CYCLIC PURSUIT WITHOUT COORDINATES: CONVERGENCE to REGULAR POLYGON FORMATIONS

Daniel Liberzon

Joint work with Maxim Arnold (now at UT Dallas) and Yuliy Baryshnikov

University of Illinois, Urbana-Champaign, U.S.A.
\[ \dot{x}_i = u_i \cos \phi_i \]
\[ \dot{y}_i = u_i \sin \phi_i \]
\[ \dot{\phi}_i = w_i \]

• Can detect presence of another agent in windshield sector \((-\alpha, \alpha)\) (unlimited sensing range)

• Assume forward speed is constant & fixed (say \(u_i \equiv 1\))

• Angular speed \(w_i \in \{-\bar{w}, 0, \bar{w}\}\)

• Agents cyclically arranged, with each agent \(i\) seeing its target agent \(i+1\) in its windshield (connectivity) – can be maintained if \(\bar{w}\) is large enough

• System behavior depends critically on windshield angle \(\alpha\)
SMALL WINDSHIELD ANGLE: RENDEZVOUS

$$\alpha < \frac{\pi}{n} \quad \text{where} \quad n = \# \text{ of agents}$$

In this case all agents converge to a single point, with perimeter $P(t)$ of the formation polygon serving as a Lyapunov function [Yu–LaValle–L, IEEE TAC, 2012]
LARGE WINDSHIELD ANGLE: REGULAR POLYGONS?

\[ \alpha > \frac{\pi}{n} \quad \text{where} \quad n = \# \text{of agents} \]

Agents diverge but tend to form regular polygons

Our goal: theoretically justify this empirically observed phenomenon
RELATION to PRIOR WORK

• Many works on convergence of multi-agent formations to regular shapes, using different tools and different modeling assumptions:
  Behroozi–Gagnon (1980s), Richardson (2001),

• Unique feature of our model (same as in Yu–LaValle–L, 2012):
  neither inter-agent distance nor heading error available for feedback

• As in Marshall et al., we will use eigenvalue properties of
  block-circulant matrices to show convergence

• But unlike in Marshall et al., here regular shape is attractive while
  formation size grows, which makes analysis different
SYSTEM EQUATIONS

Angles are subject to constraint $\sum_{i=1}^{n} \psi_i = \pi(n - 2k)$

\[
\dot{\psi}_i = \frac{(\sin \alpha_{i-1} + \sin(\psi_i + \alpha_i))/l_i}{l_i} - \frac{(\sin \alpha_i + \sin(\psi_{i+1} + \alpha_{i+1}))/l_{i+1}}{l_{i+1}}
\]

\[
\dot{l}_i = -\cos \alpha_{i-1} - \cos(\psi_i + \alpha_i)
\]

“Constant-bearing” case: assume $\alpha_i \equiv \alpha \ \forall i$, i.e., each $a_i$ maintains its target $a_{i+1}$ exactly on its windshield boundary

This is sliding regime for angular velocity $\omega_i \in \{-\bar{\omega}, 0, \bar{\omega}\}$
STATIONARY SHAPES ARE REGULAR POLYGONS

A shape is an equivalence class of polygons w.r.t. scaling & rigid motions.

A shape is stationary if it is invariant under system dynamics.

For a shape to be stationary we must have $\dot{\psi}_i \equiv 0 \ \forall i$.

Can show that then all angles must be equal: $\psi_i \equiv \psi \ \forall i$.

So all edges have the same derivative:

$\ell_i(t) = \ell_i(0) - t (\cos \alpha + \cos(\psi + \alpha))$

Example: convex regular $n$-gon

$\psi = \pi - 2\pi/n$

Need $\alpha > \pi/n$.

Case of interest here is when $\alpha$ is large enough s.t. $\cos \alpha + \cos(\psi + \alpha) < 0$.

Since $\ell_i(t) \to \infty$ and $\ell_i(t)/\ell_j(t) \to 1$, stationary shape must have equal edges: $\ell_i(t) = \ell_j(t) \ \forall i, j$. 

$\dot{\ell}_i = -\cos \alpha - \cos(\psi_i + \alpha)$

$\dot{\psi}_i = (\sin \alpha + \sin(\psi_i + \alpha))/\ell_i - (\sin \alpha + \sin(\psi_{i+1} + \alpha))/\ell_{i+1}$
Similarly to [Galloway et al., 2013], instead of edge lengths $\ell_i(t)$ consider $\rho_i(t) := \frac{\ell_i(t)}{P(t)}$ where $P(t) := \sum_{i=1}^{n} \ell_i(t)$ is perimeter.

In rescaled coords, stationary shapes (regular polygons) → stationary points, i.e., equilibria: $\psi_i \equiv \psi$, $\rho_i \equiv 1/n$ $\forall i$.

Also introduce time rescaling $\tau(t) := t/P(t)$ which allows us to write the system in $(\psi_i, \rho_i)$ coords.

If equilibria are locally exponentially stable for this new system, then regular polygons are locally attractive for original system.
EIGENVALUE ANALYSIS of LINEARIZED SYSTEM

- Rescaled coords: \((\psi_1, \rho_1, \ldots, \psi_n, \rho_n)\) where \(\rho_i(t) := \ell_i(t)/P(t)\)

- \(2n \times 2n\) Jacobian matrix at equilibria \(\psi_i = \psi, \rho_i = 1/n\) takes block-circulant form \(J = \begin{pmatrix}
A_0 & A_1 & A_2 & \cdots & A_{n-1} \\
A_{n-1} & A_0 & A_1 & \cdots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
A_1 & A_2 & \cdots & A_{n-1} & A_0
\end{pmatrix}\)
- It always has one 0 eigenvalue with eigenvector \((d, 0, \ldots, d, 0)\)
  – not an admissible direction as all angles \(\psi_i\) cannot increase

- \(k\)-th pair of eigenvalues of \(J\) is eigenvalues of \(2 \times 2\) matrix \(A_0 + A_1 \chi_k + A_2 \chi_k^2 + \cdots + A_{n-1} \chi_k^{n-1}\) where \(\chi_k := e^{2\pi ik/n}\)

- By Routh-Hurwitz criterion for polynomials with complex coefficients can show that these eigenvalues have \(\text{Re}(\lambda) < 0\) if and only if

\[
(3C^2 + AC + 1 + B) \cos(2\pi/n) < (6C^2 + A^2 + 5AC - 1 - B)
\]

where \(A := \cos \alpha, B := \cos \psi, C := \cos(\psi + \alpha)\)

Sufficient condition for local attractivity of regular polygons
DISCUSSION

We have convergence to regular polygons if $A + C < 0$ and
$$(3C^2 + AC + 1 + B)\cos\left(\frac{2\pi}{n}\right) < (6C^2 + A^2 + 5AC - 1 - B) \quad (*)$$
where $A := \cos \alpha$, $B := \cos \psi$, $C := \cos(\psi + \alpha)$

For convex regular $n$-gon we have $\psi = \pi\left(1 - \frac{2}{n}\right)$ and
can show that (*) holds for $\alpha > \pi/n$ as long as $\alpha$ is not too large
(recall that $\alpha < \pi/n$ gives rendezvous)

Example: $n = 4$
Pink region is where $A + C > 0$
Blue region is where (*) fails
White region is the ‘good’ region
Square ($\psi = \pi/2$) is locally attractive for $\alpha \in (\pi/4, \pi)$
DISCUSSION

We have convergence to regular polygons if $A + C < 0$ and
$(3C^2 + AC + 1 + B) \cos(2\pi/n) < (6C^2 + A^2 + 5AC - 1 - B)$ \hspace{1cm} (*)
where $A := \cos \alpha$, $B := \cos \psi$, $C := \cos(\psi + \alpha)$

For convex regular $n$-gon we have $\psi = \pi(1 - 2/n)$ and can show that (*) holds for $\alpha > \pi/n$ as long as $\alpha$ is not too large (recall that $\alpha < \pi/n$ gives rendezvous)

Example: $n = 5$
Pink region is where $A + C > 0$
Blue region is where (*) fails
White region is the ‘good’ region
Both pentagon ($\psi = 3\pi/5$) and pentagram ($\psi = \pi/5$) are locally attractive for some range of $\alpha$ (separatrix?)
DISCUSSION

We have convergence to regular polygons if $A + C < 0$ and

$$(3C^2 + AC + 1 + B) \cos(2\pi/n) < (6C^2 + A^2 + 5AC - 1 - B) \quad (\star)$$

where $A := \cos \alpha$, $B := \cos \psi$, $C := \cos(\psi + \alpha)$

Using 1 instead of $\cos(2\pi/n)$ in $(\star)$, we obtain

$$3C^2 + A^2 + 4AC - 2 - 2B > 0$$

which can be simplified to

$$\cos \left( 2\alpha + \frac{\psi}{2} \right) + 3 \cos \left( 2\alpha + \frac{3\psi}{2} \right) > 0$$

This sufficient condition is more conservative but works for any $n$. 
CONCLUSIONS

• Agents in cyclic pursuit with large fixed relative heading angle diverge but tend to form regular patterns

• Stationary shapes are regular polygons

• Block-circulant structure of linearization matrix gives a sufficient condition for convergence

• Need to better understand system behavior, especially far away from stationary shapes

• Extensions to non-constant heading angles and non-cyclic formations are also of interest