Reachable set estimation for switched linear systems with dwell-time switching

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HIGHLIGHTS

- No need for exponential decrease/bounded increase of the Lyapunov function.
- New notion of τ-reachable set, i.e. the set reached at time τ after the switch.
- A parametrization of the estimate of the reachable set is proposed.
- Two numerical approaches based on linear matrix inequalities are given.

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ABSTRACT

In this work we address the problem of (outer) estimation of reachable sets in switched linear systems subject to dwell-time switching. After giving some conditions that exploit the well-known properties of exponential decrease/bounded increase of the Lyapunov function (i.e. exponential decrease in between switching times and bounded increase at switching times), we overcome the need for such properties. This is done by introducing a new notion of τ-reachable set, i.e. the set that can be reached by the trajectories defined at time τ after the switch. Such extended notion of reachable set can be used to parametrize the estimate of the reachable set as a function of the distance in time from the switch. Two approaches are provided to implement such parametrization: the first approach exploits the evolution of the system in between switches via the matrix exponential of the state subsystem matrix; the second approach exploits a time-scheduled Lyapunov function. A numerical example is provided to show the effectiveness of the proposed methods and computational cost is addressed.

1. Introduction

Switched systems have emerged as an important class of hybrid systems and represent an active research area in the field of control systems [1,2], with impactful applications in networked control, multi-agent systems, cybersecurity, and many other subjects [3–5]. A switched system is composed of a family of continuous- or discrete-time subsystems and a switching rule orchestrating the switching among them. Stability and stabilization topics have been among the main concerns in the field of switched systems. An established technique to effectively deal with stability and stabilization of switched systems is the so-called multiple Lyapunov function approach [6–10], combined with slowly switching such as dwell time and average

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dwell time switching [11–13]. Reachable set estimation, which aims to derive a bounded set that contains all the state trajectories generated by a dynamic system with a prescribed class of initial state set and inputs, is another major concern for switched systems. Reachable set estimation is not only of theoretical interest in robust control theory [14,15], but also crucial to engineering verification and validation problems [16,17]. In some early work, reachable set bounding was considered in the context of state estimation and it has later received a lot of attention in parameter estimation, see [18] and references therein. Recently, ellipsoidal techniques based on Lyapunov function have attracted some attention for estimating reachable sets in different class of systems: in the framework of bounding ellipsoid, the quadratic Lyapunov function has played a fundamental role in the reachable set estimation problem, and it has been applied to time-delay systems [19–21], singular systems [22], and discrete-time switched systems [23].

However, to the best of the authors' knowledge, the reachable set estimation for continuous-time switched systems with constrained switching law is not very mature, which motivates our study in this paper. This study tackles the problem of reachable set estimation for continuous-time switched systems via a multiple Lyapunov function approach. At first we exploit the well-known properties of exponential decrease/bounded increase of the Lyapunov function (i.e. exponential decrease in between switching times and bounded increase at switching times). Then we show how the need for these properties can be overcome: this is done by introducing a new notion of τ-reachable set, i.e. the set that can be reached by the trajectories defined at time τ after the switch. Such extended notion of reachable set is used to parametrize the estimate of the reachable set as a function of the distance in time from the switch. Two approaches are provided to implement such parametrization: the first one exploits the evolution of the system in between switches via the matrix exponential of the state subsystem matrix; the second exploits a time-scheduled quadratic Lyapunov function. Both approaches can be implemented numerically by means of linear matrix inequalities. A numerical example is provided to show the effectiveness of the proposed methods. Recently, [24] has considered the problem of reachable set estimation for continuous-time switched systems with constrained switching law. Even though the bounded increase condition at switching instants is removed, [24] is still based on an exponential decrease condition in between switching instants: furthermore, differently from this work, the concepts of τ-reachable set and parametrization the estimate of the reachable set are not exploited.

The rest of the paper is organized as follows: Section 2 provides the basic ideas behind the estimation of reachable sets. Section 3 exploits the exponential decrease/bounded increase properties, while Section 4 overcomes the needs for these properties. Section 5 discusses the numerical implementation via linear matrix inequalities, while Section 6 provides a numerical example. Section 7 concludes the paper.

Notation: In this paper $\mathbb{R}$ and $\mathbb{N}_+$ represent the sets of real and positive natural numbers, while $\mathbb{S}_+$ represents the set of symmetric and positive definite matrices. The transpose of a vector x or of a matrix P is indicated with x' and P', respectively. For a symmetric matrix $P = P'$, the notation $P > 0$ means that P is positive definite. The operator $\mathrm{Tr}\, P$ represents the trace of matrix P. Finally, for switched systems, the switching signal $\sigma(\cdot)$ will take values in the set $\mathscr{M} = \{1, 2, \ldots, M\}$, with M being the number of subsystems. We also use the notation $\{(t_0, \sigma(t_0)), (t_1, \sigma(t_1)), \ldots, (t_k, \sigma(t_k)), \ldots \,|\, k \in \mathbb{N}_+\}$ to denote the switching instants and the corresponding activated subsystem.

2. Preliminaries and problem formulation

In order introduce to the main ideas behind the estimation[1] of reachable sets we start by considering the linear system

$$\dot{x}(t) = Ax(t) + Bw(t), \quad x(0) = x_0 \tag{1}$$

with $x \in \mathbb{R}^{n_x}$ being the state, $w \in \mathbb{R}^{n_w}$ an external disturbance, A and B matrices of appropriate dimensions. Let us assume the initial state x_0 belongs to the ellipsoidal set

$$x_0 \in \mathscr{X}_0 = \left\{ x_0 \in \mathbb{R}^{n_x} \,|\, x_0' R_0 x_0 \le 1, R_0 \in \mathbb{S}_+^{n_x \times n_x} \right\} \tag{2}$$

and the disturbance w satisfies the following ellipsoidal constraint

$$w \in \mathscr{W} = \left\{ w \in \mathbb{R}^{n_w} \,|\, w' R_w w \le 1, R_w \in \mathbb{S}_+^{n_w \times n_w} \right\}. \tag{3}$$

The estimation of the reachable set amounts to estimate the set of states that can be reached by starting inside $\mathscr{X}_0$ for any possible input disturbances in $\mathscr{W}$. In order to have a well-posed problem with bounded estimate, A is assumed to be Hurwitz.

The idea for estimating the reachable set is to consider a Lyapunov function $V(x)$ and find a region $\mathscr{R}_{\dot{V}}$ outside which the derivative $\dot{V}(x, w)$ of the Lyapunov function is negative definite for any $w \in \mathscr{W}$, i.e.

$$\mathscr{R}_{\dot{V}} = \left\{ x \in \mathbb{R}^{n_x} \,|\, \dot{V}(x, w) < 0, \forall w \in \mathscr{W} \right\} \tag{4}$$

where the derivative $\dot{V}(x, w)$ of the Lyapunov function depends on both x and w in view of the system dynamics in (1). In the following, the dependence of the derivative of the Lyapunov function on x and w will be omitted whenever obvious. We

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[1] The expression 'estimate of the reachable set' has to be intended as an *outer* estimate of the reachable set that contains the true reachable set.

have that, inside $\mathscr{R}_{\dot{V}}$, one cannot guarantee $\dot{V} < 0$. Let us now denote with $\mathscr{R}$ the region defined by the smallest level set of V which contains the region $\mathscr{R}_{\dot{V}}$, i.e.

$$\mathscr{R}_{\dot{V}} \subseteq \mathscr{R} = \left\{ x \in \mathbb{R}^{n_x} \mid V(x) \leq 1 \right\}. \tag{5}$$

Note that, without loss of generality we take the smallest level set to be one, since the Lyapunov function $V(x)$ can always be scaled appropriately with a positive scalar. Then, using invariance theory [25], $\mathscr{R}$ is an outer estimate of the reachable set, since, along the border of $\mathscr{R}$ every trajectory of (1) is pushed inside $\mathscr{R}$ for any $w \in \mathscr{W}$.

Summarizing, using the S-procedure [26], the estimation of the reachable set for the linear system (1) can be obtained by solving the following problem

$$\dot{V} - \lambda_1(w'R_w w - 1) - \lambda_2(1 - V) < 0$$
$$\lambda_1, \lambda_2 > 0 \tag{6}$$

which expresses the fact that $\dot{V}$ is negative definite for any $w \in \mathscr{W}$ and for x such that $V(x) \geq 1$. In addition, we need $\mathscr{X}_0 \subseteq \mathscr{R}$, i.e. $x_0'R_0 x_0 \leq V(x_0), \forall x_0 \in \mathscr{X}_0$. For a quadratic Lyapunov function $V(x) = x'Px$ and the linear system (6) the estimation of the reachable set can be written in the following LMI form

$$\max \mathrm{Tr}\, P$$

$$\text{s.t.}$$

$$\begin{bmatrix} A'P + PA + \lambda_2 P & PB \\ B'P & -\lambda_1 R_w \end{bmatrix} < 0 \tag{7}$$

$$\lambda_1, \lambda_2 > 0, \quad \lambda_1 \leq \lambda_2$$

$$R_0 \leq P$$

where the maximization of the trace of P is a way to make the outer estimate $\mathscr{R}$ as small as possible around the true reachable set.

We can now extend the previous ideas to switched linear systems in the form

$$\dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}w(t), \quad x(0) = x_0 \tag{8}$$

subject to (2) and (3), and with $\sigma \in \mathscr{M} = \{1, 2, \ldots, M\}$. In order to have a well-posed problem with bounded estimate, all matrices $A_\sigma, \sigma \in \mathscr{M}$ are assumed to be Hurwitz. Let us assume the switching signal $\sigma(\cdot)$ satisfies the following definition.

Definition 1 (*Dwell-time Switching*). For a switching signal $\sigma(\cdot)$ and time instants $t \geq \bar{t} \geq 0$, let $N(t, \bar{t})$ denote the number of discontinuities of σ in the open interval $[\bar{t}, t)$. The switching signal $\sigma(\cdot)$ is said to have dwell time τ_d if there exists a positive number τ_d such that

$$N(t, \bar{t}) \leq 1 + \frac{t - \bar{t}}{\tau_d}. \tag{9}$$

With this definition, we are now ready to formulate the problem at hand:

Problem 1 (*Reachable Set Estimation Under Dwell-time Switching*). Consider the switched linear system (8) subject to conditions (2), (3), and dwell-time switching (9). Find an outer estimate $\mathscr{R}$ of the reachable set, i.e. the level set of a Lyapunov function as in (5) such that along the border of $\mathscr{R}$ every trajectory of (8) is pushed inside $\mathscr{R}$ for any $w \in \mathscr{W}$.

3. Estimation based on exponential decrease/bounded increase properties

In stability of switched linear systems with dwell-time switching it is customary to use a multiple Lyapunov function exploiting the following properties [27]:

- *Exponential decrease*: in between switching instants, the Lyapunov function decreases exponentially;
- *Bounded increase*: at switching instants, there is a bound on the possible increase of the Lyapunov function.

Similar properties can be used to estimate the reachable set, as stated by the following two lemmas.

Lemma 1. *Consider the switched linear system* (8) *subject to conditions* (2), (3), *and* (9). *If there exist a family of positive definite Lyapunov functions* $V_i : \mathbb{R}^{n_x} \to \mathbb{R}_+, i \in \mathscr{M}$, *with* $V_i(0) = 0$, *positive scalars* $\lambda_1, \lambda_2 > 0$ *and* $\mu \geq 1$ *such that*

$$\dot{V}_i - \lambda_1(w'R_w w - 1) - \lambda_2(1 - V_i) < 0, \quad \forall i \in \mathscr{M} \tag{10a}$$

$$V_j(x(t_k)) \leq \mu V_i(x(t_k)), \quad \forall i \neq j \in \mathscr{M} \tag{10b}$$

where t_k are the switching instants, then, for the dwell time

$$\tau_d > \frac{\ln(\mu)}{\zeta} \tag{11}$$

with $0 < \zeta < \lambda_1$, the estimate of the set reachable by any initial state

$$x_0 \in \overline{\mathscr{X}}_0 = \left\{ x_0 \in \mathbb{R}^{n_x} \mid V_{\sigma(0)}(x_0) \leq \beta \right\} \supseteq \mathscr{X}_0 \tag{12}$$

with $\beta = \lambda_2/(\lambda_2 - \zeta)$ is

$$\mathscr{R} = \left\{ x(t) \in \mathbb{R}^{n_x} \mid V_{\sigma(t)}(x(t)) \leq \mu\beta \right\}. \tag{13}$$

Proof. First, define the Lyapunov function $V(x(t)) = V_{\sigma(t)}(x(t))$ (in the following the explicit dependence of this Lyapunov function on the state might be omitted when obvious, and only dependence on time will be indicated). Note that (10a) can be written as

$$\dot{V}_i - \lambda_1 w' R_w w + \lambda_1 V_i - \epsilon(1 - V_i) < 0, \quad \forall i \in \mathscr{M}$$

with $\lambda_2 = \lambda_1 + \epsilon$. After choosing $0 < \zeta < \lambda_1$ we have

$$\dot{V}_i + \zeta V_i + (\lambda_1 - \zeta)V_i - \lambda_1 w' R_w w - \epsilon(1 - V_i) \tag{14}$$

$$= \dot{V}_i + \zeta V_i + \left[(\lambda_2 - \zeta)V_i - \lambda_1 w' R_w w - \epsilon \right] \tag{15}$$

$$\leq \dot{V}_i + \zeta V_i + \left[(\lambda_2 - \zeta)V_i - \lambda_2 \right]. \tag{16}$$

Let us define $\beta = \lambda_2/(\lambda_2 - \zeta)$: in view of (16) we recognize two cases in between switches:

  (a) for $V(t) \geq \beta$ we have exponential decrease of $V(\cdot)$
  (b) for $V(t) < \beta$ then $V(\cdot)$ may be increasing.

Case (a). We assume that $V(t) \geq \beta$ for $t \in [\bar{t}, \bar{t} + T)$, where $T > 0$ is some time instant for which $V(\bar{t} + T) \geq \beta$. Then, exploiting the exponential decrease condition of (14) we have, for $\bar{t} \leq t \leq \bar{t} + T$,

$$V(t) \leq \mu^{N(t,\bar{t})} e^{-\zeta(t-\bar{t})} V(\bar{t})$$

$$\leq \mu e^{\left(\frac{\ln(\mu)}{\tau_d} - \zeta \right)(t-\bar{t})} V(\bar{t})$$

where second inequality has been obtained by using (9). We obtain $V(t) \leq \mu V(\bar{t})$ for $\tau_d > \ln(\mu)/\zeta$, which implies that there exists a time for which V enters the ball defined by β.

Case (b). If we are inside the ball defined by β, in view of (10a) we cannot exit the ball, unless a switching occurs. Furthermore, we have, at switching instants t_k,

$$V_{\sigma(t_k^+)}(x(t_k)) \leq \mu V_{\sigma(t_k)}(x(t_k))$$

which implies that a switch might bring me outside the ball β, after which, we are in case (a) again.

We conclude, by looking at the overall behavior of the Lyapunov function $V(\cdot)$, that (13) is an estimate of the reachable set, for all initial states satisfying (12). ∎

Remark 1. It is interesting to study the effect of ζ in (11) on the dwell time and on the estimate of the reachable set. We note that, for $\zeta \to 0$

$$\tau_d \to \infty, \quad \beta \to \frac{\lambda_1 + \epsilon}{\lambda_1 + \epsilon} = 1 \tag{17}$$

which gives us the maximum dwell time and the smallest estimate of the reachable set. On the other hand, for $\zeta \to \lambda_1$

$$\tau_d \to \frac{\ln(\mu)}{\lambda_1}, \quad \beta \to \frac{\lambda_1 + \epsilon}{\epsilon} > 1 \tag{18}$$

we obtain the minimum dwell time and the largest estimate of the reachable set.

A second lemma for estimation of the reachable set can now be stated, which is based also on exponential decrease/bounded increase conditions, but for the family of shifted Lyapunov functions $\tilde{V}_i = V_i - 1, i \in \mathscr{M}$.

Lemma 2. *Consider the switched linear system* (8) *subject to conditions* (2), (3), *and* (9). *If there exist a family of positive definite Lyapunov functions* $\tilde{V}_i + 1 = V_i : \mathbb{R}^{n_x} \to \mathbb{R}_+, i \in \mathcal{M}$, *with* $\tilde{V}_i(0) = -1$, *positive scalars* $\lambda_1, \lambda_2, \gamma, \phi, \delta > 0$, *and* $\mu \geq 1$ *such that*

$$\dot{\tilde{V}}_i - \lambda_1(w'R_w w - 1) + \lambda_2 \tilde{V}_i < 0, \quad \forall i \in \mathcal{M} \tag{19a}$$

$$\tilde{V}_j(x(t_k)) \leq \mu \tilde{V}_i(x(t_k)) - \gamma(\tilde{V}_i(x(t_k)) - \delta), \quad \forall i \neq j \tag{19b}$$

$$\tilde{V}_j(x(t_k)) \leq \mu\delta - \phi(\delta - \tilde{V}_i(x(t_k))), \quad \forall i \neq j \tag{19c}$$

where t_k are the switching instants, then, for the dwell time

$$\tau_d > \frac{\ln(\mu)}{\lambda_2} \tag{20}$$

the estimate of the set reachable by any initial state

$$x_0 \in \overline{\mathscr{X}}_0 = \left\{ x_0 \in \mathbb{R}^{n_x} \mid V_{\sigma(0)}(x_0) \leq 1 + \delta \right\} \supseteq \mathscr{X}_0 \tag{21}$$

is given by

$$\mathscr{R} = \left\{ x(t) \in \mathbb{R}^{n_x} \mid V_{\sigma(t)}(x(t)) \leq 1 + \mu\delta \right\}. \tag{22}$$

Proof. We first notice that (19b) implies

$$\tilde{V}_j(x(t_k)) \leq \mu \tilde{V}_i(x(t_k)) \text{ for } \tilde{V}_i(x(t_k)) \geq \delta, \quad \forall i \neq j \tag{23}$$

while (19c) implies

$$\tilde{V}_j(x(t_k)) \leq \mu\delta \text{ for } \tilde{V}_i(x(t_k)) \leq \delta, \quad \forall i \neq j. \tag{24}$$

Define the shifted Lyapunov function $\tilde{V}(x(t)) = \tilde{V}_{\sigma(t)}(x(t))$ (in the following the explicit dependence of this function on the state might be omitted when obvious, and only dependence on time will be indicated) and note that (19a) implies decrease of $\tilde{V}(\cdot)$ at exponential rate λ_2 for $\tilde{V}(t) \geq 0$. This time we define three cases

(a) for $\tilde{V}(t) \geq \delta$ we have exponential decrease of $\tilde{V}(\cdot)$ at rate λ_2, with possible jumps as in (23)
(b) for $0 \leq \tilde{V}(t) < \delta$ we have exponential decrease of $\tilde{V}(\cdot)$ at rate λ_2, with possible jumps as in (24)
(c) for $\tilde{V}(t) < 0$ then $\tilde{V}(\cdot)$ may be increasing, and we might have jumps as in (24).

Case (a). We assume that $\tilde{V}(t) \geq \delta$ for $t \in [\bar{t}, \bar{t} + T)$, where $T > 0$ is some time instant for which $\tilde{V}(\bar{t} + T) \geq \delta$. Then, exploiting the exponential decrease condition we have, for $\bar{t} \leq t \leq \bar{t} + T$,

$$\tilde{V}(t) \leq \mu^{N(t,\bar{t})} e^{-\lambda_2(t-\bar{t})} \tilde{V}(\bar{t})$$

$$\leq \mu e^{\left(\frac{\ln(\mu)}{\tau_d} - \lambda_2\right)(t-\bar{t})} \tilde{V}(\bar{t})$$

where the second inequality has been obtained by (9). We obtain $\tilde{V}(t) \leq \mu \tilde{V}(\bar{t})$ for $\tau_d > \ln(\mu)/\lambda_2$, which implies that there exists a time for which $\tilde{V}(\cdot)$ enters the ball defined by δ.

Case (b). We assume that at time $\bar{t}$ we have $0 \leq \tilde{V}(\bar{t}) < \delta$. Inside the ball defined by δ we still have exponential convergence in between switches. So, due to the bounded effect of the jump as in (19c), it suffices to find the dwell-time τ_d with which we will decay inside δ again after a jump, i.e.

$$\mu\delta e^{-\lambda_2 \tau_d} = \delta.$$

It turns out that $\tau_d > \ln(\mu)/\lambda_2$, i.e. with this dwell time we cannot leave the region defined by $\tilde{V}(t) \leq \mu\delta$.

Case (c). Finally, in this case we have that, in view of (19a), we cannot exit the ball defined by the zero level set of $\tilde{V}(\cdot)$, unless a switching occurs. In addition, at switching instants t_k,

$$\tilde{V}_{\sigma(t_k^+)}(x(t_k)) \leq \mu\delta$$

in which case we are in case (a) and (b) again.

We conclude, by looking at the overall behavior of the shifted Lyapunov function $\tilde{V}(\cdot)$, that for $\tau_d > \ln(\mu)/\lambda_2$ the estimate of the reachable set is (22) for all initial states satisfying (21).

Remark 2. The reason why Lemmas 1 and 2 are not symmetrical (Lemma 1 has only one jump condition (10b), while Lemma 2 has two jump conditions (19b) and (19c)) is that the condition $\tilde{V}_j(x) \leq \mu \tilde{V}_i(x)$ with $\mu \geq 1$ cannot be ensured in the entire state space. For this reason, we impose that $\tilde{V}_j(x) \leq \mu \tilde{V}_i(x)$ is valid only outside a certain ball defined by δ.

Remark 3. Both Lemmas 1 and 2 can be implemented numerically by means of linear matrix inequalities with quadratic Lyapunov functions. The disadvantage of Lemma 2 with respect to Lemma 1 is that it involves more multipliers. On the other hand, Lemma 2 might give some advantage since, differently from Remark 1, it may not require to enlarge the estimate of the reachable set if we decrease the dwell time.

4. Estimation without exponential decrease/bounded increase properties

The previous lemmas are based on the well-known exponential decrease/bounded increase conditions. We now provide alternative conditions which exploit the evolution of the system in between two switches. First the following definition must be given.

Definition 2 (τ-reachable Set). Consider the switched linear system (8) under conditions (2) and (3), and dwell-time switching τ_d. The τ-reachable set is the set that can be reached by the portions of trajectories of (8) defined in $[t_k + \tau, t_{k+1})$, $k \in \mathbb{N}_+$, where t_k are the switching times.

The meaning of the τ-reachable set is that, instead of looking at the entire trajectory of (8), we neglect the portions of trajectories defined in the open interval $[t_k, t_k + \tau)$, and we look at the remaining portions of trajectories defined in $[t_k + \tau, t_{k+1})$. Since we are dealing with switched linear systems with dwell-time switching τ_d, it is relevant to estimate where the trajectories are after the dwell time τ_d, i.e. have an estimate of the τ_d-reachable set. Let us denote such estimate with $\mathscr{R}_{\tau_d}$. An estimate of the τ_d-reachable set is provided by the following lemma.

Lemma 3. Consider the switched linear system (8) subject to conditions (2), (3), and (9). If there exist a family of positive definite Lyapunov functions $V_i : \mathbb{R}^{n_x} \to \mathbb{R}_+$, $i \in \mathscr{M}$, with $V_i(0) = 0$, positive scalars $\lambda_1, \lambda_2, \lambda_3, \lambda_4 > 0$ such that

$$\dot{V}_i - \lambda_1(w'R_w w - 1) - \lambda_2(1 - V_i) < 0, \quad \forall i \in \mathscr{M} \tag{25a}$$

$$\begin{aligned} V_j(x(t_k + \tau_d)) &\leq V_i(x(t_k)) \\ &+ \lambda_3(w'R_w w - 1) - \lambda_4(V_i(x(t_k)) - 1), \quad \forall i \neq j \end{aligned} \tag{25b}$$

where t_k are the switching instants, then for any initial state in the set

$$x_0 \in \overline{\mathscr{X}}_0 = \left\{ x_0 \in \mathbb{R}^{n_x} \mid V_{\sigma(0)}(x_0) \leq 1 \right\} \supseteq \mathscr{X}_0 \tag{26}$$

an estimate of the τ_d-reachable set is given by

$$\mathscr{R}_{\tau_d} = \left\{ x(t) \in \mathbb{R}^{n_x} \mid V_{\sigma(t)}(x(t)) \leq 1 \right\} \tag{27}$$

for $t \in [t_k + \tau_d, t_{k+1})$.

Proof. The proof is based on invariant set theory. Define the Lyapunov function $V(x(t)) = V_{\sigma(t)}(x(t))$. In between switches, condition (25a) guarantees that, for every disturbance satisfying (3), any state x for which $V(x) \geq 1$ satisfies $\dot{V} < 0$, $w \in \mathscr{W}$, which means that the trajectories outside $V(x) \geq 1$ are attracted inside the level set, and the trajectories already inside the level set will not leave the set.

The occurrence of a switch from subsystem i to j might cause the trajectory to leave the set $V(x) \leq 1$: however, for every point on the border of the level set $V_i(x) = 1$, condition (25b) implies that $V_j(x(t_k + \tau_d)) \leq V_i(x(t_k)) \leq 1$, i.e. we inside the level set again. This, together with uniqueness of the solution of the switched system, implies that any point inside the set $V(x) \leq 1$ will return inside the set at time τ_d after any switch. As a result (27) is an estimate of the τ_d-reachable set for any initial state in (26), which concludes the proof.

Lemma 3 does not provide directly any estimate $\mathscr{R}$ of the reachable set, but it suggests a way to estimate the reachable set. The idea is to find, for every $0 \leq T < \tau_d$ and for the same Lyapunov function in (25), the level set κ of V such that

$$\mathscr{R}_T = \left\{ x(t) \in \mathbb{R}^{n_x} \mid V_{\sigma(t)}(x(t)) \leq \kappa \right\} \tag{28}$$

for $t \in [t_k, t_k + T)$. By doing this, we are parameterizing the estimate of the reachable set as a function of T, where T denotes the distance from the last switching instants. This is formalized in the second part of Lemma 3, whose straightforward proof is omitted.

Lemma 3 (Continued). For the same family of positive definite Lyapunov functions $V_i : \mathbb{R}^{n_x} \to \mathbb{R}_+$, $i \in \mathscr{M}$, of the first part Lemma 3, if there exist positive scalars $\lambda_5, \lambda_6 > 0$ and $\kappa \geq 1$ such that

$$V_j(x(t_k + T)) \leq \kappa + \lambda_5(w'R_w w - 1) - \lambda_6(1 - V_i(x(t_k))), \quad \forall i \neq j, \ T \in [0, \tau_d) \tag{29}$$

where t_k are the switching instants, then for any initial state in the set

$$x_0 \in \overline{\mathscr{X}}_0 = \left\{ x_0 \in \mathbb{R}^{n_x} \mid V_{\sigma(0)}(x_0) \leq 1 \right\} \supseteq \mathscr{X}_0 \tag{30}$$

an estimate of the T-reachable set is given by

$$\mathscr{R}_T = \left\{ x(t) \in \mathbb{R}^{n_x} \mid V_{\sigma(t)}(x(t)) \leq \kappa \right\} \tag{31}$$

for $t \in [t_k + T, t_{k+1})$.

By exploiting the result in the second part of Lemma 3, not only the reachable set can be estimated for $T \to 0$, but it is possible to define different estimates $\mathscr{R}_T$ of the T-reachable set depending on the distance T from the switch: eventually, for $T \to \tau_d$, we have $\kappa \to 1$ and convergence of $\mathscr{R}_T$ to $\mathscr{R}_{\tau_d}$.

5. Numerical implementation of Lemma 3

The last step is how to implement the conditions in Lemma 3. In the following we propose two numerical approaches via Linear Matrix Inequalities (LMIs): the first one is based on the matrix exponential of the state subsystem matrices, while the second one is based on a time-scheduled Lyapunov function.

5.1. Matrix exponential-based approach

Take a set of disturbance inputs $w^{(s)}, s = 1, \ldots, S$, whose convex hull $\mathscr{W}_c$ satisfies $\mathscr{W} \subseteq \mathscr{W}_c$. The numerical implementation of the matrix exponential-based approach is formalized in the following theorem.

Theorem 1. *If the following is satisfied*

$$\max \sum_{m=1}^{M} \mathrm{Tr}\, P_m \tag{32a}$$

s.t.
$$\begin{bmatrix} A_i'P_i + P_iA_i + \lambda_{2_i}P_i & B_i'P_i \\ P_iB_i & -\lambda_{1_i}R_w \end{bmatrix} < 0 \tag{32b}$$

$$\lambda_{1_i}, \lambda_{2_i} > 0, \quad \lambda_{1_i} \leq \lambda_{2_i} \tag{32c}$$

$$\begin{bmatrix} e^{A_i'\tau_d}P_j e^{A_i\tau_d} - P_i + \lambda_4 P_i & e^{A_i'\tau_d}P_j\overline{B}_i w^{(s)} \\ w^{(s)\prime}\overline{B}_i'P_j e^{A_i\tau_d} & w^{(s)\prime}\overline{B}_i'P_j\overline{B}_i w^{(s)} - \lambda_{4_i} \end{bmatrix} \leq 0 \tag{32d}$$

$$\lambda_{4_i} > 0, \quad s = 1, \ldots, S \tag{32e}$$

$$R_0 \leq P_i \tag{32f}$$

where $\overline{B}_i = \int_0^{\tau_d} e^{A_i(\tau_d - s)}B_i ds$, then an outer estimate of the τ_d-reachable set is

$$\mathscr{R}_{\tau_d} = \left\{ x(t) \in \mathbb{R}^{n_x} \mid x'(t)P_{\sigma(t)}x(t) \leq 1 \right\} \tag{33}$$

for $t \in [t_k + \tau_d, t_{k+1})$. Once the family of Lyapunov function has been found via (32a)–(32f), the estimates of the T-reachable sets at different time instants T can be calculated from

$$\min \kappa \tag{34a}$$

s.t.
$$\begin{bmatrix} e^{A_i'T}P_j e^{A_iT} - \lambda_6 P_i & e^{A_i'T}P_j\check{B}_i w^{(s)} \\ w^{(s)\prime}\check{B}_i'P_j e^{A_iT} & w^{(s)\prime}\check{B}_i'P_j\check{B}_i w^{(s)} - \kappa + \lambda_{6_i} \end{bmatrix} \leq 0 \tag{34b}$$

$$\lambda_{6_i} > 0, \ \kappa \geq 1 \tag{34c}$$

where $\check{B}_i = \int_0^{T} e^{A_i(T-s)}B_i ds$ and the estimate is given by

$$\mathscr{R}_T = \left\{ x(t) \in \mathbb{R}^{n_x} \mid x'(t)P_{\sigma(t)}x(t) \leq \kappa \right\} \tag{35}$$

for $t \in [t_k + T, t_{k+1})$.

Proof. The proof follows directly from the two steps of Lemma 3, taking into account that (32) derives from (25), while (34) derives from (29). In particular, the following reasoning lies behind the LMIs (32) and (34). For a linear system we have, when the subsystem i is active in the interval $[t_k, t_k + \tau_d)$

$$x(t_k + \tau_d) = e^{A_i \tau_d} x(t_k) + \int_0^{\tau_d} e^{A_i(\tau_d - s)} B_i w ds \qquad (36)$$

We observe that, for a constant $w^{(s)}$, we can bring $w^{(s)}$ outside the integral and write (32d) as a result of condition (25b). The main intuition is that, by using the superposition principle for linear systems, if the input is a convex combination of $w^{(s)}$, then the state will also be the convex combination of the resulting outputs, which leads to evaluating different LMIs for different $w^{(s)}$. The proof is concluded by substituting (36) in (25b) and (29), and by noticing that multipliers λ_3 in (25b) and λ_5 in (29) disappear due to substituting the variable w with its realizations $w^{(s)}$, $s = 1, \ldots, S$.

Remark 4. Note that in (32) and (34) no exponential decrease/ bounded increase property of the Lyapunov function is used, but rather the evolution of the subsystems via the subsystem state matrices. With respect to this, the approach can be regarded as an extension of the famous stability condition in [28]. Also note that the idea of having different estimates $\mathcal{R}_T$, depending on the time passed after the switch, can be used with Lemmas 1 and 2 as well, with the difference that more conservative conditions will be obtained, due to the need for the exponential decrease/bounded increase properties.

5.2. Time-scheduled based approach

A second approach to the numerical solution of Lemma 3, can be derived by observing that the idea of having different estimates $\mathcal{R}_T$, $0 \leq T < \tau_d$ can be formalized via the celebrated time-scheduled Lyapunov function approach [29,30]. The crucial idea behind the time-scheduled Lyapunov function approach is that the Lyapunov function can be taken to be time-dependent in between t_k and $t_k + \tau_d$, and constant in between $t_k + \tau_d$ and t_{k+1}. Time dependence is created by partitioning the dwell time $\tau_d = \sum_{l=0}^{L} \delta_{\tau_d}^{[l]}$, by taking different 'samples' $P_i^{[l]}$, $l = 0, \ldots, L$ of the Lyapunov matrix and by interpolating them as

$$P(t) = \begin{cases} P_i^{[l]} + (P_i^{[l+1]} - P_i^{[l]}) \dfrac{t - t_k}{\delta_{\tau_d}^{[l+1]}} & \begin{aligned} t_k + \sum_{r=0}^{l-1} \delta_{\tau_d}^{[r]} &\leq t \\ &\leq t_k + \sum_{r=0}^{l} \delta_{\tau_d}^{[r]} \end{aligned} \\[2em] P_i^{[L]} & t_k + \sum_{r=0}^{L} \delta_{\tau_d}^{[r]} \leq t \leq t_{k+1} \end{cases} \qquad (37)$$

where $i = \sigma(t_k)$. The numerical implementation of the time-scheduled based approach is formalized in the following theorem.

Theorem 2. *If the following is satisfied*

$$\max \sum_{m=1}^{M} \sum_{l=1}^{L} \operatorname{Tr} P_m^{[l]} \qquad (38a)$$

s.t.

$$\begin{bmatrix} \dot{P}_i^{[l]} + A_i' P_i^{[l]} + P_i^{[l]} A_i + \lambda_{2_i} P_i^{[l]} & B_i' P_i^{[l]} \\ P_i^{[l]} B_i & -\lambda_{1_i} R_w \end{bmatrix} < 0 \qquad (38b)$$

$$\begin{bmatrix} \dot{P}_i^{[l]} + A_i' P_i^{[l+1]} + P_i^{[l+1]} A_i + \lambda_{2_i} P_i^{[l+1]} & B_i' P_i^{[l+1]} \\ P_i^{[l+1]} B_i & -\lambda_{1_i} R_w \end{bmatrix} < 0 \qquad (38c)$$

$$\begin{bmatrix} A_i' P_i^{[L]} + P_i^{[L]} A_i + \lambda_{2_i} P_i^{[L]} & B_i' P_i^{[L]} \\ P_i^{[L]} B_i & -\lambda_{1_i} R_w \end{bmatrix} < 0 \qquad (38d)$$

$$P_i^{[L]} - P_j^{[0]} \geq 0, \quad l = 0, \ldots, L-1, \quad i \neq j \qquad (38e)$$

$$\lambda_{1_i}, \lambda_{2_i} > 0, \quad \lambda_{1_i} \leq \lambda_{2_i} \qquad (38f)$$

$$R_0 \leq P_i^{[L]} \qquad (38g)$$

where $\dot{P}_i^{[l]} = \dfrac{P_i^{[l+1]} - P_i^{[l]}}{\delta_{\tau_d}^{[l+1]}}$ *and* $\bar{B}_i = \int_0^{\tau_d} e^{A_i(\tau_d - s)} B_i ds$, *then an outer estimate of the* τ_d-*reachable set is*

$$\mathcal{R}_{\tau_d} = \left\{ x(t) \in \mathbb{R}^{n_x} \mid x'(t) P_{\sigma(t)}^{[L]} x(t) \leq 1 \right\} \qquad (39)$$

for $t \in [t_k + \tau_d, t_{k+1})$. Furthermore, an outer estimate of the T-reachable set is given by

$$\mathcal{R}_T = \left\{ x(t) \in \mathbb{R}^{n_x} \mid x(t)'P_{\sigma(t)}(T)x(t) \leq 1 \right\} \tag{40}$$

for $t \in [t_k + T, t_{k+1})$.

Proof. While the matrix exponential-based approach involves two steps (32) and (34) in the spirit of Lemma 3, the explicit definition of the interpolation (37), makes it possible for the time-scheduled based approach to require a unique optimization step (38). For this peculiar characteristic, it is more straightforward to prove Theorem 2 via a special case of Lemmas 1 and 2 with $\mu = 1$: first, note that for $\mu = 1$ Lemmas 1 and 2 coincide (for Lemma 2 we can select $\gamma = 0$ and $\phi = \mu$); second, note that Lemmas 1 and 2 are valid also for non-autonomous Lyapunov functions (i.e. explicitly dependent on time, e.g. as in (37)).

After these observations, we take $V(t) = x'(t)P_{\sigma(t)}(t)x(t)$, with $P_{\sigma(t)}(t)$ as in (37), and we note that conditions (38) amount to the same conditions in Lemma 1 with $\mu = 1$: in particular, using Lemma 3 in [29] we derive that (38b) and (38c) implies (10a) in between $[t_k, t_k + \tau_d)$ (with $\dot{P}_i^{[l]}$ being part of the time derivative of the Lyapunov function (37)). In addition, (38d) implies (10a) in between $[t_k + \tau_d, t_{k+1})$. At switching times, we have that (38e) implies $V_{\sigma(t_k^+)}(x(t_k)) \leq V_{\sigma(t_k)}(x(t_k))$. The proof is concluded by noticing that the estimate of the set reachable is given by

$$\mathcal{R} = \left\{ x(t) \in \mathbb{R}^{n_x} \mid V_{\sigma(t)}(x(t)) \leq 1 \right\}$$

which, by using the interpolation (37), results in the estimate of the τ_d-reachable set (39) and the estimate of the T-reachable set (40). □

The following remarks follow:

Remark 5. As revealed by the proof of Theorem 2, the time-scheduled Lyapunov function approach provides a nice connection between Lemmas 1 and 2 (for $\mu = 1$), and Lemma 3. One might be tempted to think that solving Lemmas 1 and 2 for $\mu = 1$ requires a common Lyapunov function (in view of the condition $V_j(x(t_k)) \leq V_i(x(t_k))$, which for a common Lyapunov function automatically becomes $V_j(x(t_k)) = V_i(x(t_k))$). The time-scheduled Lyapunov function approach, however, overcomes the need for a common Lyapunov function by allowing

$$x'(t_k)P_{\sigma(t_k^+)}^{[0]}x(t_k) \leq x'(t_k)P_{\sigma(t_k)}^{[L]}x(t_k)$$

with possibly different Lyapunov functions.

Remark 6. The time-scheduled Lyapunov function approach might be computationally more expensive than the matrix exponential -based approach, since it involves more positive definite matrices $P_i^{[l]}$, $l = 0, \ldots, L$. On the other hand, it requires to select less multipliers (in particular, we do not need the multipliers λ_{4_i} in (32d)) and, furthermore, it does not require to approximate the set (3) with a convex hull $\mathcal{W}_c$ that satisfies $\mathcal{W} \subseteq \mathcal{W}_c$.

6. Numerical example

The following switched system is used to validate the proposed ideas

$$A_1 = \begin{bmatrix} -0.5 & -0.4 \\ 3 & -0.5 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} -0.5 & -3 \\ 0.4 & -0.5 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix}$$

which is a disturbance version of the celebrated switched system example with 'orthogonal' phase planes.

For Lemma 1 we used $\lambda_1 = 0.3$, $\lambda_2 = 0.4$, and $\mu = 5$. Fig. 1 shows the results of Lemma 1 with infinite dwell time ($\zeta \to 0$) and smallest estimate of reachable set (level set $\beta = 1$): we can verify that the reachable set coincides with the reachable sets of the stand-alone subsystems. Fig. 1 also shows the results of Lemma 1 with minimum dwell time ($\zeta \to \lambda_1$, $\tau_d = 5.36$) and largest estimate of reachable set (level set $\beta = 5$). Validation of Lemma 1 (for minimum dwell time) for 1000 initial conditions inside the initial set is presented in Fig. 2.

For Lemma 2 we used $\lambda_1 = 0.3$, $\lambda_2 = 0.4$, $\mu = 7.5$, $\phi = 5.0$, $\delta = 1.6$, and $\gamma = 2.5$. Fig. 3 shows the results of Lemma 2 where, differently from Lemma 1, the size of the estimate of reachable set (level set $1 + \mu\delta = 13$) is not influenced by the dwell time ($\tau_d = 5.04$). It is then interesting to compare Lemmas 1 and 2: for this example we see that Lemma 2 leads to not only a smaller dwell time, but also a smaller estimate of the reachable set than Lemma 1. Validation of Lemma 2 for 1000 initial conditions inside the initial set is presented in Fig. 4.

For Lemma 3 (using the matrix exponential-based implementation) we used $\lambda_1 = 0.3$, $\lambda_2 = 0.4$, $\lambda_4 = 0.3$, and $\lambda_6 = 0.5$. Fig. 5 shows the results of Lemma 3 for the dwell time $\tau_d = 5.04$: the estimates of T-reachable sets are given at the following

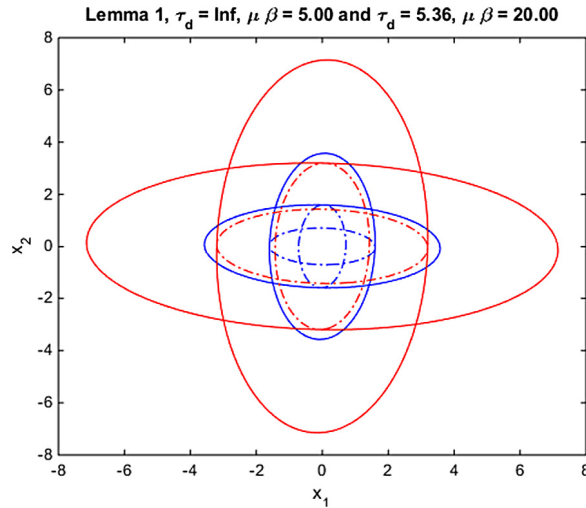**Fig. 1.** Lemma 1: infinite dwell time: $\overline{\mathcal{X}}_0$ (smaller dash-dotted), $\mathcal{R}$ (smaller solid); minimum dwell time: $\overline{\mathcal{X}}_0$ (larger dash-dotted), $\mathcal{R}$ (larger solid).
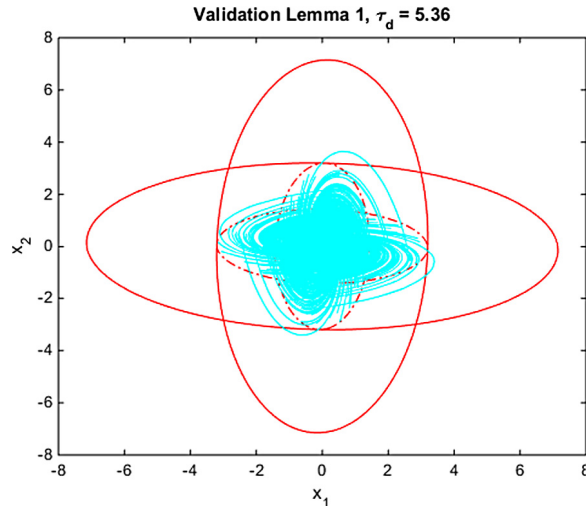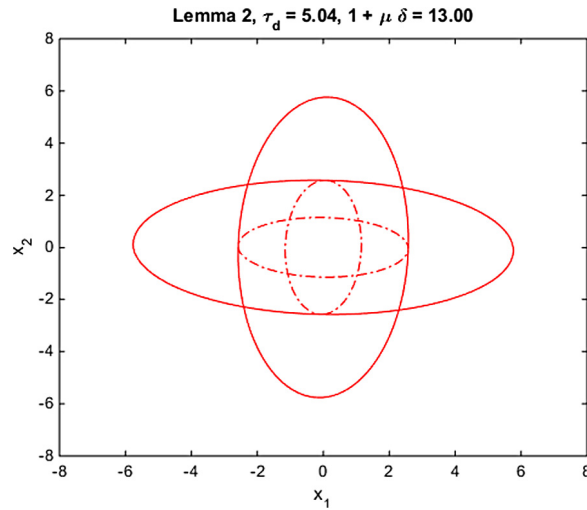

Fig. 2. Validation of Lemma 1.

time instants from the switch: 0, 1.01, 2.01, 3.02, and 4.03. The corresponding level sets are $\kappa = 5.98, 4.81, 3.20, 2.02, 1.31$, which decrease toward one as indicated by Lemma 3. The smallest set in Fig. 5 corresponds to the estimate of the τ_d-reachable set: it can be also seen that the estimates are sensibly smaller than in Lemmas 1 and 2 (note the different scale in Fig. 5 and following figures). Validation of Lemma 3 for 1000 initial conditions inside the initial set is presented in Fig. 6.

For Lemma 3 (using the time-scheduled Lyapunov function implementation) we used $\lambda_1 = 0.3$ and $\lambda_2 = 0.4$. In order to have a fair comparison with the exponential matrix-based implementation, we split the dwell time $\tau_d = 5.04$ in 5 equal parts, which correspond the same instants previously mentioned: 1.01, 2.01, 3.02, and 4.03. Fig. 7 shows the estimates of T-reachable sets: the smallest set in Fig. 7 corresponds to the estimate of the τ_d-reachable set. It can be also seen that the estimates are comparable with the ones in the matrix exponential-based implementation. Validation of this implementation of Lemma 3 for 1000 initial conditions inside the initial set is presented in Fig. 8.

Finally, Table 1 compares the approaches from a computational point of view, in terms of the decision variables and inequalities arising from the solution of Lemmas 1–3. It can be seen that Lemmas 1 and 2 are computationally less intensive (while giving more conservative results): in particular, Lemma 2 is slightly more intensive than Lemma 2 as discussed in Remark 3. Lemma 3 (in its double implementation, the matrix exponential-based or the time-scheduled based) involves more decision variables and inequalities: in particular, for the example at hand, the matrix exponential-based approach with $S = 6$ (the number of points used to approximate the disturbance set) is less costly than the time-scheduled based approach

Table 1
Computational cost of different lemmas.

	No. decision var.	No. constraints
Lemma 1	16	6
Lemma 2	20	7
Lemma 3, $S = 6$ (matrix exp.)	84	8
Lemma 3, $L = 5$ (time sched.)	116	36

Fig. 3. Lemma 2: $\overline{\mathscr{X}}_0$ (dash-dotted), $\mathscr{R}$ (solid).
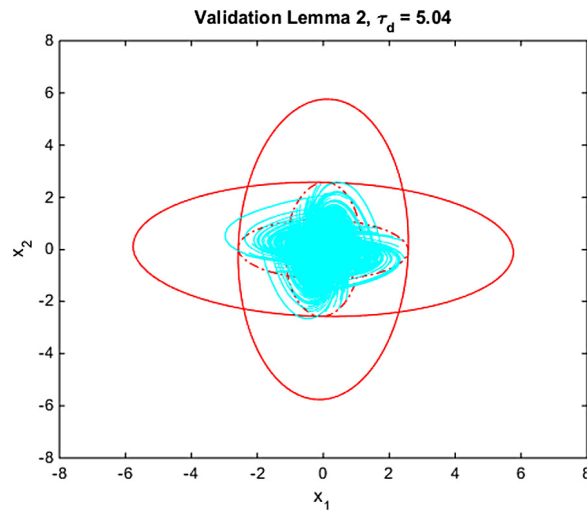
Fig. 4. Validation of Lemma 2.

with $L = 5$ (the number of points used to interpolate the Lyapunov function). This is consistent with what discussed in Remark 6. Finally, it has to be noted that the computational cost of the matrix exponential-based approach will increase with S, while the computational cost of the time-scheduled based approach will increase with L.

7. Conclusions

This work addressed the problem of (outer) estimation of reachable sets in switched linear systems subject to dwell-time switching. The main contribution of this work was to overcome the need for exponential decrease/bounded increase of the Lyapunov function (i.e. exponential decrease in between switching times and bounded increase at switching times).
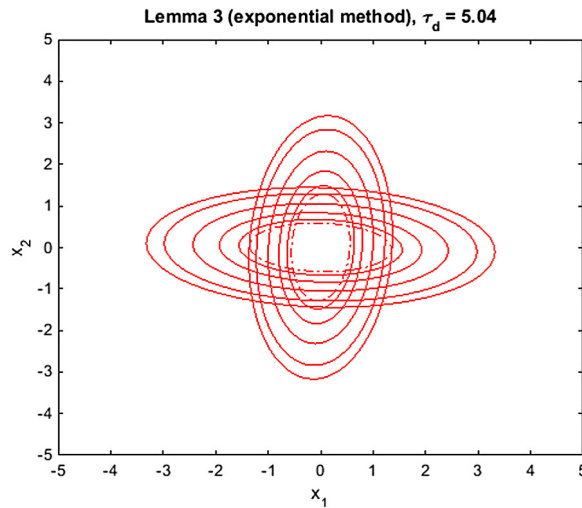
Fig. 5. Lemma 3 (exponential matrix-based): $\mathscr{R}_{\tau_d}$ (dash-dotted), $\mathscr{R}_T$ for different T (solid).
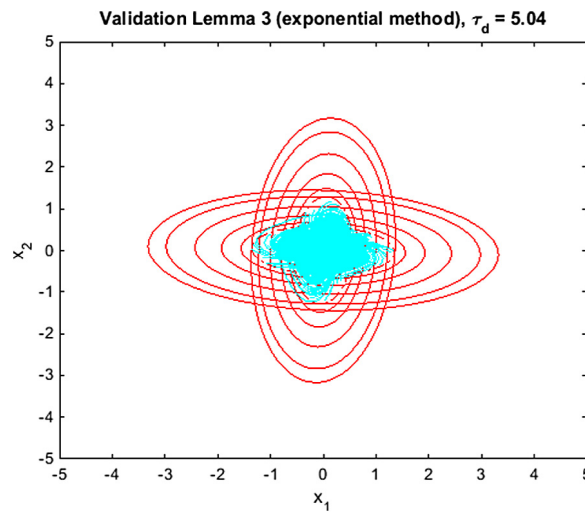

Fig. 6. Validation of Lemma 3 (exponential matrix-based).

This was done by introducing a new notion of reachable set: the τ-reachable set, i.e. the set that can be reached by the trajectories defined at time τ after the switch. Such extended notion of reachable set have been used to parametrize the estimate of the reachable set as a function of the distance from the switch. In this way one can obtain an 'envelope' of estimates depending how much time passed from the last switch. Two numerical approaches have been provided to implement such parametrization: the first approach exploits the evolution of the system in between switches via the matrix exponential of the state subsystem matrix, while the second approach exploits a time-scheduled Lyapunov function approach. Both approaches can be implemented via linear matrix inequalities, and a numerical example was provided to show the effectiveness of the proposed methods.
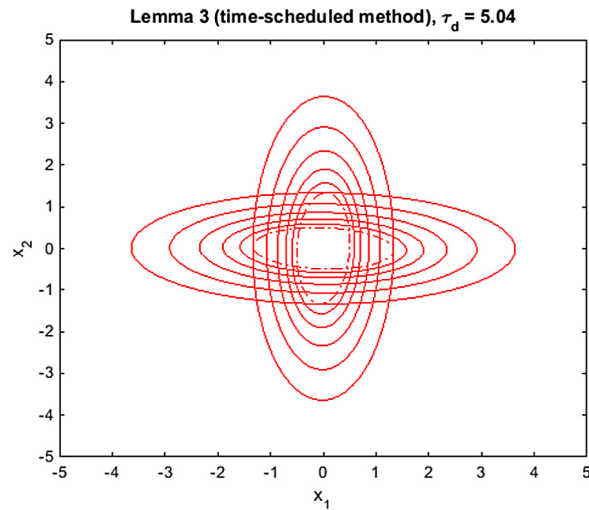
Acknowledgment

Fig. 7. Lemma 3 (time-scheduled Lyapunov): $\mathscr{R}_{\tau_d}$ (dash-dotted), $\mathscr{R}_T$ for different T (solid).
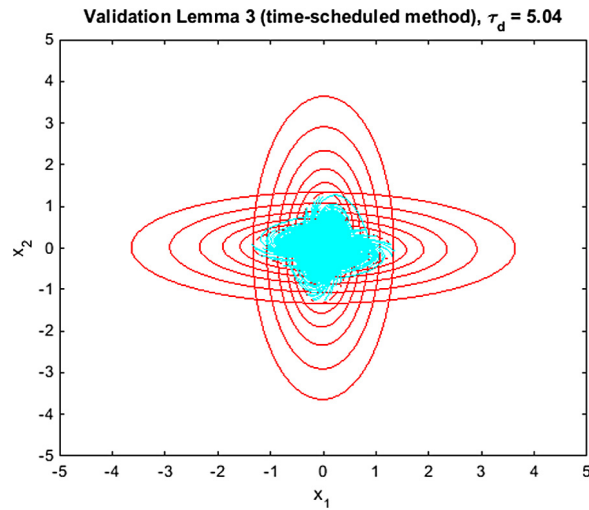

Fig. 8. Validation of Lemma 3 (time-scheduled Lyapunov).

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