 0$, and $k \geq 0$. Eq. (2) indicates that, given $\theta_k = j$, the process σ "regenerates" at time τ_k as if it starts from time 0, given $\theta_0 = j$. Moreover, σ possesses a certain Markovian property at the regeneration time τ_k; once we know the value of σ_{τ_k}, using (P2) we can determine the value θ_k, which then determines the future distribution of σ by (2). As is pointed out in Çinlar (1974), it is convenient to take π to be the identity, but this is not necessary. Also we remark that (P1) is a technical condition needed to state (P3). For the details, the readers are referred to Çinlar (1975) and Çinlar (1974).

Then, we introduce the class of switched linear systems studied in this paper.

Definition 2. Let σ be a Markov regenerative process and let $A_\lambda \in \mathbb{R}^{n\times n}$ for each $\lambda \in \Lambda$. Then, the stochastic differential equation

$$\Sigma : \frac{dx}{dt} = A_{\sigma_t} x, \tag{3}$$

where $x(0) = x_0 \in \mathbb{R}^n$ is a constant, is called a *Markov regenerative switched linear system*.

For example, a time-homogeneous Markov process r is Markov regenerative with an underlying embedded Markov renewal process being $\{(r_{t_k}, t_k)\}_{k\geq 0}$, where $0 = t_0 < t_1 < t_2 < \cdots$ are the times at which the process r changes its value. Therefore, Markov jump linear systems (Costa et al., 2013) are Markov regenerative switched linear systems. In Section 3, we will show in detail that more general classes of switched linear systems, such as semi-Markov jump linear systems (Antunes et al., 2013; Ogura & Martin, 2014) and regenerative switched linear systems (Ogura & Martin, 2015), are contained in the class of Markov regenerative switched linear systems.

Based on the embedded Markov renewal process, we can naturally define the stability of Markov regenerative switched linear systems as follows.

Definition 3. Given a positive integer m, we say that Σ is *exponentially mth mean stable* if there exist $C > 0$ and $\beta > 0$ such that $E[\|x(t)\|^m] \leq Ce^{-\beta t}\|x_0\|^m$ for every x_0 and θ_0.

Also, we here introduce the positivity of Σ.

Definition 4. We say that Σ is *positive* if $x_0 \geq 0$ implies $x(t) \geq 0$ with probability one for all θ_0 and $t \geq 0$.

We remark that, for Σ to be positive, it is clearly sufficient that all the matrices A_λ ($\lambda \in \Lambda$) are Metzler, i.e., the off-diagonal entries of each A_λ are all nonnegative (Farina & Rinaldi, 2000). However, this sufficient condition is not necessary as is shown in Ogura and Martin (2015, Example 10) for regenerative switched linear systems.

In order to state the main result of this paper, we recall the notion of induced matrices. First we define (see, e.g. Parrilo & Jadbabaie, 2008) the m-lift of $x \in \mathbb{R}^n$, denoted by $x^{[m]}$, as the real vector of length $n_m = \binom{n+m-1}{m}$ with its elements being the lexicographically ordered monomials $\sqrt{\alpha!}\,x^\alpha$ ($\alpha! := m!/(\alpha_1! \cdots \alpha_n!)$) that are indexed by all the possible exponents $\alpha = (\alpha_1, \ldots, \alpha_n)$ summing up to m. For example, for $x = [x_1, x_2]^\top$ we have $x^{[1]} = x$, $x^{[2]} = [x_1^2, \sqrt{2}x_1x_2, x_2^2]^\top$, and $x^{[3]} = [x_1^3, \sqrt{3}x_1^2x_2, \sqrt{3}x_1x_2^2, x_2^3]^\top$. Then, we define the mth induced matrix of A, denoted by $A^{[m]}$, as the $n_m \times n_m$ unique matrix (Parrilo & Jadbabaie, 2008) satisfying $(Ax)^{[m]} = A^{[m]}x^{[m]}$ for every $x \in \mathbb{R}^n$.

The next theorem is the main result of this paper.

Theorem 5. *Let $p_{ij} = P(\theta_1 = j \mid \theta_0 = i)$ for all $i, j \in \{1, \ldots, N\}$. For all $0 \leq s \leq t < \infty$, define the $\mathbb{R}^{n\times n}$-valued random variable $\Phi(t; s)$ by the differential equation $\partial\Phi/\partial t = A_{\sigma_t}\Phi(t; s)$ with the initial conditions $\Phi(t; t) = I_n$ for every $t \geq 0$. Assume that the following two conditions hold:*

(A1) *Either m is even or Σ is positive;*
(A2) *There exists $T > 0$ such that $\tau_{k+1} - \tau_k \leq T$ with probability one for every $k \geq 0$.*

Then, Σ is exponentially mth mean stable if and only if the $(Nn_m) \times (Nn_m)$ real block matrix $\mathcal{A} = [\mathcal{A}_{ij}]_{1\leq i,j\leq N}$ with the (i, j)-block $\mathcal{A}_{ij}$ being defined by

$$\mathcal{A}_{ij} = p_{ji}E[\Phi(\tau_1; 0)^{[m]} \mid \theta_0 = j, \theta_1 = i] \tag{4}$$

is Schur stable.

In Section 3, we will show that the above theorem can recover stability characterizations of Markov jump linear systems (Fang & Loparo, 2002; Ogura & Martin, 2014), semi-Markov jump linear systems (Antunes et al., 2013; Ogura & Martin, 2014), and regenerative switched linear systems (Ogura & Martin, 2015). We will also show that the theorem can give a stability condition for other switched linear systems whose stability cannot be studied using the methods in the literature.

2.1. Proof

We present a proof of Theorem 5 in this subsection. This proof consists of the following two steps. We first show, in Proposition 6, that the stability of Σ can be analyzed based on its discretized version. The proof of the proposition utilizes the technique developed in Ogura and Martin (2015) to study regenerative switched linear systems. To analyze the discretized version of the system, we then present Proposition 7, which extends Theorem 3.4 in Ogura and Martin (2014) for not necessarily positive systems.

Let Σ be a Markov regenerative switched linear system. Then, the discretized process $x_d = \{x(\tau_k)\}_{k\geq 0}$ is the solution of the discrete-time system $\mathcal{S}\Sigma : x_d(k+1) = \Phi(\tau_{k+1};\tau_k)x_d(k)$. In order to proceed, we introduce a class of switched linear systems called discrete-time semi-Markov jump linear systems (Ogura & Martin, 2014). Let $\{F_k\}_{k\geq 0}$ be a stochastic process taking values in $\mathbb{R}^{n\times n}$. The system $\Sigma_d : x_d(k+1) = F_k x_d(k)$ $(x_d(0) = x_0)$ is said to be a *discrete-time semi-Markov jump linear system* if there exists a time-homogeneous Markov chain θ taking values in $\{1,\dots,N\}$ such that, for all $k \geq 0$, $i,j \in \{1,\dots,N\}$, and a Borel subset G of $\mathbb{R}^{n\times n}$, there holds that

$$P(\theta_{k+1} = j, F_k \in G \mid \theta_k,\dots,\theta_0, F_{k-1},\dots,F_0) = P(\theta_{k+1} = j, F_k \in G \mid \theta_k), \tag{5}$$

and the conditional probability

$$P(\theta_{k+1} = j, F_k \in G \mid \theta_k = i) \tag{6}$$

does not depend on k. The mean stability and the positivity of Σ_d is defined in the following standard manner. For a positive integer m, we say that Σ_d is *exponentially mth mean stable* if there exist $C > 0$ and $\beta > 0$ such that $E[\|x_d(k)\|^m] \leq Ce^{-\beta k}\|x_0\|^m$ for all x_0 and θ_0. Also we say that Σ_d is *stochastically mth mean stable* if $\sum_{k=0}^{\infty} E[\|x_d(k)\|^m]$ is finite for all x_0 and θ_0. Finally, Σ_d is said to be *positive* if $x_0 \geq 0$ implies $x_d(k) \geq 0$ with probability one for every k and θ_0.

The next proposition relates the stability of Σ and $\mathcal{S}\Sigma$.

Proposition 6. *$\mathcal{S}\Sigma$ is a discrete-time semi-Markov jump linear system. Moreover, if (A2) holds, then the following statements are true:*

- *If Σ is exponentially mth mean stable, then $\mathcal{S}\Sigma$ is stochastically mth mean stable.*
- *If $\mathcal{S}\Sigma$ is exponentially mth mean stable, then so is Σ.*

Proof. Let $F_k = \Phi(\tau_{k+1};\tau_k)$. Also, we let $(\Omega, \mathcal{M}, P)$ denote the underlying probability space. We denote the σ-algebra of Ω generated by a set X of random variables by $\mathcal{M}(X)$. We define the following σ-algebras; $\mathcal{M}_1 = \mathcal{M}(\{\theta_k\})$, $\mathcal{M}_2 = \mathcal{M}(\{\theta_k,\dots,\theta_0,F_{k-1},\dots,F_0\})$, $\mathcal{M}_3 = \mathcal{M}(\{\theta_k,\dots,\theta_0,\sigma_s; s \leq \tau_k\})$, and $\mathcal{M}_4 = \mathcal{M}(\{\theta_k,\sigma_s; s \leq \tau_k\})$. Then, we can show

$$\mathcal{M}_1 \subset \mathcal{M}_2 \subset \mathcal{M}_3 = \mathcal{M}_4. \tag{7}$$

The first inclusion is obvious. The second inclusion is true because $F_0,\dots,F_{k-1}$ are measurable on $\mathcal{M}_3$. Finally, the last identity follows from (P2). Now, from (P3) we know that $P(\theta_{k+1} = j, F_k \in G \mid \mathcal{M}_1) = P(\theta_{k+1} = j, F_k \in G \mid \mathcal{M}_4)$. Therefore, by (7) and Ogura and Martin (2014, Lemma 1.1), we conclude that $P(\theta_{k+1} = j, F_k \in G \mid \mathcal{M}_1) = P(\theta_{k+1} = j, F_k \in G \mid \mathcal{M}_2)$, which is equivalent to (5). Also, the conditional probability (6) is independent of k by (P3) and the time-homogeneity of θ. Therefore, $\mathcal{S}\Sigma$ is a discrete-time semi-Markov jump linear system.

The proof of the second statement can be done in the same way as the proof for the implication $[2 \Rightarrow 3]$ of Ogura and Martin (2015, Theorem 12) due to (A2) and the assumption (1). Also, we can prove the third statement in the same way as the proof for $[3 \Rightarrow 1]$ of Ogura and Martin (2015, Theorem 12) by (A2). The details of the proofs are thus omitted.

Proposition 6 shows that the stability analysis of Σ could be reduced to the stability analysis of discrete-time semi-Markov jump linear systems. The next proposition gives a characterization of the exponential mth mean stability of discrete-time semi-Markov jump linear systems, extending Ogura and Martin (2014, Theorem 3.4) to the case where m is even.

Proposition 7. *For $i,j \in \{1,\dots,N\}$, let p_{ij} denote the transition probability of θ from i to j. Assume that either m is even or Σ_d is positive. Then, the following statements are equivalent.*

1. *Σ_d is exponentially mth mean stable;*
2. *Σ_d is stochastically mth mean stable;*
3. *The $(Nn_m) \times (Nn_m)$ block matrix $\mathcal{F}$ with the (i,j)-block $\mathcal{F}_{ij} = p_{ji}E[F_0^{[m]} \mid \theta_0 = j, \theta_1 = i] \in \mathbb{R}^{n_m\times n_m}$ is Schur stable.*

For the proof of this proposition, we will need the next lemma.

Lemma 8. *Assume that m is an even integer. Let K be the closed convex hull of $(\mathbb{R}^n)^{[m]} = \{x^{[m]} : x \in \mathbb{R}^n\}$ in $\mathbb{R}^{n_m}$. Then, there exists a norm $|||\cdot|||$ on $\mathbb{R}^{Nn_m}$ and $f \in \mathbb{R}^{Nn_m}$ such that $|||x||| = f^\top x$ for every $x \in K \times \cdots \times K$.*

Proof. We first show that K is a proper cone (Tam & Schneider, 2006, chap. 26), that is, K is a closed and convex cone having a nonempty interior and satisfying

$$K \cap (-K) = \{0\}, \tag{8}$$

where $-K := \{-x : x \in K\}$. K is clearly a closed and convex cone.

Let us show (8). For each $i \in \{1,\dots,n\}$, let ℓ_i denote the position of the monomial x_i^m in the vector $x^{[m]}$, i.e., we assume that $(x^{[m]})_{\ell_i} = x_i^m$. Let us show the existence of a constant $C > 0$ such that every $y \in K$ satisfies

$$y_{\ell_i} \geq 0, \ |y_\ell| \leq C\sum_{i=1}^{n} y_{\ell_i}, \tag{9}$$

for all $1 \leq i \leq n$ and $1 \leq \ell \leq n_m$. Take an arbitrary y in the convex hull of $(\mathbb{R}^n)^{[m]}$. Then there exist $x_1,\dots,x_N \in \mathbb{R}^n$ and positive numbers $c_1,\dots,c_N$ such that $y = \sum_{j=1}^N c_j x_j^{[m]}$. Without loss of generality we can assume $c_j = 1$. Then, $y_{\ell_i} = \sum_{j=1}^N (x_j)_i^m \geq 0$ because m is even. Next, let $1 \leq \ell \leq n_m$ be arbitrary and let $\alpha = (\alpha_1,\dots,\alpha_n)$ be the exponent of the monomial $(x^{[m]})_\ell$, i.e., we suppose that $(x^{[m]})_\ell = \sqrt{\alpha!}x^\alpha$. Then, the inequality of arithmetic and geometric means implies $|(x^{[m]})_\ell| \leq m^{-1}\sqrt{\alpha!}\sum_{i=1}^n \alpha_i x_i^m$. Hence, the triangle inequality shows that $|y_\ell| \leq \sum_{j=1}^N |(x_j^{[m]})_\ell| \leq m^{-1}\sqrt{\alpha!}\sum_{i=1}^n\sum_{j=1}^N (x_j)_i^m \leq C\sum_{i=1}^n y_{\ell_i}$, where $C = m^{-1}\max_\alpha(\sqrt{\alpha!})$. Therefore, every y in the convex hull of $(\mathbb{R}^n)^{[m]}$ satisfies (9). A limiting argument thus shows that (9) is satisfied by every $y \in K$. Now assume $y, -y \in K$. This implies $y_{\ell_i} = 0$ for every i and therefore $y_\ell = 0$ for every ℓ by (9). Hence (8) holds.

Finally, by Ogura and Martin (2014, Lemma 1.5), the difference $K - K = \{x - y : x, y \in K\}$ equals the whole space $\mathbb{R}^{n_m}$. This

in fact shows that the interior of K is nonempty because, in general, a closed and convex cone K satisfying (8) has a nonempty interior if and only if the difference $K - K$ coincides with the whole space (Tam & Schneider, 2006, chap. 26).

Now, since K is a proper cone, the N-direct product $K \times \cdots \times K$ in $\mathbb{R}^{Nn_m}$ is also a proper cone. Therefore there exists (Seidman, Schneider, & Arav, 2005, Section 2) a norm $|||\cdot|||$ on $\mathbb{R}^{Nn_m}$ and a vector $f \in \mathbb{R}^{Nn_m}$ having the desired property. This completes the proof.

We then prove Proposition 7:

Proof of Proposition 7. The case where Σ_d is positive is shown in Ogura and Martin (2014, Theorem 3.4). Let us assume that m is even. We shall show the cycle $[1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 1]$. It is easy to prove $[1 \Rightarrow 2]$. The implication $[2 \Rightarrow 3]$ can be proved in the same way as in the proof of Theorem 3.4 in Ogura and Martin (2014) without the positivity assumption.

Let us prove $[3 \Rightarrow 1]$ assuming m is even. Take a norm $|||\cdot|||$ on $\mathbb{R}^{Nn_m}$ and $f \in \mathbb{R}^{Nn_m}$ satisfying the linearity property described in Lemma 8. By the equivalence of the norms on a finite-dimensional linear space, we can take a constant $C_1 > 0$ such that $C_1^{-1}\|\cdot\| \leq |||\cdot||| \leq C_1\|\cdot\|$. Let us consider the stochastic process e_{θ_k}, where $e_1, \ldots, e_N$ denote the standard unit vectors in $\mathbb{R}^N$. Using the general identity $\|x^{[m]}\| = \|x\|^m$ (see Parrilo & Jadbabaie, 2008), we can show the inequality $\|x_d(k)\|^m = \|x_d(k)^{[m]}\| = \|e_{\theta_k} \otimes x_d(k)^{[m]}\| \leq C_1|||e_{\theta_k} \otimes x_d(k)^{[m]}|||$. Since $e_{\theta_k} \otimes x_d(k)^{[m]} \in K \times \cdots \times K$, the linearity of the norm $|||\cdot|||$ shows that

$$\begin{aligned} E[\|x_d(k)\|^m] &\leq C_1|||E[e_{\theta_k} \otimes x_d(k)^{[m]}]||| \\ &\leq C_1^2\|E[e_{\theta_k} \otimes x_d(k)^{[m]}]\|. \end{aligned} \tag{10}$$

On the other hand, by the identity

$$E[e_{\theta_{k+1}} \otimes x_d(k+1)^{[m]}] = \mathcal{F}E[e_{\theta_k} \otimes x_d(k)^{[m]}] \tag{11}$$

proved in Ogura and Martin (2014, Proposition 3.8), if $\rho(\mathcal{F}) < 1$ then there exist $C_2 > 0$ and $\beta > 0$ such that

$$\begin{aligned} \|E[e_{\theta_k} \otimes x_d(k)^{[m]}]\| &\leq C_2 e^{-\beta k}\|e_{\theta_0} \otimes x_0^{[m]}\| \\ &= C_2 e^{-\beta k}\|x_0^{[m]}\| \\ &= C_2 e^{-\beta k}\|x_0\|^m. \end{aligned} \tag{12}$$

Therefore, the inequalities (10) and (12) prove that Σ_d is exponentially mth mean stable. This completes the proof of Proposition 7.

Now we can readily prove Theorem 5. (A1) implies that either m is even or $\mathcal{S}\Sigma$ is positive. Therefore, by Proposition 7 and the first statement of Proposition 6, we see that the following three properties are equivalent; the exponential mth mean stability of $\mathcal{S}\Sigma$, the stochastic mth mean stability of $\mathcal{S}\Sigma$, and the Schur stability of $\mathcal{A}$. This equivalence and also the second and third statements of Proposition 6 immediately prove the main result of Theorem 5.

3. Applications

In this section, we illustrate how Theorem 5, the main result of this paper, can be used to recover various stability characterizations of switched linear systems derived in the literature (Sections 3.1–3.3), as well as to analyze the stability of systems that cannot be analyzed by current techniques in the literature (Section 3.4). Throughout this section, we will use the following notation (see Brockett, 1973). For a matrix $A \in \mathbb{R}^{n\times n}$, we define $A_{[m]} \in \mathbb{R}^{n_m \times n_m}$ as the unique real matrix such that

$$(\exp(At))^{[m]} = \exp(A_{[m]}t). \tag{13}$$

3.1. Regenerative switched linear systems

In this subsection, we discuss some implications of the stability characterization provided for regenerative switched linear systems (Ogura & Martin, 2015). We first recall the definition of regenerative processes. We say that a stochastic process σ is a regenerative process (Smith, 1955) if there exists a random variable $R_1 > 0$ such that (i) $\{\sigma_{t+R_1}\}_{t\geq 0}$ is independent of $\{\{\sigma_t\}_{t<R_1}, R_1\}$, and (ii) $\{\sigma_{t+R_1}\}_{t\geq 0}$ has the same joint distribution as $\{\sigma_t\}_{t\geq 0}$. Then, following Ogura and Martin (2015), we say that Σ given in (3) is a regenerative switched linear system if σ is a regenerative process.

Let random variables $\{R_k\}_{k\geq 1}$ be independent and identically distributed and define $\tau_k = R_1 + \cdots + R_k$ for each $k \geq 1$. Then, σ can be broken into independent and identically distributed cycles $\{\sigma_t\}_{0\leq t<\tau_1}, \{\sigma_t\}_{\tau_1\leq t<\tau_2}, \ldots$ (see Sigman & Wolff, 1993). From this observation, it is immediate to see Çinlar (1975) that σ is Markov regenerative process with its embedded Markov renewal process being (θ, τ), where θ is the constant sequence $\{1, 1, \ldots\}$. Therefore, we can define the exponential mth mean stability of a regenerative switched linear system by Definition 3. We can furthermore prove, as a corollary of Theorem 5, the following stability characterization for regenerative switched linear systems originally given in Ogura and Martin (2015, Theorem 12):

Corollary 9 (*Ogura & Martin, 2015, Theorem 12*)**.** *Let Σ be a regenerative switched linear system. Assume that* (A1) *is true and, moreover, there exists $T > 0$ such that $R_1 \leq T$. Then, Σ is exponentially mth mean stable if and only if the matrix $E[\Phi(R_1; 0)^{[m]}]$ is Schur stable.*

Proof. We first remark that, since $R_1 > 0$, assumption (1) is satisfied. Moreover, $R_1 \leq T$ guarantees that (A2) is satisfied. Since (A1) is true by the assumption in the corollary, we can apply Theorem 5. Since $\theta \equiv 1$, the matrix $\mathcal{A}$ given by (4) equals the $n_m \times n_m$ matrix $p_{11}E[\Phi(\tau_1; 0)^{[m]} \mid \theta_0 = 1, \theta_1 = 1] = E[\Phi(R_1; 0)^{[m]}]$. Therefore, Theorem 5 readily completes the proof of the corollary.

3.2. Semi-Markov jump linear systems

We now consider another class of switched linear systems called semi-Markov jump linear systems (Ogura & Martin, 2014). Assume that (θ, τ) is a Markov renewal process taking values in $\{1, \ldots, N\} \times [0, \infty)$, whose definition was given in the beginning of Section 2. Then, the stochastic process $\{\sigma_t\}_{t\geq 0}$ defined by

$$\sigma_t = \theta_k, \qquad \tau_k \leq t < \tau_{k+1}, \tag{14}$$

is called a semi-Markov process (Çinlar, 1975). Then, following Ogura and Martin (2014), we say that Σ given by (3) is a semi-Markov jump linear system if σ is a semi-Markov process. It is easy to see that σ is a Markov regenerative process with its embedded renewal process being (θ, τ). The mapping π in (P2) is taken to be the identity. Without loss of generality, we can assume (1).

The following corollary of Theorem 5, which extends Ogura and Martin (2014, Theorem 2.5) to not necessarily positive systems, immediately follows from (13) and (14):

Corollary 10. *Suppose that Σ is a semi-Markov jump linear system and assume that conditions* (A1) *and* (A2) *hold. Then, Σ is exponentially mth mean stable if and only if the $(Nn_m) \times (Nn_m)$ real block matrix $\mathcal{A} = [\mathcal{A}_{ij}]_{1\leq i,j\leq N}$ with the (i, j)-block $\mathcal{A}_{ij}$ being defined by $\mathcal{A}_{ij} = p_{ji}E[\exp((A_j)_{[m]}\tau_1) \mid \theta_0 = j, \theta_1 = i]$ is Schur stable.*

3.3. Markov jump linear systems

In this subsection, we show that Theorem 5 can recover stability characterizations (Fang & Loparo, 2002; Ogura & Martin, 2014) for a class of switched linear systems called Markov jump linear

systems (Costa et al., 2013). We say that Σ given by (3) is a Markov jump linear system if σ is a time-homogeneous Markov process. Generalizing the stability characterizations of Markov jump linear systems in Fang and Loparo (2002, Theorem 3.3) for $m = 2$ and in Ogura and Martin (2014, Theorem 5.1) for positive systems, we can show the next corollary of Theorem 5:

Corollary 11. *Suppose that Σ is a Markov jump linear system and σ can take values in $\{1, \ldots, N\}$. Assume that either m is even or $A_1, \ldots, A_N$ are Metzler. Let Q be the infinitesimal generator of Markov process σ. Then, Σ is exponentially mth mean stable if and only if the matrix $\mathcal{B}_\Sigma = Q^\top \otimes I_{n_m} + \bigoplus_{i=1}^N (A_i)_{[m]}$ is Hurwitz stable.*

Proof. Notice that, for $h > 0$ arbitrarily chosen, σ is a Markov regenerative process with the embedded Markov renewal process $(\theta, \tau) = \{(\sigma_{kh}, kh)\}_{k\geq 0}$ by the Markovian property of σ. Therefore, Σ is a Markov regenerative switched linear system. The assumption (1) is trivially true and, also, condition (A2) is satisfied with $T = h$. Moreover, the assumption of the corollary ensures that condition (A1) also holds true. Let $p_{ij}(h) = P(\theta_1 = j \mid \theta_0 = i) = P(\sigma_h = j \mid \sigma_0 = i)$. Then, by Theorem 5, Σ is exponentially mth mean stable if and only if the block matrix $\mathcal{A}(h) = [\mathcal{A}(h)_{ij}]_{1\leq i,j\leq N}$ given by $\mathcal{A}(h)_{ij} = p_{ji}(h)E[\Phi(h; 0)^{[m]} \mid \sigma_h = i, \sigma_0 = j]$ is Schur stable for every $h > 0$. We continuously extend the domain of $\mathcal{A}$ to the origin by letting $\mathcal{A}(0) = I$. The continuity of the extended $\mathcal{A}$ at the origin can be verified by the Dominant Convergence theorem. Then, to complete the proof of the corollary, it is sufficient to show that

$$\mathcal{A}(h) = \exp(\mathcal{B}_\Sigma h) \tag{15}$$

for every $h \geq 0$.

To prove (15), we need to show that

$$\mathcal{A}(h)\mathcal{A}(h') = \mathcal{A}(h + h') \quad \text{for all } h, h' \geq 0 \tag{16}$$

and

$$\mathcal{A}'(0) = \mathcal{B}_\Sigma. \tag{17}$$

Let us show (16). Notice that, by (11), the trajectory x of Σ satisfies

$$E[e_{\sigma_{t+h}} \otimes x(t + h)^{[m]}] = \mathcal{A}(h)E[e_{\sigma_t} \otimes x(t)^{[m]}]$$

for all $t \geq 0$ and $h \geq 0$. This in particular implies $\mathcal{A}(h + h')(e_{\sigma_0} \otimes x_0^{[m]}) = \mathcal{A}(h)\mathcal{A}(h')(e_{\sigma_0} \otimes x_0^{[m]})$ for all $\sigma_0 \in \{1, \ldots, N\}$ and $x_0 \in \mathbb{R}^n$. Since the set $\{e_i \otimes x_0^{[m]} : i \in \{1 \ldots, N\}$ and $x_0 \in \mathbb{R}^n\}$ spans the whole space $\mathbb{R}^{Nn_m}$ by Ogura and Martin (2014, Lemma 1.4), we obtain (16).

Then let us show (17). We compute $\mathcal{A}'_{ij}(0)$, the (i, j)-block of $\mathcal{A}'(0)$, for each pair (i, j). First assume $i \neq j$. Since $\lim_{h\to 0} E[\Phi(h; 0)^{[m]} \mid \sigma_h = i, \sigma_0 = j] = I$, by the Dominated convergence theorem, we can show $\mathcal{A}'_{ij}(0) = \lim_{h\to 0}(p_{ji}(h)I - O)/h = \lambda_{ji}I$. Next assume $i = j$. Let γ denote the number of transitions of σ on the interval $[0, h]$ when $\sigma_0 = i$. Then the event $\{\sigma_h = i, \sigma_0 = i, \gamma = 1\}$ has probability zero. Moreover, using the big-O asymptotic notation, we can show that

$$E[\Phi(h; 0)^{[m]} \mid \sigma_h = i, \sigma_0 = i, \gamma = \ell] = \begin{cases} e^{(A_i)_{[m]}h}, & \text{for } \ell = 0, \\ I + O(h), & \text{for } \ell \geq 2, \end{cases}$$

$P(\gamma = 0) = e^{\lambda_{ii}h}$, and $P(\gamma \geq 2) = O(h^2)$. Therefore

$$\frac{\mathcal{A}_{ii}(h) - \mathcal{A}_{ii}(0)}{h} = \frac{p_{ii}(h)e^{(A_i)_{[m]}h} - I}{h} + O(h) \to (A_i)_{[m]} + \lambda_{ii}I$$

as $h \to 0$. The above argument proves (17) and, therefore, completes the proof of the corollary.

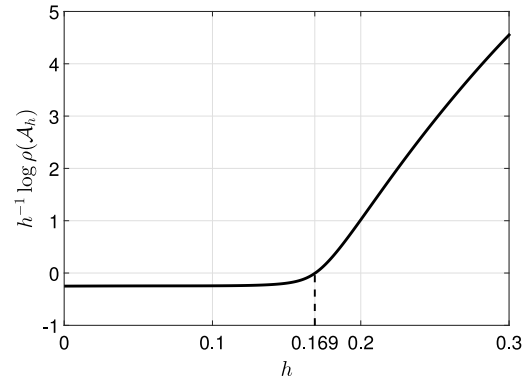


Fig. 1. $h^{-1}\log(\rho(\mathcal{A}_h))$ versus h.

3.4. Markov jump linear systems with periodic mode observation

In this subsection, we present an example of a system to which none of the results in Corollaries 9–11 are applicable. Consider the constant-gain state-feedback control of the Markov jump linear system $dx/dt = A_{P,r_t}x(t) + B_{P,r_t}u(t)$, where $A_{P,1}, \ldots, A_{P,N} \in \mathbb{R}^{n\times n}$, $B_{P,1}, \ldots, B_{P,N} \in \mathbb{R}^{n\times p}$, and r is a time-homogeneous Markov process with state space $\{1, \ldots, N\}$ and infinitesimal generator Q. If the controller can measure r at any time instant, one can consider the feedback control

$$u = K_r x \tag{18}$$

with mode-dependent gains $K_i \in \mathbb{R}^{p\times n}$ $(i = 1, \ldots, N)$. It is well known (Costa et al., 2013) that, under this ideal situation, one can find feedback gains K_i that stabilize the closed-loop system in the mean-square sense by solving linear matrix inequalities.

Following the formulation of Cetinkaya and Hayakawa (2015) in discrete-time, we here consider a more realistic situation where only the periodic samples $\{r_{kh}\}_{k\geq 0}$ with a known sampling period $h > 0$ are available to the controller. Precisely speaking, we assume that the feedback control takes the form

$$u = K_q x \tag{19}$$

where the stochastic process q is defined by $q_t = r_{kh}$ if $kh \leq t < (k + 1)h$ for all $t \geq 0$ and $k \geq 0$. We emphasize that, though we assume that only periodic samples of r are available, the infinitesimal generator Q is assumed to be known before the feedback control is designed. Defining $\sigma = (r, q)$ and $A_\sigma = A_{P,r} + B_{P,r}K_q$, we can write the closed-loop equation in the form (3), which we denote by Σ_h. Since r is time-homogeneous and Markovian, we can see that σ is a Markov regenerative process with an associated embedded Markov renewal process $\{(r_{kh}, kh)\}_{k\geq 0}$. The function π in (P2) maps the pair $(i, j) \in \{1, \ldots, N\}^2$ to i. Then, using Theorem 5, we can prove the following characterization of the exponential mean stability of Σ_h:

Corollary 12. *Assume that m is even. For each $j \in \{1, \ldots, N\}$, let $\mathcal{B}_j = Q^\top \otimes I_{n_m} + \bigoplus_{i=1}^N (A_{(i,j)})_{[m]}$. Define*

$$\mathcal{A}_h = \sum_{j=1}^N \exp(\mathcal{B}_j h)((e_j e_j^\top) \otimes I_{n_m}).$$

Then, Σ_h is exponentially mth mean stable if and only if $\mathcal{A}_h$ is Schur stable.

Proof. Since m is even, condition (A1) is satisfied. Moreover, because $\tau_{k+1} - \tau_k = h$, condition (A2) as well as (1) hold true. Therefore, by Theorem 5, it is sufficient to show that the matrix $\mathcal{A}$ given by (4) equals $\mathcal{A}_h$. To compute $\mathcal{A}_{ij}$, observe that σ confined in $[0, h)$ with the condition $r_0 = j$ is a time-homogeneous

Markov process with state space $\{(1, j), \ldots, (N, j)\}$ and infinitesimal generator Q. Therefore, from (15), $\mathcal{A}_{ij}$ equals the (i, j)-block of the matrix $\exp(\mathcal{B}_j h)$. Hence, the jth block-column of $\mathcal{A}$ equals that of $\exp(\mathcal{B}_j h)$. This proves $\mathcal{A} = \mathcal{A}_h$, as desired.

Remark 13. Note that we can use neither Corollaries 9 nor 10 to analyze the stability of Σ_h. First, it is shown in Ogura and Martin (2015, Example 7) that σ cannot be a semi-Markov process. Moreover, though σ can be realized as a regenerative process, the realization in Ogura and Martin (2015, Example 7) does not satisfy (A2) by the following reason. In the embedded renewal process (θ, τ) of the realization, the renewal times $\{\tau_k\}_{k\geq 0}$ are given as the sampling times $t = kh$ $(k = 0, 1, \ldots)$ such that $r(t) = r_0$. However, the difference $\tau_{k+1} - \tau_k$ can take an arbitrary large number and, hence, cannot be bounded by a uniform and finite number.

Let us see an example. Consider the Markov jump linear system with the following coefficient matrices:

$$A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -0.545 & 0.626 \\ 0 & -1.570 & 1.465 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -0.106 & 0.087 \\ 0 & -3.810 & 3.861 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 1,80 & -0.3925 & 4.52 \\ 3.14 & 0.100 & -0.28 \\ -19.06 & -0.148 & 1.56 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0 \\ -0.283 \\ 0.333 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 0 \\ 0.087 \end{bmatrix}, \quad B_3 = \begin{bmatrix} -0.064 \\ 0.195 \\ -0.080 \end{bmatrix},$$

and the infinitesimal generator

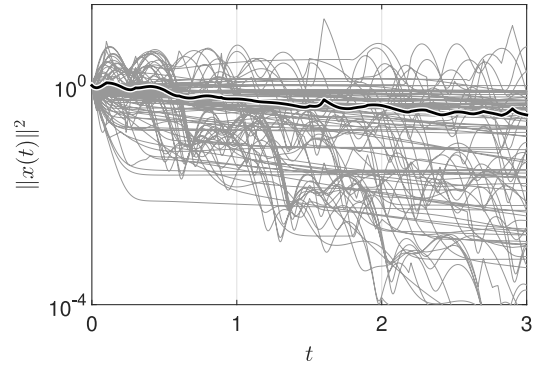
$$Q = \begin{bmatrix} -0.53 & 0.32 & 0.21 \\ 0.50 & -0.88 & 0.38 \\ 0.40 & 0.13 & -0.53 \end{bmatrix}.$$

This system, taken from Blair and Sworder (1975), models a certain economic system. We denote by Σ_0 the closed-loop system when the classical feedback control (18) is applied to this system. In Costa et al. (2013), the feedback gains that stabilize Σ_0 (in the mean square sense) with the minimum H^2 norm are obtained as

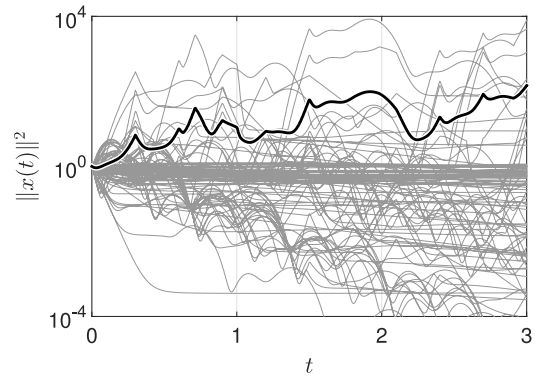
$$K_1 = \begin{bmatrix} 2.0343 & 14.5181 & -23.5917 \end{bmatrix},$$
$$K_2 = \begin{bmatrix} 1.0187 & 73.0961 & -78.7596 \end{bmatrix},$$
$$K_3 = \begin{bmatrix} 93.6651 & -11.4921 & 11.6875 \end{bmatrix}.$$

We use Corollary 12 to investigate how the stability property of Σ_0 is altered when the feedback control (18) is replaced with the one in (19) based on periodic observations. For $m = 2$ and period $h = (0.001)\ell$ $(\ell = 1, \ldots, 300)$, we compute the spectral radius of $\mathcal{A}_h$ given in Corollary 12. In Fig. 1, we show the graph of $h^{-1}\log(\rho(\mathcal{A}_h))$ as h varies. From the corollary and the graph, we determine that the system Σ_h is mean square stable if and only if $0 < h < 0.169$ [years]. This implies that, to guarantee the stability of the controlled economic system, r must be sampled with a period less than about 2 months. It is interesting to observe that, as $h \to 0$, the quantity $h^{-1}\log(\rho(\mathcal{A}_h))$ becomes close to -0.250, the maximum real part of the eigenvalue of the matrix $\mathcal{B}_{\Sigma_0}$ that characterizes the stability of the original system Σ_0 by Corollary 11. This shows that, in the limit of $h \to 0$, the stability of the original closed-loop system Σ_0 is "recovered" by Σ_h.

In Fig. 2a and 2b, we show 100 sample paths of $\|x(t)\|^2$ and their sample averages when $h = 0.1$ and $h = 0.3$, respectively. For the generation of each sample path, we take x_0 randomly from the unit sphere in $\mathbb{R}^3$ and r_0 from the uniform distribution on $\{1, 2, 3\}$. We can see that mean square stability is achieved with sampling period $h = 0.1$, while the closed-loop system exhibits instability for $h = 0.3$. Finally, for each value of h, we show sample paths of $\sigma = (r, q)$ in Fig. 3. We notice that the change of the values of q occurs only at sampling instants, shown by the dotted lines.

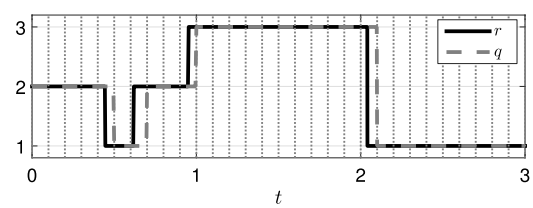


(a) $h = 0.1$ (stable).

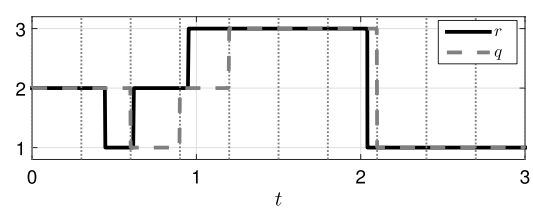


(b) $h = 0.3$ (unstable).

Fig. 2. Sample paths of $\|x(t)\|^2$ (thin lines) and their averages (thick lines) for $h = 0.1$ and 0.3.



(a) $h = 0.1$ (stable).



(b) $h = 0.3$ (unstable).

Fig. 3. Sample paths of $\sigma = (r, q)$ for $h = 0.1$ and 0.3. Solid lines, dashed lines, and dotted lines show r, q, and sampling instants, respectively.

4. Conclusion

In this paper, we have investigated the mean stability of Markov regenerative switched linear systems. The class of switched linear systems contain a wide variety of important stochastic switched linear systems that have appeared in the literature. We have shown that the mean stability of a Markov regenerative switched linear system is characterized by the spectral radius of a matrix arising from its transition matrix. A numerical example was presented to illustrate the obtained result.

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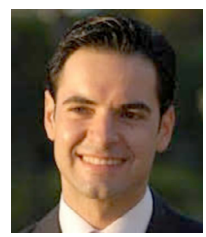

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