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# Nonlinear norm-observability and simulation of control systems[☆]

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ABSTRACT

The (bi)simulation relation has recently been attracting growing interest in the study of nonlinear control systems, in the hope that through such a relation, the behaviors and properties of a nonlinear system can be inferred from those of another system which is easier to handle. In this paper, we consider the propagation of the property of nonlinear norm-observability through a simulation relation. Given two control systems that are related by a graph simulation relation, we derive conditions under which the norm-observability of the simulating system implies the norm-observability of the simulated system. The obtained results are given in terms of set-valued functions. Several examples are included to illustrate various applications of our results.



## 1. Introduction

In many cases, high fidelity models to accurately represent a dynamical system may be too intricate for use in system analysis and control design. It is therefore desirable to have a methodology that relates "complex" models (for example, models with high nonlinearity) to "simple" ones (for example, systems being linear or mildly nonlinear), while preserving certain properties of interest relevant for analysis or synthesis. In the past decades, approaches based on (bi)simulation relations have been introduced in the study of controlled dynamical systems, exploring the possibility of connecting a system with another system whose behaviors and/or properties are easier to understand (see, e.g., [1–4]). (Bi)simulation relations are natural and important objects in control systems theory. Loosely speaking, a simulation between two dynamical systems defines a relation with the property that every trajectory of the first system can be associated with a trajectory of the second system. If the association is bidirectional, then one obtains a bisimulation relation between the two dynamical systems. The notions of simulation and bisimulation relations provide a potentially useful tool for classifying linear and nonlinear systems [2,4]. They also have interesting connections with other fundamental concepts in nonlinear systems theory such as controlled invariance [1,3,5] and feedback transformations [6]. As already stated, an important motivation for studying (bi)simulation relations is to hope to reason about certain properties across related systems. Some pertinent work includes studies on reasoning about controllability of ($C$-related) linear systems [7], reasoning about stability properties of hybrid systems [8], and the propagation of controllability properties through a simulation relation for nonlinear systems [9].

Observability is certainly one of the key concepts in control theory. In the context of nonlinear systems, various observability definitions have been proposed in the literature in order to capture the relationship between the state, the output, and the input of a system (see, e.g., [10]). The notion of norm-observability was introduced in [11] and [12]. Rather than inferring the precise value of the state, the norm-observability properties describe the ability to determine an upper bound on the norm of the state using the output and the input. As pointed out in [12], such observability properties have close ties to the important concept of input–output-to-state stability in nonlinear systems analysis [13]. The problem of determining whether a system is norm-observable, besides being interesting in itself, is particularly relevant in the context of switched nonlinear systems, as it is strongly related to the stability and supervisory control of the systems (see, e.g., [12,14,15]).

In this paper, we focus on the notion of norm-observability and examine the extent to which the norm-observability properties of nonlinear systems are preserved by simulation relations. More specifically, given two control systems that are connected by a simulation relation, our main objective is to determine conditions that allow us to propagate the norm-observability properties from

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the simulating system to the simulated system, suggesting that an observability analysis of the simulating system can shed light on the norm-observability properties of the simulated system. Currently, the main tool used to test norm-observability for nonlinear systems in the literature, to our knowledge, is the Lyapunov-like method [12]. We demonstrate by example that our results offer a new possibility for the norm-observability analysis of nonlinear systems. The notion of simulation relation embraces many different types, such as exact simulation relations, approximate simulation relations [16–19], alternating simulation relations [20, Chapters 4.3 and 9.2], contractive simulation relations [21], and graph simulation relations [6,9]. Depending on the context, some relations may be more appropriate to use than others. The simulation relations considered in the paper are the so-called graph simulation relations. As will be seen, such relations are the right tool to use to reason about nonlinear norm-observability.

*Organization:* The notions of norm-observability and graph simulation are presented in Section 2. Main results, establishing the conditions that propagate norm-observability, are proposed in Section 3. Then, several illustrative examples are given in Section 4, and a brief conclusion is drawn in the final section.

*Notation and terminology:* We use $|\cdot|$ to denote the standard Euclidean norm, and $\|z\|_I$ the essential supremum norm of a function $z(t)$ on an interval $I$. We write $B^n(r)$ for the closed ball in $\mathbb{R}^n$ with center 0 and radius $r > 0$. For a function $g : A \to B$, the *graph* of $g$, denoted by Graph$(g)$, is defined as Graph$(g) = \{(a, g(a)) : a \in A\}$. Let $X$ and $Y$ be finite-dimensional Euclidean spaces. A *set-valued function* $F$ from $X$ to $Y$ is a function that associates with any $x \in X$ a subset $F(x)$ of $Y$. If $K \subseteq X$ and if $F$ is a set-valued function from $X$ to $Y$, the image of the set $K$ under $F$ is given by $F(K) = \cup_{x\in K} F(x)$. A set-valued function $F$ is said to be *bounded* if the image of any bounded set under $F$ is bounded. We say that $F$ is *upper semicontinuous* at $x \in X$ if for any open $N$ containing $F(x)$ there exists a neighborhood $M$ of $x$ such that $F(M) \subseteq N$.

## 2. Preliminaries

### 2.1. Norm-observability notions

To make the paper reasonably self-contained, we briefly recall the definitions of norm-observability introduced in [12]. Consider the following system

$$\Sigma : \quad \dot{x} = f(x, u), \qquad y = h(x). \tag{1}$$

We assume that (see, e.g., [10]) the function $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is so that $f(\cdot, u)$ is of class $C^1$ for each fixed $u \in \mathbb{R}^m$, $f$ and $\partial f/\partial x$ are continuous on $\mathbb{R}^n \times \mathbb{R}^m$, and $f(0, 0) = 0$, and that $h : \mathbb{R}^n \to \mathbb{R}^p$ is continuous with $h(0) = 0$. By an input or control for (1), we mean a measurable function $u(\cdot) : \mathbb{R} \to \mathbb{R}^m$ which is essentially compact valued on compact intervals, i.e., for every compact interval $I \subseteq \mathbb{R}$ there exists a compact subset $K \subseteq \mathbb{R}^m$ such that $u(t) \in K$ for almost all $t \in I$ [6,9]. We denote by $\mathcal{U}^m_{\text{cpt}}$ the set of all inputs. For any $u(\cdot) \in \mathcal{U}^m_{\text{cpt}}$ and any $x_0 \in \mathbb{R}^n$, there exists a unique maximally extended solution of the initial value problem

$$\dot{x} = f(x, u(t)), \qquad x(0) = x_0.$$

Such a solution is defined on some open interval $(t^{\min}_{x_0,u}, t^{\max}_{x_0,u})$ containing 0. We assume that the system $\Sigma$ has the unboundedness observability property [22], which means that for every initial state $x_0$ and input $u$ such that $t^{\max}_{x_0,u} < \infty$, the corresponding output becomes unbounded as $t \to t^{\max}_{x_0,u}$. We recall that a function $\alpha : [0, \infty) \to [0, \infty)$ is said to be of class $\mathcal{K}_\infty$ if it is continuous, strictly increasing, unbounded, and $\alpha(0) = 0$.

**Definition 1** (*[12]*).

(a) We say that the system $\Sigma$ is *small-time norm-observable* if for every $\tau > 0$, there exist $\mathcal{K}_\infty$ functions $\gamma$ and $\chi$ such that for every $x_0 \in \mathbb{R}^n$ and for every $u \in \mathcal{U}^m_{\text{cpt}}$, it holds that

$$|x_0| \le \gamma(\|y\|_{[0,\tau]}) + \chi(\|u\|_{[0,\tau]}). \tag{2}$$

(b) We say that $\Sigma$ is *large-time norm-observable* if there exist $\tau > 0$ and two class $\mathcal{K}_\infty$ functions $\gamma$ and $\chi$ such that (2) holds for any $x_0 \in \mathbb{R}^n$ and any input $u \in \mathcal{U}^m_{\text{cpt}}$.

**Remark 1.** Roughly speaking, norm-observability imposes a bound on the norm of the initial state in terms of the norms of the output and the input. The principal difference between small-time norm-observability and large-time norm-observability is that the former requires the inequality (2) to hold for arbitrary $\tau$, while the latter requires (2) to hold for at least one $\tau > 0$. It is clear from the definition that small-time norm-observability implies large-time norm-observability. Note that the converse is, in general, not true. However, for linear systems these two notions are known to be equivalent and are both equivalent to the usual concept of observability [12].

**Remark 2.** Other equivalent definitions of small-time and large-time norm-observability can be achieved under the assumption of the unboundedness observability property for the system $\Sigma$ and its reversed-time system; see [12] for more information.

### 2.2. Graph simulation relations

Consider the system $\Sigma$ together with another system

$$\widetilde{\Sigma} : \quad \dot{z} = \tilde{f}(z, v), \qquad w = \tilde{h}(z). \tag{3}$$

Here, the function $\tilde{f} : \mathbb{R}^{\tilde{n}} \times \mathbb{R}^{\tilde{m}} \to \mathbb{R}^{\tilde{n}}$ is such that $\tilde{f}(\cdot, v)$ is a $C^1$ function for each fixed $v \in \mathbb{R}^{\tilde{m}}$, $\tilde{f}$ and $\partial\tilde{f}/\partial z$ are continuous on $\mathbb{R}^{\tilde{n}} \times \mathbb{R}^{\tilde{m}}$, and $\tilde{f}(0, 0) = 0$; and $\tilde{h} : \mathbb{R}^{\tilde{n}} \to \mathbb{R}^{\tilde{p}}$ is continuous and vanishes at 0. The following definition is patterned after that given in [6] and [9].

**Definition 2.** Given $\Sigma$ and $\widetilde{\Sigma}$, a pair of relations $(\mathcal{S}, \mathcal{R})$, where $\mathcal{S} \subseteq \mathbb{R}^n \times \mathbb{R}^{\tilde{n}}$ and $\mathcal{R} \subseteq \mathbb{R}^p \times \mathbb{R}^{\tilde{p}}$, is called a *compact graph simulation relation* of $\Sigma$ by $\widetilde{\Sigma}$ if the following conditions are satisfied:

(a) The relation $\mathcal{S}$ is the graph of a $C^2$ function $\Phi : \mathbb{R}^n \to \mathbb{R}^{\tilde{n}}$ with the following property: given any $x \in \mathbb{R}^n$ and any $u \in \mathbb{R}^m$, there exist open neighborhoods $X \subseteq \mathbb{R}^n$ of $x$ and $U \subseteq \mathbb{R}^m$ of $u$, and a compact set $V \subseteq \mathbb{R}^{\tilde{m}}$ such that for every $x' \in X$ and $u' \in U$ there is some $v' \in V$ such that
$$\frac{\partial \Phi}{\partial x}(x)\Big|_{x=x'} f(x', u') = \tilde{f}(\Phi(x'), v').$$

(b) For every $x \in \mathbb{R}^n$ we have $(h(x), \tilde{h}(\Phi(x))) \in \mathcal{R}$.

We call $\widetilde{\Sigma}$ the *simulating* system and $\Sigma$ the *simulated* system.

Note that this definition is slightly different from the one of [6] and [9] in that in condition (b) we only require the outputs $h(x)$ and $\tilde{h}(\Phi(x))$ to be related by a relation $\mathcal{R}$, rather than identical.

**Remark 3.** Intuitively, a simulation should specify that every trajectory of the simulated (or original) system can be matched by a trajectory of the simulating (or abstract) system. Certainly, one can define the concept of simulation relation by directly using this idea. But, in practice, such a definition may be inconvenient to check, especially for nonlinear systems, since it requires knowledge of the system trajectories. On the other hand, conditions (a) and (b) of Definition 2 are relatively easy to verify and suffice to guarantee that the simulating system has the capability of mimicking the behavior of the simulated system [6].

The assumptions that $\widetilde{\Sigma}$ simulates $\Sigma$ and that $\widetilde{\Sigma}$ is norm-observable are not by themselves sufficient to imply the norm-observability of $\Sigma$. As a simple counter-example, consider the linear systems

$$\Sigma: \quad \begin{aligned} \dot{x}_1 &= x_1 + u, \\ \dot{x}_2 &= -x_2, \\ y &= x_1, \end{aligned} \qquad \text{and} \qquad \widetilde{\Sigma}: \quad \begin{aligned} \dot{z} &= z + v, \\ w &= z. \end{aligned}$$

Let $\Phi : \mathbb{R}^2 \to \mathbb{R}$ be the mapping defined by $\Phi(x_1, x_2) = x_1$, and let $\mathcal{R} = \{(y, w) : y = w\}$. Then $(\mathrm{Graph}(\Phi), \mathcal{R})$ is a compact graph simulation relation of $\Sigma$ by $\widetilde{\Sigma}$, and $\widetilde{\Sigma}$ is observable, but, obviously, $\Sigma$ is not observable. Additional conditions must be introduced in order to propagate the property of norm-observability from the simulating system to the simulated system.

## 3. Main results

### 3.1. Small-time and large-time norm-observability

In this section, we first derive a small-time norm-observability result for graph simulation relations. The result makes use of the following lemma, which describes the small-time norm-observability property without using class $\mathcal{K}_\infty$ functions.

**Lemma 1.** *The system $\Sigma$ is small-time norm-observable if and only if the following conditions are satisfied:*

(a) *For every $\tau > 0$ and every $\varepsilon > 0$, there exists a $\delta > 0$ such that $\|y\|_{[0,\tau]} \le \delta$, $\|u\|_{[0,\tau]} \le \delta$, $u \in \mathcal{U}^m_{\mathrm{cpt}}$ implies $|x(0)| \le \varepsilon$.*
(b) *For every $\tau > 0$ and every $r \ge 0$, there exists an $M \ge 0$ such that $\|y\|_{[0,\tau]} \le r$, $\|u\|_{[0,\tau]} \le r$, $u \in \mathcal{U}^m_{\mathrm{cpt}}$ implies $|x(0)| \le M$.*

**Proof.** (*Sufficiency*). Suppose that conditions (a) and (b) are satisfied. Fix $\tau > 0$, and define

$$A(r) = \{\xi \in \mathbb{R}^n : \text{there exists } u \in \mathcal{U}^m_{\mathrm{cpt}}, \text{ with } \|u\|_{[0,\tau]} \le r, \text{ such that } \|y_{\xi,u}\|_{[0,\tau]} \le r\}$$

for $r \ge 0$, where $y_{\xi,u}$ is the output of $\Sigma$ corresponding to $u$ with $x(0) = \xi$. Clearly, $A(r)$ is nonempty for each $r \ge 0$. Let

$$\alpha_1(r) = \sup\{|\xi| : \xi \in A(r)\}.$$

Then $0 \le \alpha_1(r) < \infty$ for each $r \ge 0$ because of condition (b). It is easy to see that $\alpha_1$ is nondecreasing on $[0, \infty)$. Also, it can be shown that $\alpha_1(r) \to 0$ as $r \to 0$. In fact, let $\varepsilon > 0$ be given. Then by condition (a) there exists some $\delta > 0$ such that $\|u\|_{[0,\tau]} \le \delta$, $u \in \mathcal{U}^m_{\mathrm{cpt}}$, $\|y_{\xi,u}\|_{[0,\tau]} \le \delta$ implies $|\xi| \le \varepsilon$. For any $\xi \in A(\delta)$, since there exists $u \in \mathcal{U}^m_{\mathrm{cpt}}$ with $\|u\|_{[0,\tau]} \le \delta$ so that $\|y_{\xi,u}\|_{[0,\tau]} \le \delta$, it follows that $|\xi| \le \varepsilon$. Hence $\alpha_1(\delta) \le \varepsilon$, and therefore $\alpha_1(r) \le \varepsilon$ if $0 < r < \delta$. This shows that $\alpha_1(r)$ tends to 0 as $r \to 0$.

Define

$$\alpha_2(r) = \begin{cases} \dfrac{1}{r}\displaystyle\int_r^{2r} \alpha_1(s)ds & (\text{if } r > 0), \\ 0 & (\text{if } r = 0). \end{cases}$$

Then $\alpha_2$ is continuous on $[0, \infty)$ and satisfies $\alpha_2(r) \ge \alpha_1(r)$ for all $r \ge 0$. Put

$$\alpha(r) = r + \max\{\alpha_2(s) : 0 \le s \le r\}$$

for $r \ge 0$. Then $\alpha$ is of class $\mathcal{K}_\infty$. Let $\xi \in \mathbb{R}^n$ and let $u \in \mathcal{U}^m_{\mathrm{cpt}}$. If $\|y_{\xi,u}\|_{[0,\tau]} = \infty$, it is clear that $|\xi| \le \alpha(\|u\|_{[0,\tau]}) + \alpha(\|y_{\xi,u}\|_{[0,\tau]})$. If $\|y_{\xi,u}\|_{[0,\tau]} < \infty$, put $\mu = \max\{\|u\|_{[0,\tau]}, \|y_{\xi,u}\|_{[0,\tau]}\}$. Then $\xi \in A(\mu)$. Hence

$$|\xi| \le \alpha_1(\mu) \le \alpha(\mu) \le \alpha(\|u\|_{[0,\tau]}) + \alpha(\|y_{\xi,u}\|_{[0,\tau]}).$$

Then $\Sigma$ is small-time norm-observable.

The necessity follows directly from the properties of class $\mathcal{K}_\infty$ functions. □

Consider systems $\Sigma$ and $\widetilde{\Sigma}$. Let $\Phi : \mathbb{R}^n \to \mathbb{R}^{\tilde{n}}$ be a $C^2$ function and suppose that $(\mathrm{Graph}(\Phi), \mathcal{R})$ is a compact graph simulation relation of $\Sigma$ by $\widetilde{\Sigma}$. Let $F$ be the set-valued function from $\mathbb{R}^{p+m}$ to $\mathbb{R}^{\tilde{m}}$ defined by

$$F(y, u) = \{v \in \mathbb{R}^{\tilde{m}} : \text{there exists } x \in \mathbb{R}^n \text{ such that } h(x) = y \text{ and } (\partial\Phi/\partial x)f(x, u) = \tilde{f}(\Phi(x), v)\},$$

and let $G$ be the set-valued function from $\mathbb{R}^p$ to $\mathbb{R}^{\tilde{p}}$ defined by

$$G(y) = \{w \in \mathbb{R}^{\tilde{p}} : (y, w) \in \mathcal{R}\}.$$

We are now in a position to show one of the main results of this section.

**Theorem 1.** *Consider systems $\Sigma$ and $\widetilde{\Sigma}$. Suppose $(\mathrm{Graph}(\Phi), \mathcal{R})$ is a compact graph simulation relation of $\Sigma$ by $\widetilde{\Sigma}$, and*

(a) *$F(0, 0) = \{0\}$, $G(0) = \{0\}$, $\Phi^{-1}(0) = \{0\}$,*
(b) *$F$, $G$, and $\Phi^{-1}$ are upper semicontinuous at the origins in $\mathbb{R}^{p+m}$, $\mathbb{R}^p$, and $\mathbb{R}^{\tilde{n}}$, respectively,*
(c) *$F$, $G$, and $\Phi^{-1}$ are bounded.*

*If $\widetilde{\Sigma}$ is small-time norm-observable, then $\Sigma$ is small-time norm-observable.*

**Proof.** We will show that $\Sigma$ satisfies conditions (a) and (b) of Lemma 1.

Fix $\tau > 0$, and fix $\varepsilon > 0$. Since $\Phi^{-1}$ is upper semicontinuous at 0 and $\Phi^{-1}(0) = \{0\}$, we can choose $\varepsilon' > 0$ such that $\Phi^{-1}(B^{\tilde{n}}(\varepsilon')) \subseteq B^n(\varepsilon)$. Since $\widetilde{\Sigma}$ is assumed to be small-time norm-observable, there exists a $\delta' > 0$ such that $\|w\|_{[0,\tau]} \le \delta'$, $\|v\|_{[0,\tau]} \le \delta'$, $v \in \mathcal{U}^{\tilde{m}}_{\mathrm{cpt}}$ implies $|z(0)| \le \varepsilon'$. Analogously, we can choose $\delta > 0$ such that

$$F(B^{p+m}(2\delta)) \subseteq B^{\tilde{m}}(\delta') \quad \text{and} \tag{4}$$
$$G(B^p(\delta)) \subseteq B^{\tilde{p}}(\delta'). \tag{5}$$

Let $x \in \mathbb{R}^n$, $u \in \mathcal{U}^m_{\mathrm{cpt}}$, and suppose that

$$\|u\|_{[0,\tau]} \le \delta \quad \text{and} \quad \|h \circ \psi\|_{[0,\tau]} \le \delta, \tag{6}$$

where $\psi$ is the trajectory of the system $\Sigma$ corresponding to the initial condition $x$ and the input $u$. We will show that $|x| \le \varepsilon$. Note that $\psi(t)$ is well defined on $[0, \tau]$ since the system $\Sigma$ has the unboundedness observability property. Define $\tilde{\psi} = \Phi \circ \psi$. Since $(\mathrm{Graph}(\Phi), \mathcal{R})$ is a compact graph simulation relation of $\Sigma$ by $\widetilde{\Sigma}$, there exists $v \in \mathcal{U}^{\tilde{m}}_{\mathrm{cpt}}$ such that

$$\dot{\tilde{\psi}}(t) = \tilde{f}(\tilde{\psi}(t), v(t))$$

for almost all $t \in [0, \tau]$ (see [6]). In other words, $\tilde{\psi}$ is the trajectory of the system $\widetilde{\Sigma}$ corresponding to the initial condition $\Phi(x)$ and the control $v$. It is easily seen from the definition of $F$ that for almost all $t \in [0, \tau]$, we have $v(t) \in F(h(\psi(t)), u(t))$. This combined with (4) and (6) shows that $\|v\|_{[0,\tau]} \le \delta'$. Similarly, for every $t \in [0, \tau]$ we have $\tilde{h}(\tilde{\psi}(t)) \in G(h(\psi(t)))$. Hence $\|\tilde{h} \circ \tilde{\psi}\|_{[0,\tau]} \le \delta'$. It follows that $|\Phi(x)| \le \varepsilon'$, so that

$$x \in \Phi^{-1}(B^{\tilde{n}}(\varepsilon')) \subseteq B^n(\varepsilon),$$

or equivalently $|x| \le \varepsilon$, and we conclude that the system $\Sigma$ satisfies condition (a) of Lemma 1.

Next, fix $r \ge 0$. It follows from (c) that there exists an $r' \ge 0$ such that

$$F(B^{p+m}(2r)) \subseteq B^{\tilde{m}}(r') \quad \text{and} \quad G(B^p(r)) \subseteq B^{\tilde{p}}(r').$$

Choose $M' \ge 0$ so that $\|w\|_{[0,\tau]} \le r'$, $\|v\|_{[0,\tau]} \le r'$, $v \in \mathcal{U}^{\tilde{m}}_{\mathrm{cpt}}$ implies $|z(0)| \le M'$, and then choose $M \ge 0$ such that $\Phi^{-1}(B^{\tilde{n}}(M')) \subseteq B^n(M)$. Let $x \in \mathbb{R}^n$, and let $u \in \mathcal{U}^m_{\mathrm{cpt}}$. A similar argument shows that whenever $\|u\|_{[0,\tau]} \le r$ and $\|h \circ \psi\|_{[0,\tau]} \le r$, then $|x| \le M$. (Here $\psi$

denotes the trajectory of $\Sigma$ corresponding to the initial condition $x$ and the control $u$.) Hence, the system $\Sigma$ satisfies condition (b) of Lemma 1, and the theorem is proved. □

Next, we give a large-time norm-observability result for graph simulation relations. The discussion parallels that of small-time norm-observability. In a manner analogous to the small-time norm-observability property, the large-time norm-observability property can also be described without using class $\mathcal{K}_\infty$ functions, as the following lemma shows.

**Lemma 2.** *The system $\Sigma$ is large-time norm-observable if and only if there is a $\tau > 0$ with the following two properties:*

(a) *For every $\varepsilon > 0$, there is $\delta > 0$ such that $\|y\|_{[0,\tau]} \le \delta$, $\|u\|_{[0,\tau]} \le \delta$, $u \in \mathcal{U}^m_{\mathrm{cpt}}$ implies $|x(0)| \le \varepsilon$.*
(b) *For every $r \ge 0$, there is $M \ge 0$ such that $\|y\|_{[0,\tau]} \le r$, $\|u\|_{[0,\tau]} \le r$, $u \in \mathcal{U}^m_{\mathrm{cpt}}$ implies $|x(0)| \le M$.*

**Proof.** The proof is similar to that of Lemma 1. □

Using Lemma 2, the following result is established by the same arguments as Theorem 1.

**Theorem 2.** *Consider systems $\Sigma$ and $\widetilde{\Sigma}$. Suppose that $(\mathrm{Graph}(\Phi), \mathcal{R})$ is a compact graph simulation relation of $\Sigma$ by $\widetilde{\Sigma}$, and conditions (a)–(c) of Theorem 1 are satisfied. If $\widetilde{\Sigma}$ is large-time norm-observable, then $\Sigma$ is large-time norm-observable.* □

Theorems 1 and 2 tell us that under mild assumptions, the small-time and large-time norm-observability properties of a system can be inferred by analyzing its simulating system. We remark that the underlying idea of conditions (a)–(c) in Theorem 1 is quite natural. Clearly, the set-valued function $F$ relates the input and the output of the simulated system to the input of the simulating system, the set-valued function $G$ relates the output of the simulated system to that of the simulating system, and the function $\Phi$ relates the states of the simulated system and its simulating system. The idea of conditions (a) and (b) in Theorem 1 is, roughly, to guarantee the propagation of the property that the initial state should be small provided the inputs and outputs are small, and the idea of condition (c) is to ensure the propagation of the property that the initial state should be bounded provided the inputs and outputs are bounded.

### 3.2. Systems with no inputs

For the system with no inputs

$$\dot{x} = f(x), \qquad y = h(x),$$

one can define corresponding observability notions by omitting the term $\chi(\|u\|_{[0,\tau]})$ from the right side of the inequality (2); see [11] and [12]. We will also use the terminologies *small-time norm-observability* and *large-time norm-observability* for the corresponding variants of the observability notions. As a special case, our observability results for graph simulation relations are easily adapted to the situation of systems without inputs. More precisely, consider two autonomous systems

$$\Sigma_i: \quad \dot{x}_i = f_i(x_i), \qquad y_i = h_i(x_i), \qquad x_i \in \mathbb{R}^{n_i}, \\ y_i \in \mathbb{R}^{p_i}, \quad i = 1, 2. \tag{7}$$

We assume that $f_i(0) = 0$ and $h_i(0) = 0$ for $i = 1, 2$. We say that a pair of relations $(\mathcal{S}, \mathcal{R})$, with $\mathcal{S} \subseteq \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ and $\mathcal{R} \subseteq \mathbb{R}^{p_1} \times \mathbb{R}^{p_2}$, is a *graph simulation relation* of $\Sigma_1$ by $\Sigma_2$ if the following two conditions are satisfied:

(a) The relation $\mathcal{S}$ is the graph of a $C^2$ function $\Phi : \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ with the property that $(\partial\Phi/\partial x)f_1(x) = f_2(\Phi(x))$ for all $x \in \mathbb{R}^{n_1}$.
(b) The relation $\mathcal{R}$ is such that $(h_1(x), h_2(\Phi(x))) \in \mathcal{R}$ for all $x \in \mathbb{R}^{n_1}$.

Let $G$ be the set-valued function from $\mathbb{R}^{p_1}$ to $\mathbb{R}^{p_2}$ given by

$$G(y_1) = \{y_2 \in \mathbb{R}^{p_2} : (y_1, y_2) \in \mathcal{R}\}.$$

The following corollary is a straightforward adaptation of Theorems 1 and 2.

**Corollary 1.** *Consider systems $\Sigma_1$ and $\Sigma_2$ as given in (7). Suppose that $(\mathrm{Graph}(\Phi), \mathcal{R})$ is a graph simulation relation of $\Sigma_1$ by $\Sigma_2$, and*

(a) *$G(0) = \{0\}$, $\Phi^{-1}(0) = \{0\}$,*
(b) *$G$ and $\Phi^{-1}$ are upper semicontinuous at the origins in $\mathbb{R}^{p_1}$ and $\mathbb{R}^{n_2}$, respectively,*
(c) *$G$ and $\Phi^{-1}$ are bounded.*

*If $\Sigma_2$ is small-time (resp., large-time) norm-observable, then $\Sigma_1$ is small-time (resp., large-time) norm-observable.* □

**Remark 4.** The conditions proposed in Theorem 1 and Corollary 1 for reasoning about norm-observability do not imply any assumptions about the state–space dimensions of the related systems. In other words, our results are valid regardless of whether the simulating system has a smaller state–space dimension than the simulated system. Of course, in practice, the simulating system is often chosen to be of a reduced dimension. Notwithstanding, the proposed observability results may also provide valuable insight when the state–space dimension of the simulating system is not strictly less than that of the simulated system (see Examples 2 and 3 in the next section for illustrations).

## 4. Examples

This section contains four examples to illustrate various applications of the ideas discussed so far.

**Example 1.** In this example, we specialize our results for the case of linear time-invariant systems. Consider the linear system

$$\dot{x} = Ax, \qquad y = Cx, \tag{8}$$

with $x \in \mathbb{R}^n$, $y \in \mathbb{R}^p$, $A \in \mathbb{R}^{n\times n}$, and $C \in \mathbb{R}^{p\times n}$. Let $V : \mathbb{R}^n \to \mathbb{R}^{\tilde{n}}$ be a linear injection, and let $T : \mathbb{R}^p \to \mathbb{R}^{\tilde{p}}$ be a linear mapping. We identify $V$ and $T$ with matrices of sizes $\tilde{n} \times n$ and $\tilde{p} \times p$, respectively. Suppose that there exist an $\tilde{n} \times \tilde{n}$ matrix $\widetilde{A}$ and a $\tilde{p} \times \tilde{n}$ matrix $\widetilde{C}$ such that $\widetilde{A}V = VA$ and $\widetilde{C}V = TC$. Put $\mathcal{R} = \{(y, Ty) : y \in \mathbb{R}^p\}$. Then it is straightforward to see that $(\mathrm{Graph}(V), \mathcal{R})$ is a graph simulation relation of the system (8) by the system

$$\dot{z} = \widetilde{A}z, \qquad w = \widetilde{C}z, \tag{9}$$

with $z \in \mathbb{R}^{\tilde{n}}$ and $w \in \mathbb{R}^{\tilde{p}}$. We would like to point out that although the size of the system (9) is, in fact, not smaller than that of the system (8) (since $V$ is injective), the system (9) may in fact have a clear structure whose observability is easier to detect (for example, of the phase-variable canonical form). It is easily seen that $G(y) = \{Ty\}$ for every $y \in \mathbb{R}^p$, and that for $z \in \mathbb{R}^{\tilde{n}}$ the inverse image of $z$ under the mapping $V$ is $\{V^+z\}$ if $z \in \mathrm{Im}\, V$, and the empty set otherwise. (Here $V^+ = (V^TV)^{-1}V^T$ is the Moore–Penrose pseudoinverse of $V$.) Clearly, conditions (a)–(c) of Corollary 1 are satisfied. Thus, observability of the system (9) implies observability of the system (8).

We remark that the propagation of observability can be achieved in the opposite direction (namely, from the system (8) to the system (9)), with the additional assumptions that $T$ is injective, and that

$$\bigcap_{i=1}^{\tilde{n}} \mathrm{Ker}\,(\widetilde{C}\widetilde{A}^{i-1}) \subseteq \mathrm{Im}\,V. \tag{10}$$

In fact, if (10) holds, then to each $a \in \cap_{i=1}^{\tilde{n}} \mathrm{Ker}\,(\widetilde{C}\widetilde{A}^{i-1})$ there is associated a $b \in \mathbb{R}^n$ such that $Vb = a$. From this and induction we have

$$TCA^{i-1}b = \widetilde{C}\widetilde{A}^{i-1}Vb = 0 \quad \text{for} \quad i = 1, 2, 3, \ldots,$$

so that $b \in \cap_{i=1}^{n} \mathrm{Ker}\,(CA^{i-1})$ (since $T$ is injective). Hence

$$\bigcap_{i=1}^{n} \mathrm{Ker}\,(CA^{i-1}) = 0 \quad \Rightarrow \quad \bigcap_{i=1}^{\tilde{n}} \mathrm{Ker}\,(\widetilde{C}\widetilde{A}^{i-1}) = 0.$$

That is, observability of (8) implies observability of (9). Furthermore, when the system (9) is observable, (10) is obviously satisfied. We then summarize that, under the assumption that $T$ is injective, the system (9) is observable if and only if the system (8) is observable and the condition (10) is satisfied. A similar result appeared in the previous literature [23], where the result was obtained in the context of the *inclusion principle*. Here, we come to the result in a different setting and provide more detailed and complete information on observability propagation. (For example, we show that the observability propagation from (9) to (8) can be achieved directly without preassuming that $T$ is injective; see [23] for more details.) □

**Example 2.** It is apparent that our results may provide a new strategy for the norm-observability analysis of nonlinear systems. As an example, consider the dynamical model of the single-link manipulator with flexible joint and negligible damping [24]

$$\begin{aligned}
\dot{x}_1 &= x_2,\\
\dot{x}_2 &= -a\sin x_1 - b(x_1 - x_3),\\
\dot{x}_3 &= x_4,\\
\dot{x}_4 &= c(x_1 - x_3) + du,\\
y_1 &= x_1, \quad y_2 = x_2, \quad y_3 = x_3,
\end{aligned} \tag{11}$$

where $x_1$ is the link angle and $x_2$ the corresponding angle velocity, $x_3$ is the motor angle and $x_4$ the corresponding velocity, $u$ is the input torque, and $a$, $b$, $c$, and $d$ are positive constants. We assume $x_1$, $x_2$, and $x_3$ as measurable variables. The state equation of (11) can be transformed into the form

$$\begin{aligned}
\dot{z}_i &= z_{i+1}, \qquad i = 1, 2, 3,\\
\dot{z}_4 &= -(b + c)z_3 + a(z_2^2 - c)\sin z_1 - az_3\cos z_1 + bdu
\end{aligned}$$

through the global diffeomorphism

$$z = \Phi(x) = \begin{bmatrix} x_1 \\ x_2 \\ -a\sin x_1 - b(x_1 - x_3) \\ -ax_2\cos x_1 - b(x_2 - x_4) \end{bmatrix}$$

(see, e.g., [25]). Define a new input $v$ such that

$$v = a(z_2^2 - c)\sin z_1 - az_3\cos z_1 + bdu,$$

and let

$$\begin{aligned}
\mathcal{R} \;&=\; \big\{(y, w) \in \mathbb{R}^3 \times \mathbb{R}^3 : w_1 = y_1,\; w_2 = y_2,\\
w_3 \;&=\; -a\sin y_1 - b(y_1 - y_3)\big\}.
\end{aligned}$$

It can be easily checked that (Graph($\Phi$), $\mathcal{R}$) is a compact graph simulation relation of the system (11) by the system

$$\begin{aligned}
&\dot{z}_1 = z_2, \quad \dot{z}_2 = z_3, \quad \dot{z}_3 = z_4,\\
&\dot{z}_4 = -(b + c)z_3 + v,\\
&w_1 = z_1, \quad w_2 = z_2, \quad w_3 = z_3,
\end{aligned}$$

which is observable. In order to use Theorems 1 and 2, we compute

$$F(y, u) = \big\{a(y_2^2 + a\cos y_1 - c)\sin y_1 + ab(y_1 - y_3)\cos y_1 + bdu\big\},$$

and

$$G(y) = \big\{(y_1, y_2, -a\sin y_1 - b(y_1 - y_3))\big\}.$$

We see that conditions (a)–(c) of Theorem 1 are satisfied. As a result, the system (11) is (small-time and large-time) norm-observable. □

**Example 3.** In this example, we show how it is possible to use the results to detect the *non-observability* property of a nonlinear system. Consider the system

$$\begin{aligned}
\dot{x}_1 &= x_2 + u_1,\\
\dot{x}_2 &= (\sin x_3)u_1,\\
\dot{x}_3 &= x_1^2\cos x_4 + u_2,\\
\dot{x}_4 &= x_1 + u_1 + x_2x_3u_2,\\
y_1 &= x_1, \quad y_2 = x_2.
\end{aligned} \tag{12}$$

We define new inputs $v_1$ and $v_2$ such that

$$v_1 = x_2 + u_1 \quad \text{and} \quad v_2 = x_1^2\cos x_4 + u_2,$$

respectively, so that the system (12) is in the form

$$\begin{aligned}
\dot{x}_1 &= v_1,\\
\dot{x}_2 &= -x_2\sin x_3 + (\sin x_3)v_1,\\
\dot{x}_3 &= v_2,\\
\dot{x}_4 &= x_1 - x_2 - x_1^2x_2x_3\cos x_4 + v_1 + x_2x_3v_2,\\
y_1 &= x_1, \quad y_2 = x_2.
\end{aligned} \tag{13}$$

If $I$ is the identity mapping on $\mathbb{R}^4$ and if $\mathcal{R} = \{(y, y) : y \in \mathbb{R}^2\}$, then it is easy to see that (Graph($I$), $\mathcal{R}$) is a compact simulation relation of the system (13) by the system (12). The set-valued functions $F$ and $G$ can be computed as

$$F(y, v) = \big\{(v_1 - y_2, v_2 - y_1^2\cos t) : t \in \mathbb{R}\big\} \quad \text{and}$$
$$G(y) = \big\{y\big\},$$

which satisfy conditions (a)–(c) of Theorem 1. We observe that the system (13) is not (small-time and large-time) norm-observable, because it is not possible to collect information on $x_4$ by measuring $y_1$, $y_2$, $v_1$, and $v_2$. Then the contrapositives of Theorems 1 and 2 imply that the system (12) is not (small-time and large-time) norm-observable. □

**Example 4.** Let us give a very simple example of deducing the norm-observability of the simulated system from the simulating system, whose state–space dimension is smaller than that of the simulated system. Consider the following two systems $\Sigma$ and $\widetilde{\Sigma}$:

$$\Sigma: \quad \begin{aligned}
\dot{x}_1 &= x_1 - 2x_2 + 2x_1x_2,\\
\dot{x}_2 &= x_1 + x_2 - x_1^2,\\
y &= h(x),
\end{aligned}$$

$$\widetilde{\Sigma}: \quad \dot{z} = 2z, \qquad w = \tilde{h}(z),$$

where $h : \mathbb{R}^2 \to \mathbb{R}^2$ is continuous and satisfies

$$h(x) = x \text{ if } |x| < 1 \text{ or } |x| > 2, \text{ and } h(x) \neq 0 \text{ if } 1 \leq |x| \leq 2,$$

and $\tilde{h} : \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying

$$\tilde{h}(z) = z \text{ if } |z| < 1 \text{ or } |z| > 2.$$

Let $\Phi : \mathbb{R}^2 \to \mathbb{R}$ be the mapping $\Phi(x) = x_1^2/2 + x_2^2$, and let

$$\begin{aligned}
\mathcal{R} = &\big\{(x, x_1^2/2 + x_2^2) : |x| < 1 \text{ or } |x| > 2\big\}\\
&\cup \big\{(h(x), \tilde{h}(x_1^2/2 + x_2^2)) : 1 \leq |x| \leq 2\big\}.
\end{aligned}$$

It is a matter of simple calculation to see that $(\mathrm{Graph}(\Phi), \mathcal{R})$ is a graph simulation relation of $\Sigma$ by $\widetilde{\Sigma}$. Clearly, $\Phi^{-1}$ is bounded, upper semicontinuous at the origin, and equal to $\{0\}$ at the origin. The function $G$ in this case is

$$G(y) = \begin{cases} \left\{y_1^2/2 + y_2^2\right\} \cup \left\{\tilde{h}(x_1^2/2 + x_2^2) : h(x) = y \text{ and } 1 \le |x| \le 2\right\} \\ \quad (\text{if } |y| < 1 \text{ or } |y| > 2), \\ \left\{\tilde{h}(x_1^2/2 + x_2^2) : h(x) = y \text{ and } 1 \le |x| \le 2\right\} \\ \quad (\text{if } 1 \le |y| \le 2). \end{cases}$$

Since $h(x) \neq 0$ whenever $1 \le |x| \le 2$, we have $G(0) = \{0\}$. Since $\tilde{h}$ is continuous, $\tilde{h}(x_1^2/2 + x_2^2)$ is bounded when $1 \le |x| \le 2$, so that $G$ is bounded. It remains to show that $G$ is upper semicontinuous at 0. This follows by showing that $G(y) = \{y_1^2/2 + y_2^2\}$ for sufficiently small $|y|$. Let $\delta = \inf\{|h(x)| : 1 \le |x| \le 2\}$. Then $\delta > 0$ (since $h$ is continuous and $h(x) \neq 0$ for $1 \le |x| \le 2$), and thus $G(y) = \{y_1^2/2+y_2^2\}$ if $|y| < \min\{1, \delta\}$, as required. So Corollary 1 tells us that the norm-observability of $\Sigma$ can be inferred by analyzing $\widetilde{\Sigma}$. Note that it suffices to consider the system $\widetilde{\Sigma}$ with the output function $\tilde{h}(z) = z$, which is clearly observable. This allows us to conclude that the system $\Sigma$ is norm-observable. □

## 5. Conclusions

We have examined to what extent nonlinear systems that are connected by a graph simulation relation share the norm-observability properties. Several results have been derived, which fit into the paradigm that the simulated system is norm-observable if there is a graph simulation relation relating the two systems and if the simulating system is norm-observable. The proposed results guarantee the possibility that checking norm-observability of the simulating system is sufficient for checking that of the simulated system, and therefore, may offer new avenues to explore the norm-observability properties of nonlinear control systems.

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