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Complete Lyapunov functions of control systems

Josiney A. Souza

Departamento de Matemática, Universidade Estadual de Maringá, Cx. Postal 331, 87020-900, Maringá-PR, Brazil

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ABSTRACT

In this paper, the notion of complete Lyapunov function of control systems is introduced. The purpose is to determine a continuous real-valued function that describes the global structure of the system. The existence of complete Lyapunov functions is proved for certain classes of affine control systems on compact manifolds.

1. Introduction

The global analysis of dynamical systems describes the possible limit behavior of the trajectories of the system and how the limit sets are related. It is well-known that Lyapunov functions are important tools for studying the dynamical behavior of a flow or autonomous differential equation. The Lyapunov theorems for an autonomous system can be used to prove the stability of the equilibrium of an autonomous dynamical system (as reference source we mention Colonius–Kliemann [1], Conley [2], Khalil [3] and Robinson [4]). Certain notion of (smooth) control Lyapunov functions are very useful in control theory supplying sufficient criteria for feedback stabilization, control, and tracking for various classes of nonlinear systems (e.g., Freeman–Kokotović [5]). The structure of perturbed and controlled systems can also be characterized by using semicontinuous control Lyapunov functions (see [6]).

In Conley's theory the global structure of a dynamical system on compact metric space can be described via generalized Lyapunov functions. The Morse components of a Morse decomposition are connected by orbits which go through decreasing levels of some Lyapunov function. A complete Lyapunov function for a Morse decomposition $\mathcal{M} = \{\mathcal{C}_1, \ldots, \mathcal{C}_n\}$ of a flow σ on a compact metric space X is a continuous real valued function $L_{\mathcal{M}} : X \to \mathbb{R}$ which is strictly decreasing on orbits outside $\bigcup_{i=1}^{n} \mathcal{C}_i$ and such that, for each critical value c, the set $L_{\mathcal{M}}^{-1}(c)$ is a Morse component. For each Morse decomposition $\mathcal{M}$ there exists a complete Lyapunov function $L_{\mathcal{M}}$ (see [7, Section 4]). More restrictively, a complete Lyapunov function for the flow σ on X is a continuous real valued function $L : X \to \mathbb{R}$, which is strictly decreasing on orbits outside the chain recurrence set $\mathfrak{R}$ of σ and such that the set $L(\mathcal{R})$ of critical values of L is nowhere dense in $\mathcal{R}$ and, for each critical value c, the set $L^{-1}(c)$ is a maximal chain transitive set (chain transitive component). The existence of a complete Lyapunov function for a flow on a compact metric space is ensured (e.g. Conley [2, Chapter II, Section 6.4]). For an elaboration of this point of view, see also [4]. The Conley theorems for flows which are directly related to Lyapunov functions have been extended by Rybakowski [8] for semiflows on compact metric spaces, and by Patrão [7] and Patrão and San Martin [9] for semiflows on compact Hausdorff spaces.

In the present paper, we extend the notion of complete Lyapunov function to control systems. As in Conley theory of flows, this type of Lyapunov function associates to the concepts of Morse decomposition and chain recurrence of control systems. We prove the existence of complete Lyapunov functions for certain classes of affine control systems on compact manifolds.

2. Limit behavior and chain transitivity

In this section, some basic properties and results of control systems are described. We recall the global analysis of control systems in terms of semigroup actions.

Consider the following class of control systems

$$\dot{x}(t) = X(x(t), u(t))$$
$$u \in \mathcal{U}_{pc} = \{u : \mathbb{R} \to U : u \text{ piecewise constant}\}$$

on a connected d-dimensional C^∞-manifold M, where $U \subset \mathbb{R}^n$. Assume that, for each $u \in \mathcal{U}_{pc}$ and $x \in M$, the preceding equation has a unique solution $\varphi(t, x, u)$, $t \in \mathbb{R}$, with $\varphi(0, x, u) = x$, and the vector fields $X(\cdot, u)$, $u \in U$, are complete.

E-mail address: jasouza3@uem.br.

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Set $F = \{X(\cdot, u) : u \in U\}$. The system group $\mathcal{G}$ and the system semigroup $\mathcal{S}$ are defined, respectively, as

$$\mathcal{G} = \left\{ e^{t_n Y_n} e^{t_{n-1} Y_{n-1}} \cdots e^{t_0 Y_0} : Y_j \in F, t_j \in \mathbb{R}, n \in \mathbb{N} \right\},$$
$$\mathcal{S} = \left\{ e^{t_n Y_n} e^{t_{n-1} Y_{n-1}} \cdots e^{t_0 Y_0} : Y_j \in F, t_j \geq 0, n \in \mathbb{N} \right\}.$$

For $t > 0$ we define the sets

$$\mathcal{S}_{\leq t} = \left\{ e^{t_n Y_n} e^{t_{n-1} Y_{n-1}} \cdots e^{t_0 Y_0} : Y_j \in F, t_j \geq 0, \sum_{j=0}^{n} t_j \leq t, n \in \mathbb{N} \right\},$$
$$\mathcal{S}_{\geq t} = \left\{ e^{t_n Y_n} e^{t_{n-1} Y_{n-1}} \cdots e^{t_0 Y_0} : Y_j \in F, t_j \geq 0, \sum_{j=0}^{n} t_j \geq t, n \in \mathbb{N} \right\}.$$

Note that $\mathcal{S} = \mathcal{S}_{\leq t} \cup \mathcal{S}_{\geq t}$. We set $\mathcal{F} = \left\{ \mathcal{S}_{\geq t} : t > 0 \right\}$. This family is a directed set when ordered by reverse inclusion. In other words, $\mathcal{F}$ is a time-dependent filter basis on the subsets of $\mathcal{S}$ (that is, $\emptyset \notin \mathcal{F}$, and given $t, s > 0$, $\mathcal{S}_{\geq t+s} \subset \mathcal{S}_{\geq t} \cap \mathcal{S}_{\geq s}$). The concept of chain recurrence for the control system coincides with the concept of $\mathcal{F}$-chain recurrence (see [10] for details).

Definition 2.1. The *ω-limit set* of $X \subset M$ is defined as

$$\omega(X) = \bigcap_{t>0} \operatorname{cls}\left(\mathcal{S}_{\geq t} X \right),$$

and the *ω*-limit set* of X as

$$\omega^*(X) = \bigcap_{t>0} \operatorname{cls}\left(\mathcal{S}_{\geq t}^{-1} X \right).$$

It is easily seen that $\omega(X) = \bigcap_{n\in\mathbb{N}} \operatorname{cls}\left(\mathcal{S}_{\geq n} X\right)$ and $\omega^*(X) = \bigcap_{n\in\mathbb{N}} \operatorname{cls}\left(\mathcal{S}_{\geq n}^{-1} X\right)$. Limit sets for control systems on compact manifolds are nonempty and compact, because $\mathcal{F}$ is a filter basis on the subsets of $\mathcal{S}$. In general, nonempty ω-limit sets are forward invariant and nonempty ω^*-limit sets are backward invariant (see [10, Propositions 2.10 and 2.13]).

Example 2.1. Consider in $M = \mathbb{R}^2$ the control system

$$x'(t) = \begin{pmatrix} -u(t) & 1 \\ -1 & 0 \end{pmatrix} x(t), \quad u \in \mathcal{U}_{cp}, \qquad U = [0, 1].$$

For $u \equiv 0$ the system moves on circles centered at 0; for $u > 0$ the system moves on spirals centered at 0. If $\|x\| \neq 0$, it follows that $\omega(x) = \{y \in M : \|y\| \leq \|x\|\}$ and $\omega^*(x) = \varnothing$. Note that the limit set $\omega(x)$ is forward invariant but not backward invariant.

The following definitions reproduce basic concepts from Conley theory of dynamical systems.

Definition 2.2. An *attractor* for the control system on the manifold M is a set $\mathcal{A} \subset M$ which admits a neighborhood N such that $\omega(N) = \mathcal{A}$. A *repeller* is a set $\mathcal{R} \subset M$ which has a neighborhood V with $\omega^*(V) = \mathcal{R}$. The neighborhoods N and V are called *attractor neighborhood* of $\mathcal{A}$ and *repeller neighborhood* of $\mathcal{R}$, respectively. We consider both the empty set and M as attractors and repellers.

The following result is proved in [10, Proposition 3.1] and adapted for control systems.

Proposition 2.1. *Assume that the manifold M is compact.*

1. *For each attractor $\mathcal{A}$ with attractor neighborhood N there is $t > 0$ such that* $\operatorname{cls}\left(\mathcal{S}_{\geq t} N\right) \subset \operatorname{int}(N)$.
2. *For each repeller $\mathcal{R}$ with repeller neighborhood V there is $t > 0$ such that* $\operatorname{cls}\left(\mathcal{S}_{\geq t}^{-1} V\right) \subset \operatorname{int}(V)$.

Let $\mathcal{A}$ be an attractor for the control system. The set

$$\mathcal{A}^* = M \setminus \{x \in M : \omega(x) \subset \mathcal{A}\}$$

is called *complementary repeller* of $\mathcal{A}$.

Definition 2.3. Assume that the manifold M is compact. Let $\emptyset = \mathcal{A}_0 \subset \mathcal{A}_1 \subset \cdots \subset \mathcal{A}_n = M$ be an increasing sequence of attractors of the control system on M, and let $\mathcal{C}_i = \mathcal{A}_i \cap \mathcal{A}_{i-1}^*$, $i = 1, \ldots, n$. The ordered collection $\mathcal{M} = \{\mathcal{C}_1, \ldots, \mathcal{C}_n\}$ is called a *Morse decomposition* of the control system. Each component $\mathcal{C}_i$ is called a *Morse set*. A Morse decomposition $\mathcal{M} = \{\mathcal{C}_1, \ldots, \mathcal{C}_n\}$ is said to be *finer* than a Morse decomposition $\mathcal{M}' = \{\mathcal{C}_1', \ldots, \mathcal{C}_m'\}$ if for each Morse set $\mathcal{C}_j'$ there is a Morse set $\mathcal{C}_i$ with $\mathcal{C}_i \subset \mathcal{C}_j'$. A Morse decomposition is called the *finest* Morse decomposition if it is finer than all Morse decompositions.

The notion of Morse decomposition on noncompact manifolds will appear later [11].

We may assume the following hypothesis.

Definition 2.4. The family $\mathcal{F}$ satisfies the *translation hypothesis* if:

1. for all $g \in \mathcal{S}$ and $t > 0$, there is $s > 0$ such that $\mathcal{S}_{\geq s} \subset \mathcal{S}_{\geq t} g$.
2. for all $g \in \mathcal{S}$ and $t > 0$, there is $s > 0$ such that $\mathcal{S}_{\geq s} \subset g \mathcal{S}_{\geq t}$.

The item 1. of the translation hypothesis implies the Hypothesis H_3 defined in [10, Section 2]: for all $s \in \mathcal{S}$ and $A \in \mathcal{F}$ there exists $B \in \mathcal{F}$ such that $B \subset As$. This hypothesis allows to extend Conley's results from the setting of dynamical systems on compact metric spaces to the setting of semigroup actions on compact spaces. If the manifold M is compact, the complementary repeller $\mathcal{A}^*$ of an attractor $\mathcal{A}$ is really a repeller of the control system and coincides with the set $\{x \in M : \omega(x) \cap \mathcal{A} = \emptyset\}$. Moreover, for $x \in M \setminus (\mathcal{A} \cup \mathcal{A}^*)$, one has $\omega^*(x) \subset \mathcal{A}^*$ and $\omega(x) \subset \mathcal{A}$ (see [10, Section 3] for details). This fact reproduces the notion of attractor–repeller pair of the Conley theory. The item 2. of translation hypothesis means that the family $\mathcal{F}^* = \left\{ \mathcal{S}_{\geq t}^{-1} : t > 0 \right\}$ also satisfies Hypothesis H_3. It assures that limit sets are invariant by the system, which implies attractors and repellers are invariant sets. As a consequence, a Morse decomposition of the control system is a collection of compact invariant sets.

Another relevance of the translation hypothesis is the correspondence between the chain control sets as defined in [1] and the maximal chain transitive sets as defined in [10]. We recall that an (ε, T)-chain from x to y consists of a sequence of points $x_0 = x, \ldots, x_n = y \in M$, times $t_0, \ldots, t_{n-1} > T$ and control functions $u_0, \ldots, u_{n-1}$ such that

$$d(\varphi(t, x_i, u_i), x_{i+1}) < \varepsilon$$

for all $i = 0, \ldots, n-1$. For $x \in M$, we set

$$\Omega(x) = \{y \in M : \text{there is an } (\varepsilon, T)\text{-chain from } x \text{ to } y, \text{ for all } \varepsilon, T > 0\},$$
$$\Omega^*(x) = \{y \in M : \text{there is an } (\varepsilon, T)\text{-chain from } y \text{ to } x, \text{ for all } \varepsilon, T > 0\}.$$

A point $x \in M$ is chain recurrent if $x \in \Omega(x)$. The set of all chain recurrent points for the control system is called the *chain recurrence set*, and is denoted by $\mathfrak{R}$.

A *chain control set* for the control system is a set $E \subset M$ which satisfies

a. For all $x, y \in E$, $x \in \Omega(y)$ and $y \in \Omega(x)$.
b. For all $x \in E$ there is a control function u such that $\varphi(t, x, u) \in E$ for all $t \in \mathbb{R}$;
c. E is maximal with these two properties.

A *maximal chain transitive set* for the control system is a set $C \subset M$ which satisfies

i. For all $x, y \in C$, $x \in \Omega(y)$ and $y \in \Omega(x)$.
ii. C is maximal with this property.

In other words, $C = \Omega(x) \cap \Omega^*(x)$ for all $x \in C$.

The following theorem on chain recurrence is proved in [10, Proposition 4.10 and Theorem 4.1] and [12, Proposition 4.2 and Theorem 5.2], which extends the results from Conley theory of dynamical systems to control systems.

Theorem 2.2. *Assume that the manifold M is compact and the family $\mathcal{F}$ satisfies the translation hypothesis.*

1. *For $x \in M$, $\Omega(x)$ is the intersection of all attractors containing $\omega(x)$, and $\Omega^*(x)$ is the intersection of all complementary repellers $\mathcal{A}^*$, where $\mathcal{A}$ is an attractor such that $x \notin \mathcal{A}$.*
2. *The maximal chain transitive sets of the control system are compact invariant sets.*
3. *The chain recurrence set $\mathfrak{R}$ of the control system coincides with the intersection*
$$\bigcap \left\{\mathcal{A} \cup \mathcal{A}^* : \mathcal{A} \text{ is an attractor}\right\}.$$
4. *There exists the finest Morse decomposition for the control system if and only if the chain recurrence set is a union of a finite number of maximal chain transitive sets.*

It is readily seen that a chain control set E is contained in a maximal chain transitive set C. If M is compact and the family $\mathcal{F}$ satisfies the translation hypothesis, we have $E = C$, since C is invariant from Theorem 2.2. Therefore, under translation hypothesis, the concepts of chain control set and maximal chain transitive set are equivalent.

A theoretical condition that yields translation hypothesis occur is $\mathcal{S}$ being total in $\mathcal{G}$, that is, $\mathcal{G} = \mathcal{S} \cup \mathcal{S}^{-1}$. Indeed, let $\mathcal{G}^+$ denote the semigroup

$$\mathcal{G}^+ = \left\{ e^{t_n Y_n} e^{t_{n-1} Y_{n-1}} \cdots e^{t_0 Y_0} : Y_j \in F, t_j \in \mathbb{R}, \sum_{j=0}^{n} t_j \geq 0, n \in \mathbb{N} \right\}.$$

Assume that the system semigroup $\mathcal{S}$ coincides with $\mathcal{G}^+$. Then $\mathcal{S}$ is total in $\mathcal{G}$. For $e^{tY} \in \mathcal{S}$ and $T > 0$, take $s > 0$ such that $s \geq T + t$. We have

$$\mathcal{S}_{\geq s} = \mathcal{S}_{\geq s} e^{-tY} e^{tY} \subset \mathcal{S}_{\geq T} e^{tY} \quad \text{and}$$
$$\mathcal{S}_{\geq s} = e^{tY} e^{-tY} \mathcal{S}_{\geq s} \subset e^{tY} \mathcal{S}_{\geq T}.$$

Therefore, the family $\mathcal{F}$ satisfies the translation hypothesis. We refer to [13] for discussions on total semigroups in Lie groups. However, there are larger classes of interesting control systems whose family $\mathcal{F}$ satisfies the translation hypothesis, although the system semigroup is not total. Let us see an example.

Example 2.3. Let $M = G$ be a Lie group and $\mathfrak{g}$ the Lie algebra of G. Let $\mathfrak{a} \subset \mathfrak{z}(\mathfrak{g})$ be a vector subspace in the center $\mathfrak{z}(\mathfrak{g})$ of $\mathfrak{g}$. Fix a nonzero vector field $X \in \mathfrak{g}$. Consider the control system on G determined by the vector fields in $F = \{X + Y : Y \in \mathfrak{a}\}$. We have the system semigroup

$$\mathcal{S} = \left\{\exp t_n (X + Y_n) \cdots \exp t_0 (X + Y_0) : t_j \geq 0, Y_j \in \mathfrak{a}\right\}$$

where exp is the exponential map of $\mathfrak{g}$ into G. The family $\mathcal{F}$ satisfies the translation hypothesis. In fact, for $T > 0$ and $\exp t(X + Y) \in \mathcal{S}$, we take $s > 0$ such that $s \geq T + t$. For $\exp \tau (X + Z) \in \mathcal{S}_{\geq s}$, we have

$$\begin{aligned}
\exp \tau (X + Z) &= \exp \tau (X + Z) \exp -t (X + Y) \exp t (X + Y) \\
&= \exp ((\tau - t) X + \tau Z - tY) \exp t (X + Y) \\
&= \exp (\tau - t) \left(X + \frac{\tau}{\tau - t} Z - \frac{t}{\tau - t} Y \right) \\
&\quad \times \exp t (X + Y) \in \mathcal{S}_{\geq T} \exp t (X + Y).
\end{aligned}$$

Hence, $\mathcal{S}_{\geq s} \subset \mathcal{S}_{\geq T} \exp t (X + Y)$. Analogously, $\mathcal{S}_{\geq s} \subset \exp t (X + Y) \mathcal{S}_{\geq T}$. Nevertheless, $\mathcal{S}$ is no total in $\mathcal{G}$. Indeed, the group system is the subgroup of G

$$\mathcal{G} = \left\{\exp t_n (X + Y_n) \cdots \exp t_0 (X + Y_0) : Y_j \in \mathfrak{a}, t_j \in \mathbb{R}, n \in \mathbb{N}\right\}.$$

Then $\exp(\mathfrak{a}) \subset \mathcal{G}$, but $\exp(\mathfrak{a})$ does not lie in $\mathcal{S} \cup \mathcal{S}^{-1}$.

Translation hypothesis also occurs in bilinear control systems with commuting matrices, as follows.

Example 2.4. Let $M = \mathbb{R}^d$, $U = \{u \in \mathbb{R}^n : a \leq \|u\| \leq b\}$ with $a > 0$, and $A_1, \ldots, A_n \in \mathbb{R}^{d \times d}$ which are pairwise commutative. Consider the bilinear control system

$$\dot{x}(t) = X(x(t), u(t)) = \sum_{i=1}^{n} u_i(t) A_i(x(t))$$

on $\mathbb{R}^d$. For $u = (u_1, \ldots, u_n) \in U$ and $t \geq 0$, we have $e^{tX_u} = e^{t(u_1 A_1 + \cdots + u_n A_n)}$. The family $\mathcal{F}$ satisfies the translation hypothesis. In fact, for $t > 0$ and $e^{sX_v} \in \mathcal{S}$, $v = (v_1, \ldots, v_n)$, it is enough to find $T > 0$ such that $\mathcal{S}_{\geq T} \subset \mathcal{S}_{\geq t} e^{sX_v}$, since $\mathcal{S}$ is abelian. Take $T > 0$ such that $\frac{a}{b} T - s > t$. For $e^{rX_u} \in \mathcal{S}_{\geq T}$, $u = (u_1, \ldots, u_n)$, we have

$$\begin{aligned}
e^{rX_u} e^{-sX_v} &= e^{(ru_1 - sv_1) A_1 + \cdots + (ru_n - sv_n) A_n} \\
&= e^{\frac{\|ru - sv\|}{b} \frac{b(ru_1 - sv_1)}{\|ru - sv\|} A_1 + \cdots + \frac{b(ru_n - sv_n)}{\|ru - sv\|} A_n} \\
&= e^{\frac{\|ru - sv\|}{b} X_w}
\end{aligned}$$

where $w = \left(\frac{b(ru_1 - sv_1)}{\|ru - sv\|}, \ldots, \frac{b(ru_n - sv_n)}{\|ru - sv\|} \right) \in U$ and $\frac{\|ru - sv\|}{b} \geq \frac{|ra - sb|}{b} \geq \frac{a}{b} T - s > t$. Hence, $e^{rX_u} e^{-sX_v} \in \mathcal{S}_{\geq t}$. It implies the inclusion $\mathcal{S}_{\geq T} e^{-sX_v} \subset \mathcal{S}_{\geq t}$, hence $\mathcal{S}_{\geq T} \subset \mathcal{S}_{\geq t} e^{sX_v}$.

3. Lyapunov functions

In this section we introduce the notion of complete Lyapunov function for control systems. As in dynamical systems, the global structure of a control system can be described via Lyapunov functions.

Consider the following class of affine control systems

$$\dot{x}(t) = X(x(t), u(t)) = X_0(x(t)) + \sum_{i=1}^{n} u_i(t) X_i(x(t)),$$
$$u \in \mathcal{U}_{pc} = \{u : \mathbb{R} \to U : u \text{ piecewise constant}\}$$

on a compact connected d-dimensional C^∞-manifold M, where $U \subset \mathbb{R}^n$ is compact and convex, and $X_0, \ldots, X_n$ are C^∞-vector fields on M. Assume that, for each $u \in \mathcal{U}_{pc}$ and $x \in M$, the preceding equation has a unique solution $\varphi(t, x, u)$, $t \in \mathbb{R}$, with $\varphi(0, x, u) = x$, and the vector fields $X(\cdot, u)$, $u \in U$, are complete. Let $\mathcal{U}$ be the closure of $\mathcal{U}_{pc}$ with respect to the weak* topology of $L_\infty(\mathbb{R}, \mathbb{R}^n)$. Then $\mathcal{U}$ is compact and the following function

$$\varphi : \mathbb{R} \times M \times \mathcal{U} \to M, \quad (t, u, x) \mapsto \varphi(t, x, u)$$

is continuous. The control range U is identified with the set of the constant control functions in $\mathcal{U}$. Since U is weak* closed, it is a compact subset of $\mathcal{U}$.

Remark 1. Let $\mathcal{A}$ be an attractor for the control system and N be an open neighborhood of $\mathcal{A}$ with $\mathrm{cls}(N) \cap \mathcal{A}^* = \emptyset$. We choose an open repeller neighborhood V of $\mathcal{A}^*$ such that $N \cap V = \emptyset$. From Proposition 2.1 there is $t' > 0$ such that $\mathrm{cls}\left(\mathcal{S}_{\geq t'}^{-1} V\right) \subset V$. Then $N' = M \setminus \mathrm{cls}\left(\mathcal{S}_{\geq t'}^{-1} V\right)$ is an attractor neighborhood of $\mathcal{A}$. Since $N \subset M \setminus V \subset N'$, it follows that N is also an attractor neighborhood of $\mathcal{A}$. In particular, there is $t > 0$ such that $\mathrm{cls}\left(\mathcal{S}_{\geq t} N\right) \subset N$.

In order to introduce the notion of complete Lyapunov function for control systems, we need the following preorder in the system semigroup.

Definition 3.1. For $g_1, g_2 \in \mathcal{S}$, let $g_1 \geq g_2$ iff $g_1 = g_2$ or $g_1 \in \mathcal{S}g_2$; $g_1 > g_2$ iff there is $t > 0$ such that $g_1 \in \mathcal{S}_{\geq t} g_2$.

The relation $\leq$ in Definition 3.1 is the reverse of the well-known Green's $\mathcal{L}$-preorder of semigroup theory: $g_1 \leq_{\mathcal{L}} g_2$ iff $g_1 = g_2$ or $g_1 \in \mathcal{S}g_2$ ([14]).

Definition 3.2. A *Lyapunov function* associated to an attractor–repeller pair $(\mathcal{A}, \mathcal{A}^*)$ is a real valued function $L_{\mathcal{A}} : M \to [0, 1]$ such that $L_{\mathcal{A}}^{-1}(0) = \mathcal{A}$, $L_{\mathcal{A}}^{-1}(1) = \mathcal{A}^*$ and $L_{\mathcal{A}}$ is strictly decreasing on orbits in $C(\mathcal{A}, \mathcal{A}^*) = M \setminus (\mathcal{A} \cup \mathcal{A}^*)$, that is, if $g_1 > g_2$ and $x \in C(\mathcal{A}, \mathcal{A}^*)$, then $L_{\mathcal{A}}(g_1 x) < L_{\mathcal{A}}(g_2 x)$.

The set $C(\mathcal{A}, \mathcal{A}^*)$ in Definition 3.2 is called the *set of connecting orbits* of the attractor–repeller pair $(\mathcal{A}, \mathcal{A}^*)$. The following theorem on the existence of Lyapunov functions for attractor–repeller pairs generalizes the Conley theorem for dynamical systems on compact metric spaces (see [2, Chapter II, Section 5]).

Theorem 3.1. *Let $\mathcal{A}$ be an attractor of the control system on the compact manifold M and assume that its complementary repeller $\mathcal{A}^*$ is closed. There is a Lyapunov function $L_{\mathcal{A}}$ for $(\mathcal{A}, \mathcal{A}^*)$.*

Proof. Since $\mathcal{A}$ and $\mathcal{A}^*$ are disjoint closed sets and M is a perfectly normal space, there is a continuous function $f : M \to [0, 1]$ such that $f^{-1}(0) = \mathcal{A}$ and $f^{-1}(1) = \mathcal{A}^*$. We define $h : M \to [0, 1]$ by $h(x) = \sup\{f(gx) : g \in \mathcal{S}\}$. If $g_1 \geq g_2$, then $g_1 = kg_2$ for some $k \in S$. Then, we have

$$\begin{aligned} h(g_2 x) &= \sup\{f(gg_2 x) : g \in \mathcal{S}\} \\ &\geq \sup\{f(gkg_2 x) : g \in \mathcal{S}\} \\ &= \sup\{f(gg_1 x) : g \in \mathcal{S}\} \\ &= h(g_1 x). \end{aligned}$$

Hence, h is nonincreasing on orbits. Let us verify h is continuous. It is enough to show that $h^{-1}([0, \varepsilon))$ and $h^{-1}((\varepsilon, 1])$ are open sets in M for every $0 < \varepsilon < 1$, since $h^{-1}((a, b)) = h^{-1}([0, b)) \cap h^{-1}((a, 1])$, for any basic open set $(a, b) \subset [0, 1]$. For $0 < \varepsilon < 1$, let $x \in h^{-1}((\varepsilon, 1])$. Then, there is $g \in \mathcal{S}$ such that $f(gx) > \varepsilon$. Hence, $g^{-1}\left(f^{-1}((\varepsilon, 1])\right) \subset h^{-1}((\varepsilon, 1])$ is an open neighborhood of x. Thus, $h^{-1}((\varepsilon, 1])$ is an open set in M. Now, let $x \in h^{-1}([0, \varepsilon))$. Then, $h(x) < \varepsilon$ and $\mathcal{S}x \subset f^{-1}([0, h(x)])$. By taking $\delta > 0$ with $h(x) < \delta < \varepsilon$, we have $f^{-1}([0, \delta))$ is an open neighborhood of $\mathcal{A}$ and $f^{-1}([0, \delta)) \cap \mathcal{A}^* = \emptyset$. From Remark 1, there is $t > 0$ such that

$$\mathcal{S}_{\geq t} f^{-1}([0, \delta)) \subset f^{-1}([0, \delta)).$$

If $y \in \mathcal{S}_{\geq t} f^{-1}([0, \delta))$ and $g \in \mathcal{S}$, then

$$gy \in g\mathcal{S}_{\geq t} f^{-1}([0, \delta)) \subset \mathcal{S}_{\geq t} f^{-1}([0, \delta)) \subset f^{-1}([0, \delta)).$$

Hence,

$$h(y) = \sup\{f(gy) : g \in \mathcal{S}\} \leq \delta < \varepsilon,$$

that is, $y \in h^{-1}([0, \varepsilon))$. Thus, $\mathcal{S}_{\geq t} f^{-1}([0, \delta)) \subset h^{-1}([0, \varepsilon))$. Now, since $h(x) < \delta$, we have $\mathcal{S}_{\leq t} x \subset f^{-1}([0, \delta))$, and hence $[0, t] \times \{x\} \times \mathcal{U}_{pc} \subset \varphi^{-1}\left(f^{-1}([0, \delta))\right)$. As $\mathcal{U}_{pc}$ is dense in $\mathcal{U}$, we can obtain an open neighborhood V of x such that $V \subset f^{-1}([0, \delta))$ and $[0, t] \times V \times \mathcal{U}_{pc} \subset \varphi^{-1}\left(f^{-1}\left([0, \varepsilon')\right)\right)$ for some ε', with $\delta \leq \varepsilon' < \varepsilon$. Hence, $\mathcal{S}_{\leq t} V \subset f^{-1}\left([0, \varepsilon')\right)$. Then, we have

$$\mathcal{S}V = \left(\mathcal{S}_{\leq t} \cup \mathcal{S}_{\geq t}\right) V \subset f^{-1}\left([0, \varepsilon')\right),$$

and thus $V \subset h^{-1}([0, \varepsilon))$. Therefore, $h^{-1}([0, \varepsilon))$ is an open set in M, and hence h is continuous. Because $f^{-1}(0) = \mathcal{A}$, it is easily seen that $h^{-1}(0) = \mathcal{A}$. If $x \in \mathcal{A}^* = f^{-1}(1)$, then $f(gx) = 1$ for all $g \in \mathcal{S}$, hence $\mathcal{A}^* \subset h^{-1}(1)$. On the other hand, if $h(y) = 1$ and $y \notin \mathcal{A}^*$, then $\omega(y) \subset \mathcal{A}$, hence there is a subnet $(g_i y)$ of the net $(gy)_{g \in \mathcal{S}}$ converging to some point $z \in \mathcal{A}$. Since h is continuous, it follows that $h(g_i y) \to h(z) = 0$, but as h is nonincreasing on orbits, we have $h(g_i y) \geq h(y) = 1$, for all i, which is a contradiction. Hence, $y \in \mathcal{A}^*$, and therefore $h^{-1}(1) = \mathcal{A}^*$. Now we define the function $l : M \times \mathcal{U} \to [0, 1]$ by

$$l(x, u) = \int_0^{+\infty} e^{-t} h(\varphi(t, x, u))\, dt.$$

Since h and φ are continuous, it follows that l is continuous. As $h^{-1}(0) = \mathcal{A}$ and $h^{-1}(1) = \mathcal{A}^*$, we have $l^{-1}(0) = \mathcal{A} \times \mathcal{U}$ and $l^{-1}(1) = \mathcal{A}^* \times \mathcal{U}$. Because h is nonincreasing on orbits, we have

$$\begin{aligned} l(\varphi(s, x, u), u \cdot s) &= \int_0^{+\infty} e^{-t} h(\varphi(t, \varphi(s, x, u), u \cdot s))\, dt \\ &= \int_0^{+\infty} e^{-t} h(\varphi(s, \varphi(t, x, u), u \cdot t))\, dt \\ &\leq \int_0^{+\infty} e^{-t} h(\varphi(t, x, u))\, dt \\ &= l(x, u) \end{aligned}$$

for all $s > 0$ and $x \in M$. If $x \in C(\mathcal{A}, \mathcal{A}^*)$, there is $\varepsilon > 0$ such that $h(x) \geq \varepsilon$. Since $\omega(x) \subset \mathcal{A}$, there is $T > 0$ such that $\mathcal{S}_{\geq T} x \subset h^{-1}([0, \varepsilon))$. For $u \in \mathcal{U}$, it follows that

$$t' = \sup\{t > 0 : h(\varphi(t, x, u)) \geq \varepsilon\}$$

is finite and $h\left(\varphi\left(t', x, u\right)\right) \geq \varepsilon$. If $s > 0$, the function $t \in (0, +\infty) \mapsto h(\varphi(t + s, x, u)) - h(\varphi(t, x, u))$ is not identically zero, because $h\left(\varphi\left(t' + s, x, u\right)\right) - h\left(\varphi\left(t', x, u\right)\right) < 0$. Hence

$$l(\varphi(s, x, u), u \cdot s) - l(x, u) = \int_0^{+\infty} e^{-t} \left(h(\varphi(t + s, x, u)) - h(\varphi(t, x, u))\right) dt$$

is strictly negative, that is, $l(\varphi(s, x, u), u \cdot s) < l(x, u)$. Finally, the Lyapunov function $L_{\mathcal{A}} : M \to [0, 1]$ for $(\mathcal{A}, \mathcal{A}^*)$ is defined as

$$L_{\mathcal{A}}(x) = \sup_{u \in \mathcal{U}} l(x, u).$$

Indeed, since $l^{-1}(0) = \mathcal{A} \times \mathcal{U}$ and $l^{-1}(1) = \mathcal{A}^* \times \mathcal{U}$, we have $L_{\mathcal{A}}^{-1}(0) = \mathcal{A}$ and $L_{\mathcal{A}}^{-1}(1) = \mathcal{A}^*$. Let us verify $L_{\mathcal{A}}$ is continuous. We choose ε such that $0 < \varepsilon < 1$. If $x \in L_{\mathcal{A}}^{-1}([0, \varepsilon))$, we can take ε' such that $L_{\mathcal{A}}(x) < \varepsilon' < \varepsilon$. Then $l(x, u) < \varepsilon'$ for all $u \in \mathcal{U}$. Since l is continuous, there is a neighborhood V of x in M such that $V \times \mathcal{U} \subset l^{-1}\left(\left[0, \varepsilon'\right)\right)$. If $y \in V$, then $l(y, u) < \varepsilon'$ for all $u \in \mathcal{U}$, hence $L_{\mathcal{A}}(y) \leq \varepsilon' < \varepsilon$. It follows that $V \subset L_{\mathcal{A}}^{-1}([0, \varepsilon))$, and therefore $L_{\mathcal{A}}^{-1}([0, \varepsilon))$ is open. If $x \in L_{\mathcal{A}}^{-1}((\varepsilon, 1])$, there is some u such that $l(x, u) > \varepsilon$. Then there is a neighborhood V of x with $V \times \{u\} \subset l^{-1}((\varepsilon, 1])$. If $y \in V$, we have $l(y, u) > \varepsilon$, hence $L_{\mathcal{A}}(y) > \varepsilon$. It follows that $V \subset L_{\mathcal{A}}^{-1}((\varepsilon, 1])$, and therefore $L_{\mathcal{A}}^{-1}((\varepsilon, 1])$ is open. Thus, $L_{\mathcal{A}}$ is continuous. It remains to show that $L_{\mathcal{A}}$ is strictly decreasing on orbits in $C(\mathcal{A}, \mathcal{A}^*)$. Let $s > 0$, $e^{sX_v} \in \mathcal{S}$, and $x \in C(\mathcal{A}, \mathcal{A}^*)$. Since $L_{\mathcal{A}}\left(e^{sX_v} x\right) = l(\varphi(s, x, v), u_0)$ for some $u_0 \in \mathcal{U}$, and $\varphi(t, \varphi(s, x, v), u_0) = \varphi(t + s, x, w)$, where $w \in \mathcal{U}$ is the s-concatenation of v and u_0, we have

$$\begin{aligned} L_{\mathcal{A}}\left(e^{sX_v} x\right) &= l(\varphi(s, x, v), u_0) \\ &= \int_0^{+\infty} e^{-t} h(\varphi(t, \varphi(s, x, v), u_0))\, dt \\ &= \int_0^{+\infty} e^{-t} h(\varphi(t + s, x, w))\, dt \end{aligned}$$

$$= \int_0^{+\infty} e^{-t} h\left(\varphi\left(t, \varphi\left(s, x, w\right), w \cdot s\right)\right) dt$$
$$= l\left(\varphi\left(s, x, w\right), w \cdot s\right)$$
$$< l\left(x, w\right)$$
$$\leq L_{\mathcal{A}}\left(x\right).$$

Therefore, $L_{\mathcal{A}}$ is a Lyapunov function for $(\mathcal{A}, \mathcal{A}^*)$.

Now we go into the investigation of the existence of a complete Lyapunov function for the control system. First, we define complete Lyapunov function for a Morse decomposition. □

Definition 3.3. A *complete Lyapunov function* for a Morse decomposition $\mathcal{M} = \{\mathcal{C}_1, \ldots, \mathcal{C}_n\}$ of the control system is a continuous real valued function $L_{\mathcal{M}} : M \to \mathbb{R}$, which is strictly decreasing on orbits outside $\bigcup_{i=1}^n \mathcal{C}_i$ and such that, for each Morse set $\mathcal{C}_i$, there is $c_i \in \mathbb{R}$ such that $L_{\mathcal{M}}^{-1}(c_i) = \mathcal{C}_i$. The set $L_{\mathcal{M}}\left(\bigcup_{i=1}^n M_i\right)$ is called the critical value set of $L_{\mathcal{M}}$.

From the existence of Lyapunov function for attractor–repeller pairs we can prove the existence of complete Lyapunov function for Morse decompositions, as follows.

Theorem 3.2. *Let $\mathcal{M} = \{\mathcal{C}_1, \ldots, \mathcal{C}_n\}$ be a Morse decomposition of the control system on the compact manifold M and let $\emptyset = \mathcal{A}_0 \subset \mathcal{A}_1 \subset \cdots \subset \mathcal{A}_n = M$ be the increasing sequence of attractors such that $\mathcal{C}_i = \mathcal{A}_i \cap \mathcal{A}_{i-1}^*$, $i = 1, \ldots, n$. Assume that $\mathcal{A}_i^*$ is closed for all i. There is a complete Lyapunov function for $\mathcal{M}$.*

Proof. Define $L_{\mathcal{M}} = \Sigma_{i=1}^n 3^{-i} L_{\mathcal{A}_i}$, where $L_{\mathcal{A}_i}$ is the Lyapunov function for $\left(\mathcal{A}_i, \mathcal{A}_i^*\right)$ given by Theorem 3.1. The function $L_{\mathcal{M}}$ is a complete Lyapunov function for $\mathcal{M}$. In fact, it is easily seen that $L_{\mathcal{M}}$ is continuous. If $x \in M \setminus \bigcup_{i=1}^n \mathcal{C}_i$, we take $j = \min\{i : x \in \mathcal{A}_i\}$. Since $x \notin \mathcal{A}_j \cap \mathcal{A}_{j-1}^*$, we have $x \notin \mathcal{A}_{i-1}^*$. Hence, $x \notin \mathcal{A}_{j-1} \cup \mathcal{A}_{j-1}^*$. It follows that, for any $s > 0$, $L_{\mathcal{M}}\left(e^{sX_u}x\right) < L_{\mathcal{M}}(x)$, since $L_{\mathcal{A}_{j-1}}\left(e^{sX_u}x\right) < L_{\mathcal{A}_{j-1}}(x)$. Thus $L_{\mathcal{M}}$ is strictly decreasing on orbits outside $\bigcup_{i=1}^n \mathcal{C}_i$. Given a critical value $c \in L_{\mathcal{M}}\left(\bigcup_{i=1}^n \mathcal{C}_i\right)$, we take $y \in L_{\mathcal{M}}^{-1}(c)$. Then, $L_{\mathcal{M}}(y) = L_{\mathcal{M}}\left(\mathcal{C}_j\right)$ for some M_j, that is, $L_{\mathcal{M}}(y) = \Sigma_{i=1}^{j-1} 3^{-i}$. It follows that $L_{\mathcal{A}_j}(y) = 0$ and $L_{\mathcal{A}_{j-1}}(y) = 1$, and hence $y \in \mathcal{A}_j \cap \mathcal{A}_{j-1}^* = \mathcal{C}_j$. Therefore, $L_{\mathcal{M}}^{-1}(c) = \mathcal{C}_j$, and the proof is completed. □

Finally, we define complete Lyapunov function for the control system.

Definition 3.4. A *complete Lyapunov function* for the control system is a continuous real valued function $L : M \to \mathbb{R}$, which is strictly decreasing on orbits outside the chain recurrence set $\mathfrak{R}$ and such that the set $L(\mathfrak{R})$ of critical values of L is nowhere dense in $\mathbb{R}$ and, for each critical value c, the set $L^{-1}(c)$ is a maximal chain transitive set.

From now on, we assume that the family $\mathcal{F} = \left\{\mathcal{S}_{\geq t} : t > 0\right\}$ satisfies the translation hypothesis. By Theorem 2.2, if the finest Morse decomposition exists, a complete Lyapunov function for the control system is the complete Lyapunov function for the finest Morse decomposition given by Theorem 3.2. Otherwise, since the manifold M is compact, there are at most countably many attractor–repeller pairs for the control system in M, and the existence of a complete Lyapunov function for the control system is guaranteed. The proof is similar to Conley's proof for flow on compact metric space (see [2, Chapter II, Section 6.4]). We define $L = \Sigma_{n=1}^{\infty} 3^{-n} L_{\mathcal{A}_n}$ where $L_{\mathcal{A}_1}, L_{\mathcal{A}_2}, \ldots$ are the Lyapunov functions for the attractor–repeller pairs of the control system. Then, L is a continuous function on M. From Theorem 2.2, if $x \notin \mathfrak{R}$, then $x \notin \mathcal{A}_n \cup \mathcal{A}_n^*$ for some attractor $\mathcal{A}_n$. It follows that L is strictly decreasing on orbits outside the chain recurrence set $\mathfrak{R}$. Since each $L_{\mathcal{A}_n}$ is either 0 or 1 at a point of $\mathfrak{R}$, each critical value in $L(\mathfrak{R})$ lies in the "middle third" Cantor set, hence $L(\mathfrak{R})$ is nowhere dense in $\mathbb{R}$. Now, if E is a maximal chain transitive set and $x, y \in E$, then $x \in \Omega(y)$ and $y \in \Omega(x)$. By Theorem 2.2, it follows that each attractor containing x also contains y. Hence, $L(E) = c$ for some critical value c. On the other hand, if $z \in L^{-1}(c)$, then $L(z) = L(x)$. If $\mathcal{A}_n$ is an attractor containing x, then $L_{\mathcal{A}_n}(x) = 0$, and hence $L_{\mathcal{A}_n}(z) = 0$. If $\mathcal{A}_m$ is an attractor such that $x \in \mathcal{A}_m^*$, then $L_{\mathcal{A}_m}(z) = L_{\mathcal{A}_m}(x) = 1$. Hence, z lies in each attractor containing x and in each complementary repeller containing x. By Theorem 2.2 again, it follows that $z \in \Omega(x) \cap \Omega^*(x) = E$. Therefore, $L^{-1}(c) = E$. These statements demonstrate the following theorem.

Theorem 3.3. *Assume that the translation hypothesis is satisfied. There is a complete Lyapunov function for the control system on the compact manifold M.*

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