Transmission of a continuous signal via one-bit capacity channel*

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December 13, 2013

Abstract

We study the problem of coding signals for the transmission via digital communication channels with limited capacity. We suggest a coding algorithm for transmission of a currently observed continuous time signal via a one-bit capacity channel that is capable of sending a single binary signal only for each measurement of the underlying process.

Keywords: coding, communication bit-rate constraints, one-bit capacity channel
MSC subject classification: 94A12, 94A40.

1 Introduction

We study the problem of transmission of a currently observed continuous time signal via limited capacity digital communications channel. It is assumed that the dynamics of the underlying continuous time signal is unpredictable and has unknown range. In particular, the signal is not necessary continuous, and unexpected jumps may occur. We consider the situation where the channel capacity is insufficient to send sufficiently accurate approximations of the current measurements in real time. Therefore, the observed measurements have to be coded, transmitted in the coded form, and decoded. This problem may arise, for example, for remote control of underwater vehicles, since communication is severely limited underwater (see Stitwell and Bishop [3]).

*This work was supported by ARC grant of Australia DP120100928 to the author.

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We consider an extreme case of one-bit capacity channel that can transmit a single binary signal for a single measurement of the underlying process. This channel connects two subsystems of a dynamical system. The first subsystem, that is called Coder, receives the real-valued measurements and converts them into a binary symbolic sequence which is sent over the communication channel. For each measurement, only one single bit signal can be sent. The second subsystem (Decoder) receives this symbolic sequence and converts it into a real-valued state estimate.

The system described in the present papers represents a modification of the systems from Wong and Brockett [4] and Dokuchaev and Savkin [1], where limited capacity digital channels were studied in stochastic setting. In Wong and Brockett [4], the filtering problem was considered for the case of bounded random disturbances. In Dokuchaev and Savkin [1], a filtering problem was studied for the case of non-decreasing Gaussian disturbances. In the present paper, we suggest a coding algorithm that allows to use a just one-bit capacity channel and for continuous time signal. We investigate the error dependence on the frequency of the samples measurements.

The remainder of this paper proceeds as follows. In Section 2, we introduce the class of systems under consideration formulate the main results. Section 3 presents an illustrative example. Section 4 contains brief discussion and suggest future developments. The proofs of all the results are given in Appendix.

2 Problem statement and the main result

Let $x(t)$ be a given continuous time state process observed at times $t_k = k\delta$, $k = 0, 1, 2, ...$, where $\delta > 0$ is given.

Suppose estimates of the current state $x(t)$ are required at a distant location, and are to be transmitted via a digital communication channel such that only one bit of data may be sent at each time t_k. For this task, we consider a system which consists of the coder, the transmission channel, and the decoder. Using an observation of $x(t_k)$, the coder produces a one-bit word h_k which is transmitted via the channel and then received by the decoder; the decoder produces an estimate $y(t)|_{[0,t_k]}$ which depends only on $h_1, ..., h_k$. In other words, the process $x(t)$ is supposed to be sampled at times t_k, coded, transmitted via the channel and then decoded. The block diagram of this system is shown in Figure 1. It is important that,

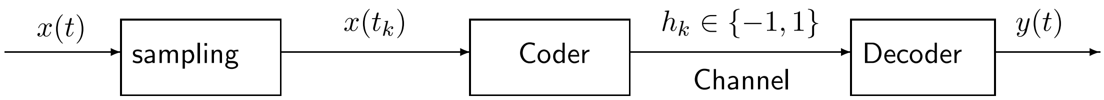

Figure 1: Block diagram of the estimator.

for each sample $x(t_k)$, only one bit of information can be transmitted. The corresponding algorithm is suggested below.

Transmission algorithm

Let numbers y_0, $M_0 > 0$, $\bar{M} > 0$, and $a \in (1, 2]$ be given parameters that are built in the coder and decoder. The algorithm can be described as follows.

(i) Sample values $x(t_k)$ are taken;

(ii) The coder computes a sequence $\{(y_k, M_k)\}_{k\geq 1} \subset \mathbf{R}^2$ and produces a sequence of binary words $\{h_k\}$ consequently for $k = -1, 0, 1, 2, ...$ by the following rule: $h_{-1} = 1$, and

$$h_k = \begin{cases} 1, & \text{if} \quad y_k < x(t_k) \\ -1, & \text{if} \quad y_k > x(t_k) \\ -h_{k-1}, & \text{if} \quad y_k = x(t_k), \end{cases}$$

where

$$y_k = y_{k-1} + h_{k-1} M_{k-1} \delta, \quad k = 1, 2, ...$$

$$M_k = \begin{cases} aM_{k-1}, & \text{if } k \notin \mathcal{T} \text{ and } k-1 \notin \mathcal{T} \\ M_{k-1}, & \text{if } k \notin \mathcal{T} \text{ and } k-1 \in \mathcal{T} \\ \max(a^{-1} M_{k-1}, \bar{M}), & \text{if} \quad k \in \mathcal{T}. \end{cases}$$

and where $\mathcal{T} = \{k \geq 1 : h_{k-1} h_k < 0\}$.

(iii) The binary symbol h_t is transmitted via the channel.

(iv) The coder computes the same sequence $\{(y_k, M_k)\}_{k\geq 1} \subset \mathbf{R}^2$ using the received values $\{h_k\}$ by the same rule as the coder.

(v) Finally, the decoder computes estimate $y(t)$ of the process $x(t)$ as

$$
\begin{aligned}
& y(t) = y_k + h_k M_k (t - t_k), \quad t \in [t_k, t_{k+1}], \quad k = 0, 1, 2, ..., \\
& y_0 = \bar{x}_0.
\end{aligned}
$$

Consider an algorithm introduced above. Let $\tau(0) = \inf\{m \in \mathcal{T}\}$. Further, for $s \geq 0$, set

$$
\tau(s) = \inf\{m > s : \ m \in \mathcal{T}\}.
$$

Let $D > 0$ be given.

Let $\mathcal{X}_D$ be the set of processes $x(t) : [0, +\infty) \to \mathbf{R}$ such that

$$
\sup_{k\geq 0} \sup_{t\in[t_k,t_{k+1}]} |x(t) - x(t_k)| \leq D\delta.
$$

Theorem 1 *For any $x \in \mathcal{X}_D$, the following holds:*

(i) $\tau(0) \leq \bar{\theta}$, where

$$
\bar{\theta} = \inf\{m \geq 0 : \ M_0(1 + a + a^2 + \cdots + a^m)\delta \geq |y_0 - x(0)| + mD\delta \}.
$$

(ii) Assume that $\bar{M} \geq 2D$. Then, if $s \in \mathcal{T}$ and $M_{s-1} \geq 2aD$, then $\tau(s) - s \leq 3$ and

$$
\begin{aligned}
& \sup_{s\geq k\leq \tau(s)} |x(t_k) - y(t_k)| \leq (M_{s-1} + D)\delta, \\
& \sup_{s\geq k\leq \tau(s)} \sup_{t\in[t_k,t_{k+1}]} |x(t) - y(t)| \leq (M_{s-1} + 2D)\delta, \qquad (1)
\end{aligned}
$$

and

$$
\min_{\rho\in\mathcal{T},\, s<\rho\leq s+4} M_\rho \leq \max(M_{s-1}/a, \bar{M}). \qquad (2)
$$

(iii) Assume that $\bar{M} \geq 2D$. Then there exists $\eta \in [\tau(0), \tau(0)+1, \tau(0)+2, ...]$ such that $M_\eta = \bar{M}$ and $|x(t_\eta) - y(t_\eta)| \leq (a\bar{M} + D)\delta$. In this case, $M_k \in \{\bar{M}, a\bar{M}\}$ for all $k \geq \eta$, and $M_k = \bar{M}$ for all $k \geq \eta$ such that $k \in \mathcal{T}$. In addition,

$$\begin{aligned}
&\sup_{k:\ t_k > \eta} |x(t_k) - y(t_k)| \leq (a\bar{M} + D)\delta, \\
&\sup_{k:\ t_k > \eta} \sup_{t \in [t_k, t_{k+1}]} |x(t) - y(t)| \leq (a\bar{M} + 2D)\delta.
\end{aligned} \tag{3}$$

The proofs of the theorem is given in Appendix.

3 Illustrative examples

We illustrate applications of the algorithm described above with the following numerical examples.

We considered $t \in [0, 2]$, $\delta = t_{k+1} - t_k = 0.04$ and $\delta = t_{k+1} - t_k = 0.02$. In all cases described below, transmission of the corresponding coded signals requires to transmit 50 bits only for $\delta = 0.04$ and 100 bits only for $\delta = 0.02$.

In those experiments, we used $a = 1.6$.

According to Theorem 1, the algorithm produces an estimate $y(t)$ that approximate $x(t)$ closely enough on the time interval $[t_\eta, T]$ such that $D \leq \bar{M}$ for $x(t)|_{[0,T]}$. The first steps $k = 0, 1, ..., \eta$ are used to bring the value $y(t)$ to a close proximity of $x(t)$. The error on the time interval $[0, t_\eta]$ can be significant, if the distance $|y_0 - \mu(0)|$ is large. This situation is illustrated with Figure 1. This Figure 1 shows a example of a continuous process $x(t)$ and the corresponding processes $y(t)$ approximating $x(t)$ for the case of $\delta = 0.04$ and $\delta = 0.02$ respectively, in the situation where $y(0) = 1$ and $x(0) = 3$, with $\bar{M} = 4D = 4\delta$.

Further, for a piecewise smooth processes with jumps, the algorithm requires certain number of steps to restore this proximity after a jump. If the underlying process $x(t)$ has a jump at time $t = T$, i.e., $|x(T - \delta) - x(T)| > D$ which may occur if $x(T) \neq x(T - 0)$, then the properties of the estimate $y(t)$ will be such as described in Theorem 1, for the time interval $[T, T_1]$ instead of $[0, T]$, where T_1 is the first jump after T. This means that thee are some time periods of increased error after the jumps of $x(t)$. This situation is illustrated with Figure 2. This Figure 2 shows a example of a discontinuous piecewise continuous process

$x(t)$ and the corresponding processes $y(t)$ approximating $x(t)$ for the case of $\delta = 0.04$ and $\delta = 0.02$ respectively, in the situation where $y(0) = 3.3$ and $x(0) = 3.5$, with $\bar{M} = 4D = 4\delta$.

Figure 3 illustrates the impact of selecting too small $\bar{M}$. By Theorem 1, the process $y(t)$ reconstructed by the decoder oscillates about the underlying process, if $\bar{M} \geq 2D$, i.e., if $\bar{M}$ is selected to be large enough. In this case, the error have the order $\bar{M}$ and does not vanish even for constant $x(t)$. If $\bar{M}$ is selected to be small, then the error at time t_m will be also small for constant or very smooth processes $x(t)$ such that $|x(t_k) - x(t_{k+1}| << D$, $k = m_0, m+1, ..., m$. However, M_m will be also small in this case. A significant error may arise if M_m is small and $|x(t_s) - x(t_{s+1}| \sim D$ for $s = m+1, ..., s_1$, since it would require a number of steps to achieve a sufficient size of M_s such that the oscillation of $y(t_s)$ about $x(t)$ described in Theorem 1 can be restored. This situation is illustrated with Figure 3. This figure shows a example of a discontinuous piecewise continuous process $x(t)$ and the corresponding processes $y(t)$ approximating $x(t)$ for the case of $\delta = 0.04$ and $\delta = 0.02$ respectively, in the situation where $y(0) = 4$ and $x(0) = 3.5$, with $\bar{M} = D/4 = \delta/4$.

We used MATLAB In these numerical examples.

4 Discussion and future developments

(i) For the case of vector process $x(t) = (x_1(t), ..., x_n(t))$, the method described above can be modified as the following. The signal $\{h_i\}$ has to be formed as a sequence

$$(h_{0,1}, ..., h_{0,n}, h_{1,1}, ..., h_{1,n}, ..., h_{k,1}, ..., h_{k,n}, ...),$$

where a subsequence $(h_{0,m}, h_{1,m}, h_{2,1}, .., h_{km}, ...)$ represent a binary signal formed for the for the mth component of x according to the algorithm described above.

(ii) The suggested algorithm is robust with respect to the errors caused by missed or misread signals h_k for the models where either the decoder is always aware that a signal was missed or possibly misread, or the coder always knows which signals were delivered. Obviously, there are models where these conditions are not satisfied. It could be interesting to find a way to modify an algorithm such that it will be robust with respect to transmission errors when these conditions are not satisfied.

(iii) An useful non-causal version of this algorithm can be developed for a case where an entire signal $x(t)|_{[0,T]}$ is known and has to be coded in a most economic way, for some given $[0,T]$. In this case, a continuous time function can be coded via the sequence of binary symbols, with the rate of one binary symbol for each sampling point, rather than a real number for each sampling point. This way, the same quality of approximation can be achieved with lesser quantity of bits than representation of a process via truncated Fourier series or splines. We leave it for the future research.

Appendix: Proof of Theorem 1

To prove statement (i), it suffices to observe that

$$\inf_{k\le m}|y(t_k)-x(t_k)|\le |y_0-x(0)|+mD\delta-M_0(1+a+a^2+\cdots+a^m)\delta.$$

Let us prove statement (ii). For certainty, we assume that $h_{s-1}=-1$. Since $s\in\mathcal{T}$, it follows that $h_s=1$. If $s+1\in\mathcal{T}$ or $s+2\in\mathcal{T}$ then $\tau(s)-s\le 3$. Suppose that $s+1\notin\mathcal{T}$, $s+1\notin\mathcal{T}$, and $s+3\notin\mathcal{T}$. We have that $x(t_{s-1})\in[y_{s-1},y_s]$. Hence

$$x(t_{s+3})\in[y_{s-1}-4D\delta, y_s+4D\delta].$$

Since $s+1\notin\mathcal{T}$ and $s+2\notin\mathcal{T}$, it follows that $x(t_{s+3})\in[y_{s-1},y_{s-1}+K\delta)$. On the other hand,

$$y(t_{s+3})=y_{s-1}-M_{s-1}\delta+M_s\delta+M_{s+1}\delta+M_{s+2}\delta\ge y_{s-1}+2M_s\delta\ge y_{s-1}+4D\delta.$$

It follows that $s+3\in\mathcal{T}$.

Let us prove (3). We have to consider the cases where $\tau(s)=s+1,s+2,s+3$ separately. Let us assume again that $h_{s-1}=-1$ and $h_s=1$.

We have that $M_{s-1}\ge 2aD$ and $x(t_s)\in[y_{s-1}+D\delta,y_s]$.

Let us assume that $\tau(s)=s+1$. In this case, $x(t_{s+1})\le y_{s+1}$,

$$x(t_{s+1})\in[y_s-D\delta,y_{s+1}],\quad x(t_s)\in[y_s,y_{s+1}+D\delta],,$$

and

$$x(t)\in[y_s-D\delta,y_{s+1}+D\delta],\qquad t\in[t_s,t_{s+1}],$$

$$y(t)=y_s+M_s(t-t_s)\delta,\quad t\in[t_s,t_{s+1}].$$

Hence

$$|x(t_i) - y(t_i)| \le (M_{s-1} + D)\delta, \quad i = s-1, s, s+1,$$
$$|x(t) - y(t)| \le (M_{s-1} + 2D)\delta, \quad t \in [t_s, t_{s+1}].$$

Let us assume that $\tau(s) = s+2$. In this case, $x(t_{s+2}) \le y_{s+2}$, $x(t_{s+1}) > y_{s+1}$,

$$x(t_{s+2}) \in [y_{s+1} - D\delta, y_{s+2}], \quad x(t_{s+1}) \in [y_{s+1}, y_{s+2} + D\delta],$$
$$x(t_s) \in [y_{s+1} - D\delta, \min(y_{s-1} + D\delta, y_{s+2} + 2D],$$

and

$$x(t) \in [y_{s+1} - D\delta, y_{s+2} + D\delta], \quad t \in [t_{s+1}, t_{s+2}],$$
$$x(t) \in [y_{s+1} - D\delta, \min(y_{t-s} + 2D\delta, y_{s+2} + D\delta)], \quad t \in [t_s, t_{s+1}],$$
$$y(t) = y_{s+i} + M_{s+i}(t - t_{s+i})\delta, \quad t \in [t_{s+i}, t_{s+i+1}], \quad i = 0, 1, 2.$$

Hence

$$|x(t_i) - y(t_i)| \le (M_{s-1} + D)\delta, \quad i = s, s+1, s+2,$$
$$|x(t) - y(t)| \le (M_{s-1} + 2D)\delta, \quad t \in [t_s, t_{s+2}].$$

Let us assume that $\tau(s) = s+3$. In this case, $x(t_{s+3}) \le y_{s+3}$, $x(t_{s+2}) > y_{s+2}$, $x(t_{s+1}) > y_{s+1}$,

$$x(t_{s+3}) \in [y_{s+2} - D\delta, y_{s+3}],$$
$$x(t_{s+2}) \in [y_{s+2}, \min(y_{s-1} + 3D\delta, y_{s+3} + D\delta],$$
$$x(t_{s+1}) \in [\max(y_{s+1}, y_{s+2} - D\delta), \min(y_{s-1} + 2D\delta, y_{s+3} + 2D)],$$
$$x(t_s) \in [y_{s+1} - D\delta, \min(y_{s-1} + D\delta, y_{s+2} + 2D)],$$

and

$$x(t) \in [y_{s+2} - D\delta, y_{s+3} + D\delta], \quad t \in [t_{s+2}, t_{s+3}],$$
$$x(t) \in [y_{s+1} - D\delta, y_{s-1} + 3D\delta], \quad t \in [t_{s+1}, t_{s+2}],$$
$$x(t) \in [y_{s+1} - D\delta, y_{s+2} + D\delta], \quad t \in [t_s, t_{s+1}],$$
$$y(t) = y_{s+i} + M_{s+i}(t - t_{s+i})\delta, \quad t \in [t_{s+i}, t_{s+i+1}], \quad i = 0, 1, 2, 3.$$

Hence

$$|x(t_i) - y(t_i)| \le (M_{s-1} + D)\delta, \quad i = s, s+1, s+2, s+3$$
$$|x(t) - y(t)| \le (M_{s-1} + 2D)\delta, \quad t \in [t_s, t_{s+3}].$$

Then statement (ii) follows.

Let us prove statement (iii). Existence of η follows from (i), where it is shown that $\tau(0) < +\infty$, and from (2). Further, let us observe that, in the sequence $(h_{\tau+1}, h_{\tau+2}, h_{\tau+3}, ...)$, there are no quadruple occurrences of the same symbol, i.e., for all $m \ge \tau$,

$$(h_{m+1}, h_{m+2}, h_{m+3}, h_{m+4}) \ne \pm(1,1,1,1). \tag{4}$$

We will use the induction method. Assume that the statement holds for $k \in [\eta, m]$, where $m \in \mathcal{T}$. It suffices to show that there exists $m_0 \in \{m+1, m+2, m+3\} \cap \mathcal{T}$ such that the statement holds for $k \in \{m+1, ..., m_0\}$. For certainty, we assume that $h_m = 1$. This means that $M_m = \bar{M}$ and $h_{m-1} = -1$.

- Assume that $h_{m+1} = -1$. It follows that $M_{m+1} = \bar{M}$ and $m+1 \in \mathcal{T}$.
- Assume that $(h_{m+1}, h_{m+2}) = (1, -1)$. It follows that $(M_{m+1}, M_{m+2}) = (\bar{M}, \bar{M})$ and $m+2 \in \mathcal{T}$.
- Assume that $(h_{m+1}, h_{m+2}) = (1, 1)$. It follows from (4) that $h_{m+3} = -1$. Hence $(M_{m+1}, M_{m+2}, M_{m+3}) = (\bar{M}, a\bar{M}, \bar{M})$ and $m+3 \in \mathcal{T}$.

In addition, (1) holds for (θ, τ) replaced by (m, m_0). By induction, the proof of (iv) follows. This completes the proof of Theorem 1. $\square$

References

[1] Dokuchaev, N.G. and Savkin, A.V. A new class of hybrid dynamical systems: state estimators with bit-rate constraints. *International Journal of Hybrid Systems.* **1** (2001), No 1, pp.33-50.

[2] G. Nair, R.J. Evans, State estimation under bit-rate constraints. In Proc. 37th IEEE Conf. Decision and Control, 1998.

[3] D.J. Stilwell, B.E. Bishop, Platoons of underwater vehicles, IEEE Control Systems Magazine., **20**(6) (2000) 45-52.

[4] W.S. Wong, R.W. Brockett, Systems with finite communication bandwidth constraints - Part 1: state estimation problems. *IEEE Trans. Automat. Contr.*, **42**(9), (1997) 1294-1299.

Figure 1: Example of a continuous process $x(t)$ and the corresponding processes $y(t)$ approximating $x(t)$ with $\delta = 0.04$ and $\delta = 0.02$ respectively, with $\bar{M} = 4\delta$. Transmission of the corresponding coded signals requires to transmit 50 bits and 100 bits respectively.

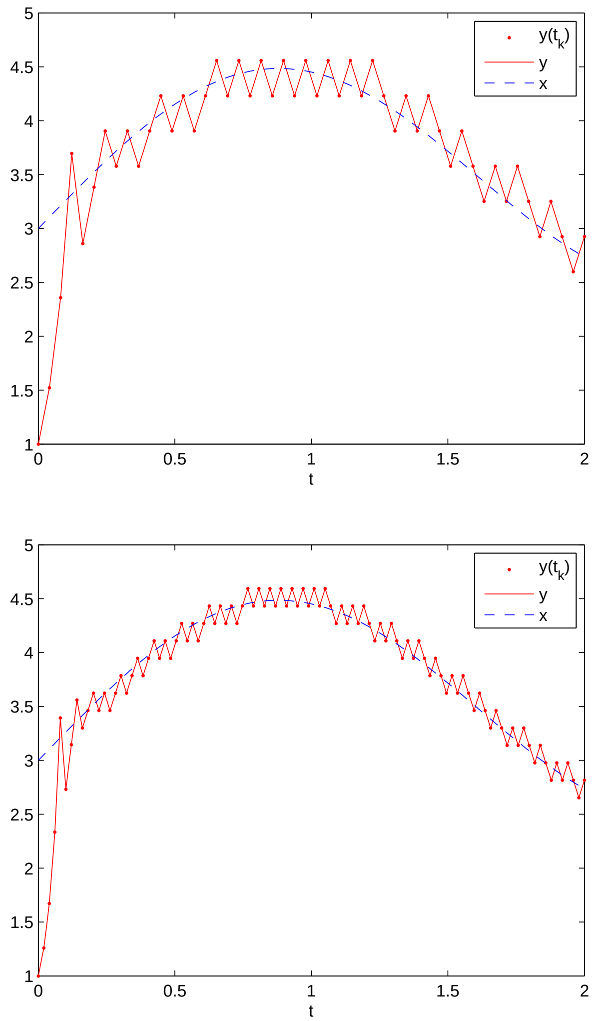

Figure 2: Example of a discontinuous process $x(t)$ and the corresponding processes $y(t)$ approximating $x(t)$ with $\delta = 0.04$ and $\delta = 0.02$ respectively, with $\bar{M} = 4\delta$. Transmission of the corresponding coded signals requires to transmit 50 bits and 100 bits respectively.

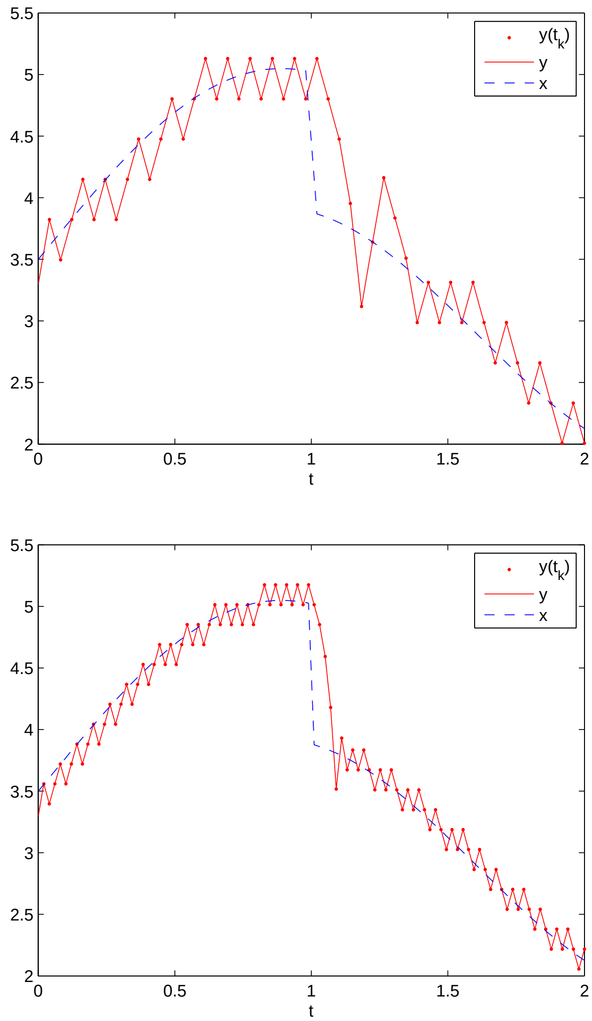

Figure 3: Example of a discontinuous process $x(t)$ and the corresponding processes $y(t)$ approximating $x(t)$ with $\delta = 0.04$ and $\delta = 0.02$ respectively, with $\bar{M} = \delta/4$. Transmission of the corresponding coded signals requires to transmit 50 bits and 100 bits respectively.