1. A control system $\dot{x} = Ax + Bu$, $y = Cx$ transforms, under a change of basis $x = P\bar{x}$ in the state space, to a control system $\dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}u$, $y = \bar{C}\bar{x}$ (note that the input u and output y stay the same, only the state x changes).

a) Derive the formulas for the new matrices $\bar{A}, \bar{B}, \bar{C}$ in terms of the original matrices A, B, C.

b) Verify that the new system has the same transfer function (or transfer matrix) as the original one.

Solution:

a) $\bar{x} = P^{-1}x$, so $\dot{\bar{x}} = P^{-1}\dot{x} = P^{-1}(Ax + Bu) = P^{-1}AP\bar{x} + P^{-1}Bu$, and $y = Cx = CP\bar{x}$.

Answer: $\bar{A} = P^{-1}AP$, $\bar{B} = P^{-1}B$, $\bar{C} = CP$.

b)
$$\begin{aligned}\bar{G}(s) &= \bar{C}(Is - \bar{A})^{-1}\bar{B} = CP(Is - P^{-1}AP)^{-1}P^{-1}B = CP\big[P^{-1}(Is - A)P\big]^{-1}P^{-1}B \\ &= C\cancel{PP^{-1}}(Is - A)^{-1}\cancel{PP^{-1}}B = G(s)\end{aligned}$$

2. Consider the system
$$\dot{x} = \begin{pmatrix} a & 1 \\ 0 & -1 \end{pmatrix} x + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u$$
where $x \in \mathbb{R}^2$, $u \in \mathbb{R}$, and $a \in \mathbb{R}$ is an unknown parameter. Suppose this system is known to have the following two properties:

(i) If we turn off the control (set $u(t) = 0$ for all t), the solutions asymptotically converge to the origin from all initial states.

(ii) From every initial state it is possible to reach the origin *in finite time* using some control.

Based on this information, determine all possible values of the parameter a.

Solution:

Property (i) tells us that $\dot{x} = Ax$ must be asymptotically stable, hence A must be Hurwitz. The eigenvalues of A are a and -1, so a must be negative.

Property (ii) tells us that the system must be controllable. The controllability matrix is
$$\mathcal{C}(A, B) = \begin{pmatrix} 1 & a+1 \\ 1 & -1 \end{pmatrix}$$
and this is singular when $a + 1 = -1$, which means $a = -2$ must be excluded.

Answer: all $a < 0$ except $a = -2$.

3. In class we defined the *unobservable subspace* of an LTV system $\dot{x} = A(t)x$, $y = C(t)x$ to be the nullspace (kernel) of the observability Gramian which is defined to be the matrix
$$M = \int_{t_0}^{t_1} \Phi^T(t, t_0)C^T(t)C(t)\Phi(t, t_0)dt$$

where Φ is the state transition matrix for $A(t)$ and t_0, t_1 are given times.

a) Prove that for an LTI system $\dot{x} = Ax$, $y = Cx$ the unobservable subspace equals the nullspace of the observability matrix $\mathcal{O}(A, C)$.

b) Briefly explain the meaning of the unobservable subspace in the context of recovering x from y (again for the LTI case).

Solution:

a) For the LTI system we have

$$M = \int_{t_0}^{t_1} e^{A(t-t_0)} C^T C e^{A(t-t_0)} dt$$

If a vector $z \in \mathbb{R}^n$ is in the nullspace of M, then

$$0 = z^T M z = \int_{t_0}^{t_1} z^T e^{A(t-t_0)} C^T C e^{A(t-t_0)} z \, dt = \int_{t_0}^{t_1} |C e^{A(t-t_0)} z|^2 dt$$

which means that $Ce^{As}z$ must equal 0 for all $s \in [0, t_1 - t_0]$. Thus the function $s \mapsto Ce^{As}z$ and all its derivatives with respect to s must vanish at $s = 0$: $Cz = CAz = \cdots = CA^{n-1}z = 0$. This implies that z must be in the nullspace of the observability matrix

$$\mathcal{O}(A, C) = \begin{pmatrix} C \\ CA \\ \cdots \\ CA^{n-1} \end{pmatrix}$$

Conversely, if z is in the nullspace of $\mathcal{O}(A, C)$ then reversing the above steps we see that it is in the nullspace of M.

b) Some examples of correct statements:

– Initial states in the unobservable subspace cannot be distinguished from the origin.

– Initial states whose difference lies in the unobservable subspace are indistinguishable from one another.

– The initial condition (and hence the entire state trajectory) can be recovered from the output up to a component lying in the unobservable subspace (which cannot be recovered).