1. A control system $\dot{x} = Ax + Bu$, $y = Cx$ transforms, under a change of basis $x = P\bar{x}$ in the state space, to a control system $\dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}u$, $y = \bar{C}\bar{x}$ (note that the input $u$ and output $y$ stay the same, only the state $x$ changes).

a) Derive the formulas for the new matrices $\bar{A}, \bar{B}, \bar{C}$ in terms of the original matrices $A, B, C$.

**Solution:**

a) $\bar{x} = P^{-1}x$, so $\dot{\bar{x}} = P^{-1}\dot{x} = P^{-1}(Ax + Bu) = P^{-1}AP\bar{x} + P^{-1}Bu$, and $y = Cx = CP\bar{x}$.

Answer: $\bar{A} = P^{-1}AP$, $\bar{B} = P^{-1}B$, $\bar{C} = CP$.

b) $\bar{G}(s) = \bar{C}(Is - \bar{A})^{-1}\bar{B} = CP(Is - P^{-1}AP)^{-1}P^{-1}B = CP[Is - (Is - A)P]^{-1}P^{-1}B$

$= CP PP^{-1}(Is - A)^{-1}PP^{-1}B = G(s)$

2. Consider the system

$$\dot{x} = \begin{pmatrix} a & 1 \\ 0 & -1 \end{pmatrix} x + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u$$

where $x \in \mathbb{R}^2$, $u \in \mathbb{R}$, and $a \in \mathbb{R}$ is an unknown parameter. Suppose this system is known to have the following two properties:

(i) If we turn off the control (set $u(t) = 0$ for all $t$), the solutions asymptotically converge to the origin from all initial states.

(ii) From every initial state it is possible to reach the origin in finite time using some control.

Based on this information, determine all possible values of the parameter $a$.

**Solution:**

Property (i) tells us that $\dot{x} = Ax$ must be asymptotically stable, hence $A$ must be Hurwitz. The eigenvalues of $A$ are $a$ and $-1$, so $a$ must be negative.

Property (ii) tells us that the system must be controllable. The controllability matrix is

$$C(A, B) = \begin{pmatrix} 1 & a + 1 \\ 1 & -1 \end{pmatrix}$$

and this is singular when $a + 1 = -1$, which means $a = -2$ must be excluded.

Answer: all $a < 0$ except $a = -2$.

3. In class we defined the unobservable subspace of an LTV system $\dot{x} = A(t)x$, $y = C(t)x$ to be the nullspace (kernel) of the observability Gramian which is defined to be the matrix

$$M = \int_{t_0}^{t_1} \Phi^T(t, t_0)C^T(t)C(t)\Phi(t, t_0)dt$$
where $\Phi$ is the state transition matrix for $A(t)$ and $t_0, t_1$ are given times.

a) Prove that for an LTI system $\dot{x} = Ax$, $y = Cx$ the unobservable subspace equals the nullspace of the observability matrix $O(A, C)$.

b) Briefly explain the meaning of the unobservable subspace in the context of recovering $x$ from $y$ (again for the LTI case).

**Solution:**

a) For the LTI system we have

$$M = \int_{t_0}^{t_1} e^{A(t-t_0)} C^T e^{A(t-t_0)} dt$$

If a vector $z \in \mathbb{R}^n$ is in the nullspace of $M$, then

$$0 = z^T M z = \int_{t_0}^{t_1} z^T e^{A(t-t_0)} C^T e^{A(t-t_0)} z \ dt = \int_{t_0}^{t_1} |C e^{A(t-t_0)} z|^2 \ dt$$

which means that $C e^{As} z$ must equal 0 for all $s \in [0, t_1 - t_0]$. Thus the function $s \mapsto C e^{As} z$ and all its derivatives with respect to $s$ must vanish at $s = 0$: $C z = CAz = \cdots = C A^{n-1} z = 0$. This implies that $z$ must be in the nullspace of the observability matrix

$$O(A, C) = \begin{pmatrix} C \\ CA \\ \cdots \\ C A^{n-1} \end{pmatrix}$$

Conversely, if $z$ is in the nullspace of $O(A, C)$ then reversing the above steps we see that it is in the nullspace of $M$.

b) Some examples of correct statements:

- Initial states in the unobservable subspace cannot be distinguished from the origin.
- Initial states whose difference lies in the unobservable subspace are indistinguishable from one another.
- The initial condition (and hence the entire state trajectory) can be recovered from the output up to a component lying in the unobservable subspace (which cannot be recovered).