



## Technical communique

Overcoming a fundamental time-domain performance limitation by nonlinear control<sup>☆</sup>B.G.B. Hunnekens<sup>a</sup>, N.v.d. Wouw<sup>a,b,c</sup>, D. Nešić<sup>d</sup><sup>a</sup> Eindhoven University of Technology, Department of Mechanical Engineering, P.O. Box 513, NL 5600 MB Eindhoven, The Netherlands<sup>b</sup> Department of Civil, Environmental & Geo-Engineering, University of Minnesota, Minneapolis, USA<sup>c</sup> Delft Center for Systems and Control, Delft University of Technology, Delft, The Netherlands<sup>d</sup> University of Melbourne, Department of Electrical and Electronic Engineering, Parkville, VIC 3010, Australia

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## ABSTRACT

It is well-known that fundamental performance limitations exist when using linear feedback control for linear systems. In this note, we present an example of a nonlinear control strategy that can achieve a time-domain performance specification that cannot be obtained by any linear controller. In particular, we present a variable-gain control approach that meets an overshoot requirement that cannot be met by any linear controller.

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## 1. Introduction

It is well-known that fundamental performance limitations exist when using linear time-invariant (LTI) feedback controllers for LTI single-input-single-output (SISO) plants (Freudenberg, Middleton, & Stefanopoulou, 2000; Middleton, 1991; Seron, Braslavsky, & Goodwin, 1997). These fundamental limitations may relate to fundamental frequency-domain limitations, such as the waterbed effect or Bode's gain–phase relationship, or time-domain limitations, such as restrictions on rise-time, overshoot and settling time of the closed-loop system.

In order to overcome these fundamental limitations, related to the usage of linear feedback controllers, or balance related performance trade-offs in a more desirable manner, the use of nonlinear control strategies has been studied extensively in the literature. Examples are the works on reset control strategies (Beker, Hollot, & Chait, 2001; Clegg, 1958; Nešić, Teel, & Zaccarian, 2011; Zhao,

Nešić, Tan, & Wang, 2013; Zheng, Chait, Hollot, Steinbuch, & Norg, 2000) split-path nonlinear filters (Foster, Gieseking, & Waymeyer, 1966; van Loon, Hunnekens, Heemels, van de Wouw, & Nijmeijer, in press), switched controllers (Feuer, Goodwin, & Salgado, 1997; Lau & Middleton, 2003), or variable-gain controllers (Chen, Lee, Peng, & Venkataramanan, 2003; Heertjes & Leenknecht, 2010; Hunnekens, van de Wouw, Heertjes, & Nijmeijer, 2015; Lin, Pachter, & Ban, 1998; van de Wouw, Pastink, Heertjes, Pavlov, & Nijmeijer, 2008; Zheng, Guo, & Wang, 2005), which all aim at improving closed-loop performance compared to that obtained by linear feedback controllers.

All these works contain interesting performance-improving results, and the benefits of several control strategies have also been validated on industrial applications (Chen et al., 2003; Heertjes & Leenknecht, 2010; Hunnekens et al., 2015; van de Wouw et al., 2008; Zheng et al., 2000, 2005). However, to the best knowledge of the authors, there exists only one example of a nonlinear control strategy that explicitly shows that certain performance specifications can be met that cannot be obtained by any linear controller. This example involves reset control, for which in Beker et al. (2001) and Zhao et al. (2013) it has been shown that certain fundamental time-domain limitations can be overcome by resetting controller states.

In this note, we present a second example of a nonlinear control strategy that can achieve performance specifications not attainable by any linear controller. More specifically, we will study

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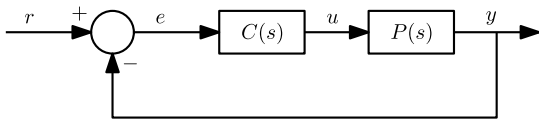


Fig. 1. Linear control scheme with linear plant  $P(s)$  and controller  $C(s)$ .

a fundamental tradeoff for linear plants with a real unstable pole, which, given a certain rise-time specification, will exhibit a minimal amount of overshoot when controlled by any linear controller (Seron et al., 1997). Using a so-called phase-based variable-gain controller (Armstrong, Guitierrez, Wade, & Joseph, 2006; Xu, Hollerbach, & Ma, 1995), we show that this fundamental limitation can be overcome. In particular, we show that an overshoot specification can be attained that is not attainable by any linear feedback controller.

The remainder of this note is organized as follows. In Section 2, we briefly revisit a fundamental time-domain limitation for linear plants with an unstable real pole. In Section 3, we present the phase-based variable-gain control strategy and show, using a simulation example, that a time-domain specification can be met using this nonlinear control strategy that cannot be met by any linear feedback controller. Conclusions are presented in Section 4.

## 2. A fundamental time-domain limitation for linear systems

Consider the linear feedback configuration in Fig. 1, which consists of a linear time-invariant (LTI) single-input-single-output (SISO) plant  $P(s)$ ,  $s \in \mathbb{C}$ , linear feedback controller  $C(s)$ , reference  $r$ , output  $y$ , tracking error  $e := r - y$  and control action  $u$ . It is well-known that there exist fundamental performance limitations in the design of linear feedback controllers  $C(s)$  for these linear SISO LTI plants  $P(s)$ , see e.g. Freudenberg et al. (2000), Middleton (1991) and Seron et al. (1997). The term *fundamental* relates to the fact that the performance limitations are independent of the design choices for the linear feedback controller  $C(s)$ .

In this note, we focus on a fundamental time-domain limitation for plants  $P(s)$  which have an unstable pole at  $s = p > 0$ . If the closed-loop system in Fig. 1 is subject to a unit step-reference  $r(t) = 1$ , for  $t \in \mathbb{R}_{\geq 0}$  ( $r(t) = 0$ ,  $t < 0$ ), a certain fundamental limitation exists between the rise-time and amount of overshoot of the closed-loop system. In order to make the latter statement mathematically more precise, consider the following definitions of rise-time and amount of overshoot.

**Definition 1** (Seron et al., 1997). The rise-time of the closed-loop system is defined as:

$$t_r := \sup_{\delta} \left\{ \delta : y(t) \leq \frac{t}{\delta} \text{ for all } t \in [0, \delta] \right\}. \quad (1)$$

**Definition 2** (Seron et al., 1997). The overshoot  $y_{os}$  of the closed-loop system is defined as the maximum value by which the output  $y(t)$  exceeds the final set-point value  $r = 1$ :

$$y_{os} := \sup_{t \geq 0} (-e(t)). \quad (2)$$

A graphical interpretation of the definition of rise-time and overshoot is given in Fig. 2. In words, this means that the rise time  $t_r$  is defined as the largest value for which the response  $y(t)$  is still below the line  $t/t_r$ , for all  $t \leq t_r$ .

Now, a fundamental time-domain limitation can be formulated in the result below.

**Corollary 3** (Seron et al., 1997). Suppose that  $P(s)$  in Fig. 1 has a real pole at  $s = p > 0$  in the open right-half-plane. If the closed-loop system is stabilized by any linear time-invariant controller  $C(s)$ , then

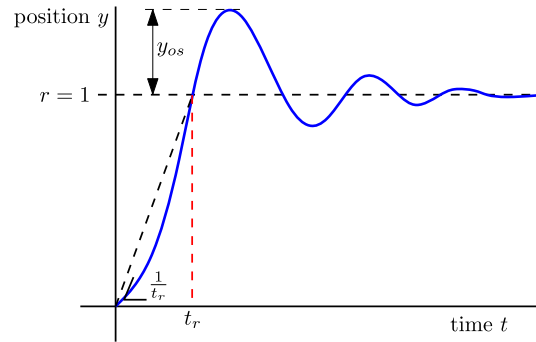


Fig. 2. Definition of rise-time  $t_r$  according to (1) and overshoot  $y_{os}$  according to (2).

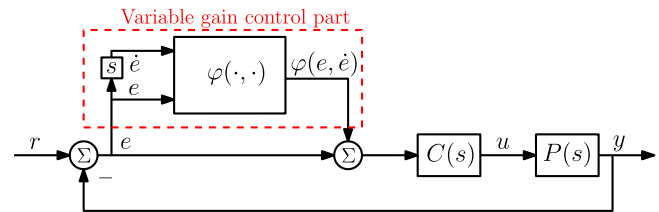


Fig. 3. Phase-based variable-gain control scheme with variable-gain element  $\varphi(e, \dot{e})$ .

its step-response  $y(t)$  must exhibit overshoot, and satisfy the following inequality:

$$y_{os} \geq \frac{(pt_r - 1)e^{pt_r} + 1}{pt_r} \geq \frac{pt_r}{2}. \quad (3)$$

**Proof.** The proof can be found in Seron et al. (1997).

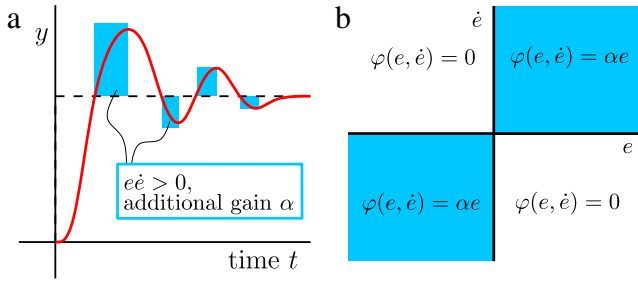
Note that both the lower-bounds for the overshoot  $y_{os}$  in (3) are monotonic in the rise time  $t_r$ . Therefore, Corollary 3 expresses the fact that if the closed-loop system is 'slow', i.e., it has a large rise time  $t_r$ , the step response will present a large amount of overshoot if there are open-loop unstable real poles. In practice, it is reasonable to assume that, a certain lower-bound for the rise-time of a closed-loop system with unstable real open-loop poles may exist, for example due to physical actuator constraints or bandwidth limitations in the system. This lower-bound for the rise-time results (via (3)) in an explicit lower bound on the amount of overshoot that the system will exhibit when using a linear feedback controller  $C(s)$ , no matter how the controller  $C(s)$  is designed/tuned.

In Section 3, we present a type of nonlinear controller which can overcome this fundamental time-domain performance limitation.

## 3. A nonlinear controller overcoming a fundamental time-domain limitation

### 3.1. Phase-based variable-gain control

Consider the nonlinear control strategy as shown in Fig. 3, which represents a so-called variable-gain control (VGC) scheme. The term *variable-gain* controller is used since the controller configuration allows the use of a variable amount of controller gain through the function  $\varphi(e, \dot{e})$ . Here, we will focus on a *phase-based* variable-gain controller, which applies additional gain based on information on the error  $e$  and time-derivative of the error  $\dot{e}$ , see e.g. Armstrong et al. (2006) and Xu et al. (1995), as opposed to *magnitude-based* variable-gain control, which modulates the gain based only on the magnitude of the error  $e$ , see Heertjes and Leenknecht (2010), Hunnekens et al. (2015) and van de Wouw et al. (2008).



**Fig. 4.** (a) Illustration of rationale behind phase-based variable-gain controller, (b) graphical illustration of phase-based nonlinearity  $\varphi(e, \dot{e})$  defined in (4).

The rationale behind the phase-based VGC is as follows:

- if the error  $e$  is moving away from zero (i.e.  $e\dot{e} > 0$ ), additional controller gain would be useful in order to quickly steer the error in the correct direction again;
- if the error is already moving towards zero (i.e.  $e\dot{e} \leq 0$ ), no additional controller gain is added since the error is already moving in the right direction,

see Fig. 4(a) for a graphical illustration of this rationale. In order to incorporate this rationale in the variable-gain control scheme in Fig. 3, the nonlinearity  $\varphi(e, \dot{e})$  is chosen as follows:

$$\varphi(e, \dot{e}) = \begin{cases} \alpha e & \text{if } e\dot{e} > 0 \\ 0 & \text{if } e\dot{e} \leq 0, \end{cases} \quad (4)$$

see Fig. 4(b) for a graphical illustration of the function  $\varphi(e, \dot{e})$ . Practically, it is advised to use a low-pass filtered version of  $\dot{e}$  in order to avoid high noise-sensitivity due to the numerical differentiation of the signal  $e$ .

### 3.2. Example of overcoming a fundamental limitation

Here, we present an example that illustrates that the variable-gain control strategy discussed above can overcome the fundamental time-domain limitation in Corollary 3. Consider the following plant:

$$P(s) = \frac{0.05s + 1}{(s - p)(0.01s + 1)^2}, \quad (5)$$

which has a real unstable pole at  $s = p = 1 > 0$ . By means of loop-shaping, a stabilizing linear controller can be designed using the full Nyquist stability criterion (Franklin, Powell, & Emami-Naeini, 2005, Section 6.3), which dictates that the Nyquist plot of the open-loop transfer function  $P(s)C(s)$  should make one counterclockwise encirclement of the  $-1$  point. This is achieved by the following linear controller

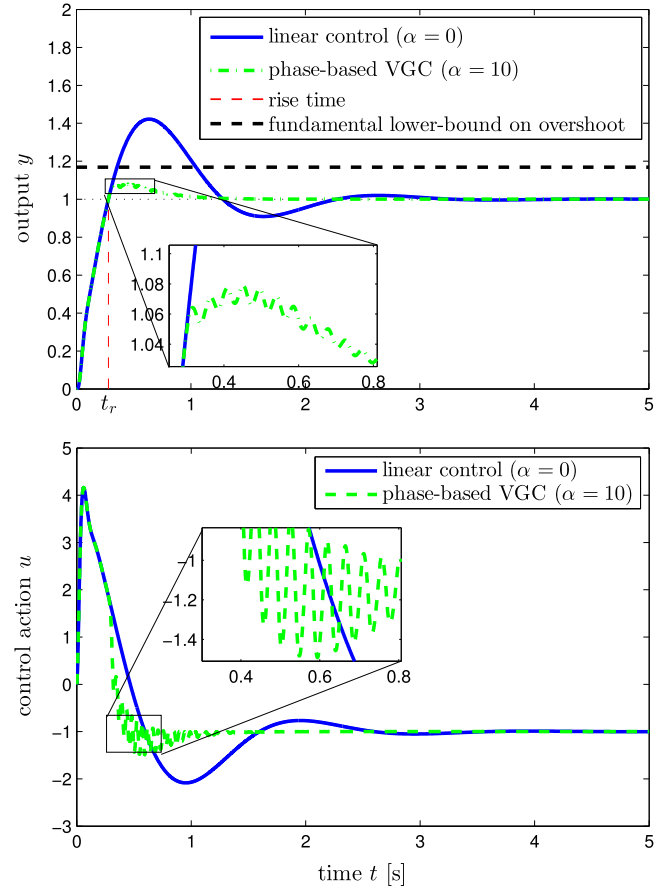
$$C(s) = k_p \underbrace{\frac{s + 2\pi f_i}{s}}_{\text{integral action}} \underbrace{\frac{1}{\frac{1}{(2\pi f_{lp})^2 s^2} + \frac{2\beta_{lp}}{2\pi f_{lp}} s + 1}}_{\text{2nd-order low-pass}}, \quad (6)$$

with gain  $k_p = 4$ , low-pass frequency  $f_{lp} = 10$  Hz, damping  $\beta_{lp} = 0.7$ , and integral action with  $f_i = 0.5$  Hz.

A step-response of the closed-loop system with plant  $P(s)$  as in (5) and controller  $C(s)$  as in (6) is shown in Fig. 5. Using this controller, the rise-time  $t_r$ , according to Fig. 2, can be determined to be  $t_r = 0.28$  s. Given this rise-time, the fundamental lower-bound for the amount of overshoot using any linear controller can be computed (using the first inequality in (3)) to be

$$y_{os} \geq 0.169, \quad (7)$$

which is also plotted in Fig. 5. Although the estimated lower-bound in (3) is somewhat conservative, the amount of overshoot achieved

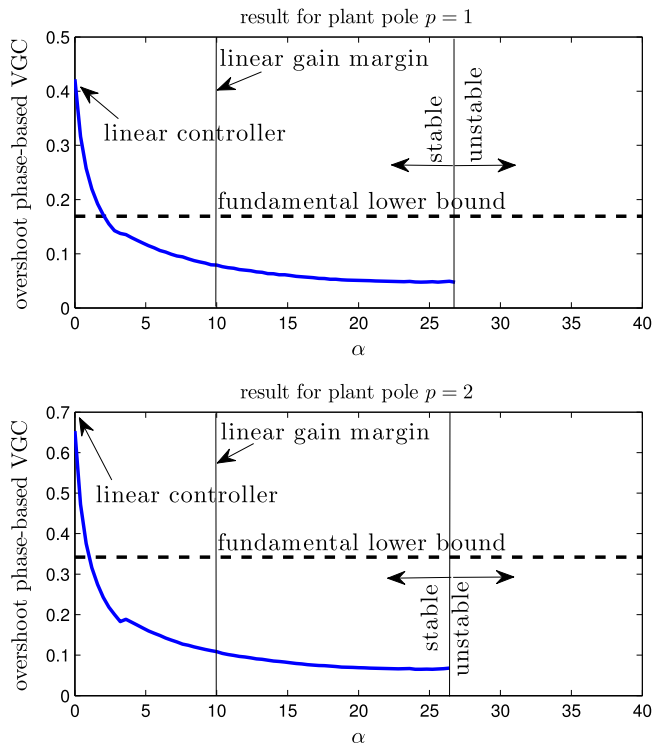


**Fig. 5.** Step-response of the closed-loop system (top) using linear control and using phase-based variable-gain control (with  $\alpha = 10$ ) and the corresponding control action  $u$  (bottom).

by the controller in (6) ( $y_{os} = 0.422$ ) indeed exceeds this lower-bound, see Fig. 5.

Now, let us consider the phase-based variable-gain controller as shown in Fig. 3 (note that the situation  $\alpha = 0$  corresponds to the linear controller  $C(s)$ ). For an additional gain of  $\alpha = 10$ , see (4), the step-response is shown in Fig. 5. Note that the phase-based VGC behaves identical to the linear controller up to the point where the error crosses zero for the first time (i.e., when  $y = 1$  is reached for the first time) since up to that point  $e\dot{e} \leq 0$ , see also Fig. 4(a). Hence, the rise-time of the variable-gain controller is identical to the linear controller  $C(s)$ . However, the variable-gain controller attains an amount of overshoot  $y_{os} = 0.079$  which is far below the lower-bound in (7), which holds for any linear controller with rise-time  $t_r = 0.28$  s. This clearly shows the potential of nonlinear control in general and of variable-gain controllers in particular in overcoming fundamental performance limitations of linear systems and attaining performance specifications not attainable by any linear controllers.

**Remark 4.** Note that in this paper, we focus on performance of the closed-loop system, rather than a formal closed-loop stability analysis of the phase-based variable-gain control scheme. However, stability results for a class of phase-based variable-gain control systems are available in e.g. Armstrong et al. (2006). These stability analysis techniques induce conservatism that may lead to conservative bounds on  $\alpha$ , thereby seemingly preventing the harvesting of the transient performance benefit of such controllers. Therefore, the focus of the current paper is on the performance of the closed-loop system and simulation-based stability analysis (see Fig. 6), rather than a formal closed-loop stability analysis of the phase-based variable-gain control scheme.



**Fig. 6.** Amount of overshoot of the phase-based VGC as a function of the additional gain  $\alpha$ , with plant pole  $p = 1$  (upper plot) and plant pole  $p = 2$  (lower plot).

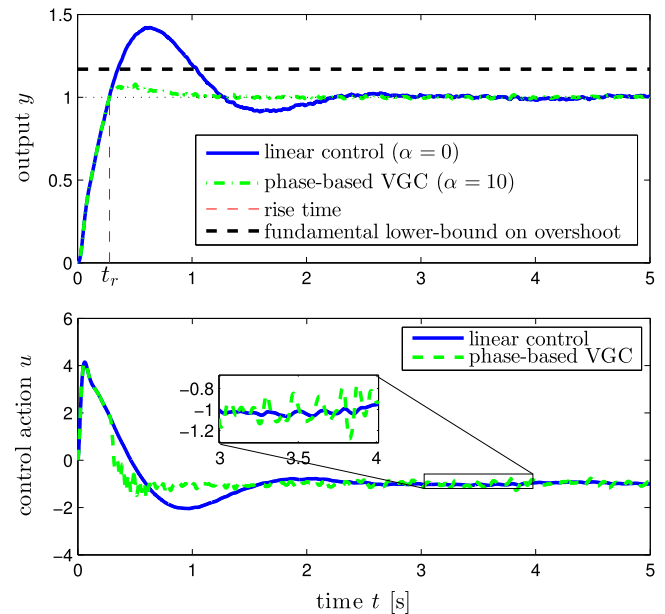
In the lower-plot of Fig. 5 the control action  $u$  is shown. Note that the change in control action after  $t = 0.28$  s is exactly what limits the amount of overshoot for the phase-based VGC. Moreover, note that the control action  $u$  is smooth (no discontinuities) due to the low-pass filter that is contained in  $C(s)$  in (6). The oscillatory behavior in the control signal for the phase-based VGC is due to the fact that  $\dot{e}$  changes sign multiple times in the simulation, see the upper-plot in Fig. 5.

In order to study the effect of the additional gain  $\alpha$  in more detail, consider the simulation results in the upper plot of Fig. 6, which shows the amount of overshoot of the closed-loop system as a function of the additional gain  $\alpha$  in  $\varphi(e, \dot{e})$  in (4), for the plant with pole at  $p = 1$ . Stable step-responses are obtained for all  $\alpha \in [0, 27]$  and, for all  $\alpha \in [2.1, 27]$ , an overshoot below the fundamental lower-bound (7) is achieved.

It is interesting to note that up to  $\alpha = 27$  a stable step-response is obtained using the phase-based VGC, whereas if this gain would be applied continuously (i.e. using  $\alpha C(s)$  as a linear controller) an unstable linear closed-loop system would result, since the gain margin of the system with linear controller can be determined to be 9.9, see Fig. 6. Therefore, in a formal stability analysis of the closed-loop system, the variable-gain element  $\varphi(e, \dot{e})$  should explicitly be accounted for, as discussed in Armstrong et al. (2006).

In order to study the robustness of the performance gain in the presence of plant uncertainties, consider the result in the lower plot of Fig. 6, where only the plant pole has been shifted to  $p = 2$ . Because the plant  $P(s)$  is different, the amount of overshoot and the fundamental lower-bound is different. However, the qualitative closed-loop behavior is the same (with an overshoot not attainable by any linear controller), which indicates that there is some robustness to model uncertainties.

As a last simulation experiment, we consider the situation in which an input plant disturbance acts on the plant. A white noise disturbance has been added to the plant input  $u(s)$ , the results are depicted in Fig. 7. The effect of the input disturbance is visible from the response  $y(t)$ , but still the phase-based variable controller



**Fig. 7.** Step-response of the closed-loop system (top) using linear control and using phase-based variable-gain control (with  $\alpha = 10$ ) and the corresponding control action  $u$  (bottom) under the influence of plant input noise.

can achieve an overshoot specification not attainable by any linear controller. From the control action  $u$  we see that the phase-based variable-gain controller is more sensitive to the noise, because it reacts to the changes in the  $(e, \dot{e})$ -quadrants, see Fig. 4. If in a practical situation this increased sensitivity is problematic, it can be counteracted by using a low-pass filter in the controller  $C(s)$ , but likely at the expense of some loss in performance.

## 4. Conclusions

Linear control systems are subject to certain fundamental performance-tradeoffs. Different types of variable-gain control strategies have been studied and used in practical/industrial applications in the last decades (Armstrong et al., 2006; Hunnekens et al., 2015; Lin et al., 1998; Su, Sun, & Duan, 2005; van de Wouw et al., 2008; Zheng et al., 2005) in order to improve the performance compared to linear systems. Still, to the best knowledge of the authors, this note gives the first explicit example of a variable-gain controller achieving a performance specification that is not achievable by any linear controller. In particular, we have shown that phase-based variable-gain control can achieve an overshoot performance specification that cannot be achieved by any linear controller.

The authors hope that the results in this note, in addition to the successful applications of nonlinear and variable-gain controllers in literature, will inspire others to research and apply nonlinear controllers for linear systems in order to improve performance beyond the reach of linear control.

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