ON MULTIPLIERS FOR LAGRANGE PROBLEMS.*

By E. J. McShane.

The proof of the Lagrange multiplier rule for the solutions of variational problems of Lagrange or Bolza type has been presented in a number of places; for example,[1] Bliss [1, 2], Morse and Myers [4]. The proof of the Weierstrass condition was until recently made only under the rather strong hypothesis of normality on every sub-arc. In 1932 Graves [3] generalized the theorem to apply to all normal problems, and stated a necessary condition involving the $\mathcal{E}$-function which is satisfied by certain anormal problems.

There is always an infinite aggregate of systems $[\lambda_0, \lambda_1(x), \cdots, \lambda_m(x)]$ which serve as Lagrange multipliers to give the analogues of the Euler equations, although this aggregate may reduce merely to the multiples of one such set. The object of the present note is to show that for every minimizing curve this aggregate contains a particularly desirable sub-aggregate, consisting of sets $[\lambda_0, \lambda_1(x), \cdots, \lambda_m(x)]$ with $\lambda_0 \geqq 0$ for which the analogues of the Du Bois-Reymond equations and transversality conditions and the Weierstrass condition all hold; from which it follows that the analogues of the Euler equations, the Clebsch condition and the Weierstrass-Erdmann corner condition must hold. The proof is not widely different from Bliss' proof of the multiplier rule, and makes no use whatever of the concept of normality.

Instead of spending several pages in setting forth the statement of the problem and the preliminary theorems, we shall regard this paper as an addendum to the paper of Bliss[2] [1]. We shall suppose that the reader has that paper at hand, and is familiar with its contents as far as the bottom of page 691. (However, we make no use of §§ 6, 7). The present paper begins at the end of the last complete sentence on page 691; the numbering of our equations begins, therefore, with (47). All page reference and all references to equations with numbers below (47) are to be understood as references to the paper of Bliss. One very slight change, however, is convenient. We shall ask that in the last half of page 691 the subscript $p+1$ be everywhere replaced by an arbitrary integer l.

1. A family of comparison arcs. A set (x, y, Y') is *admissible* if it

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[1] Numbers in square brackets refer to the very brief bibliography at the end of this paper.

[2] As a result, our theorem is stated only for Lagrange problems. The extension to Bolza problems offers no difficulty.

belongs to the neighborhood[3] $\mathfrak{N}$ of page 676 and the matrix $\| \phi_{\alpha y'_i}(x, y, Y') \|$ has rank m. Suppose that $[X, Y']$ is such that $(X, y(X))$ is not an end-point or corner of E_{12} and $(X, y(X), Y')$ is admissible. It is possible to adjoin linear functions $\bar{\phi}_\beta(y')$, $\beta = m + 1, \cdots, n$ to the functions $\phi_\alpha(x, y, y')$ in such a way that the matrix

$$\left\| \begin{matrix} \phi_{\alpha y'_i}(X, y(X), Y') \\ \bar{\phi}_{\beta y'_i}(Y') \end{matrix} \right\| \tag{47}$$

is non-singular.

By standard theorems on differential equations, for all values of b in a neighborhood of $(0, \cdots, 0)$ there is a curve $y_i = Y_i(x, b)$ which satisfies the differential equations

$$\begin{aligned} &\bar{\phi}_\beta(Y'(x, b)) = \bar{\phi}_\beta(Y'), && (\beta = m + 1, \cdots, n), \\ &\phi_\alpha(x, Y(x, b), Y'(x, b)) = 0, && (\alpha = 1, \cdots, m), \end{aligned} \tag{48}$$

and which has the initial value

$$Y_i(X, b) = y_i(X, b). \tag{49}$$

The functions $Y_i(x, b)$ are continuous together with their partial derivatives of first order for x near X and b near $(0, \cdots, 0)$.

Now let e be any small non-negative number. Using the enlarged system of functions ϕ_i of page 678, we write the equations

$$\begin{aligned} &\phi_\beta(x, y, y') = \phi_\beta(x, y(x, b), y'(x, b)), && (\beta = m + 1, \cdots, n). \\ &\phi_\alpha(x, y, y') = 0, && (\alpha = 1, \cdots, m), \end{aligned} \tag{50}$$

Suppose first that E_{12} has no corners. If e is sufficiently small, equation (50) will have a unique set of solutions $\bar{y}_i(x, b, e)$ such that

$$\bar{y}_i(X - e, b, e) = Y_i(X - e, b), \qquad (i = 1, \cdots, n). \tag{51}$$

This solution will be defined on the interval $(x_1(b), x_2(b))$, and $\bar{y}_i$ and $\bar{y}'_i$ will be continuous together with their partial derivatives of first order as to b and e for (b, e) near $(0, \cdots, 0)$. As already mentioned on page 679, if E_{12} has corners we apply the theorems on differential equations successively to the intervals of x between corners, choosing initial values at the values of x defining corners in such a way that the functions (51) are continuous.

Now we define the curve $y = y(x, b, e)$ by the equations

$$\begin{aligned} &y(x, b, e) = \bar{y}(x, b, e), \ x_1(b) \leqq x < X - e, \\ &y(x, b, e) = Y(x, b), \ X - e \leqq x < X, \\ &y(x, b, e) = y(x, b), \ X \leqq x \leqq x_2(b). \end{aligned} \tag{52}$$

[3] $\mathfrak{N}$ is an open set containing the elements $(x, y(x), y'(x))$ of E_{12}; a set can be admissible without having Y' "near" $y'(x)$.

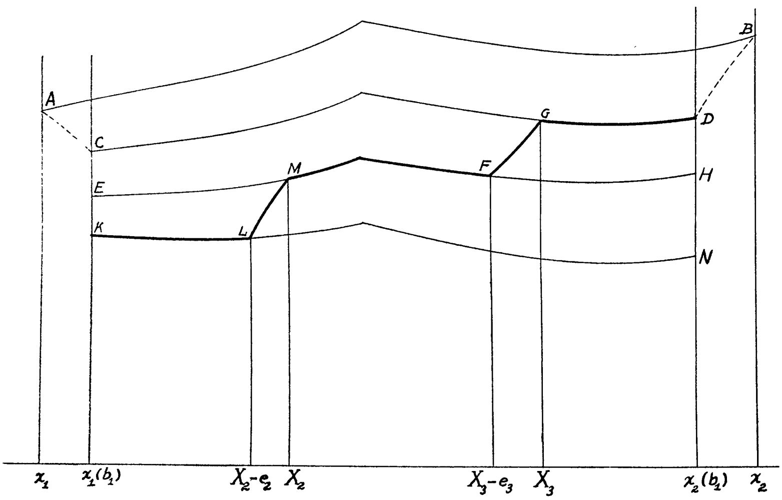
It is evident from the preceding remarks that this is continuous, and that the functions

$$I(b, e) = \int_{x_1(b)}^{x_2(b)} f(x, y(x, b, e), y'(x, b, e))\, dx \tag{53}$$

and

$$\psi_\mu(x_1(b), y(x_1(b), b, e), x_2(b), y(x_2(b), b, e))$$

are continuously differentiable for all b near $(0, \cdots, 0)$ and all small non-negative e. In fact, if we regard the integral $I(b, e)$ in (53) as the sum of three integrals, corresponding respectively to the three subintervals in the definition (52), we see that $I(b, e)$ is continuously differentiable for all (b, e)



AB:	$y = y(x)$	*EFGD*:	$y = y(x, b_1, e_3)$
CD:	$y = y(x, b_1)$	*LM*:	$y = Y(x, b_1, e_3)$
FG:	$y = Y(x, b_1)$	*KLN*:	$y = \bar{y}(x, b_1, e_2, e_3)$
EFH:	$y = \bar{y}(x, b_1, e_3)$	*KLMFGD*:	$y = y(x, b_1, e_2, e_3)$

near $(0, 0)$, irrespective of sign. However, if $e < 0$ equations (52) fail to define a single-valued continuous function.

If instead of a single set $[X, Y']$ we have several such sets, say

$$[X_k, Y'_k] \equiv [X'_k, Y'_{1,k}, \cdots, Y'_{n,k}], \quad (k = l + 1, \cdots, s, X_{l+1} < \cdots < X_s),$$

we can iterate the above construction, first using the greatest X_k, then the next greatest, and so on to the least of them. The corresponding parameters e are denoted by e_k, $k = l + 1, \cdots, s$. The resulting type of curve is shown by the heavy curve in the figure, which is drawn for the case $l = 1$, $s = 3$.

The partial derivatives of I with respect to the b_i, at $b = e = 0$, have already been computed, and are expressed in equation (45). If we introduce the notation

$$\eta_{i,k}(x) = \partial \bar{y}_i(x, b, e)/\partial e_k, \quad (b = e = 0;\ k = l+1, \cdots, s, x_1 \leqq x \leqq x_2),$$

we readily find from equations (50) and (11) that the corresponding functions ζ_k vanish identically. Computing the partial derivative of I with respect to e_k for $b = e = 0$, we find

$$(54) \quad \partial I(b, e)/\partial e_k = -f(X_k, y(X_k), y'(X_k)) + \int_{x_1}^{X_k} (f_{y^i}\eta_{i,k} + f_{y'_i}\eta'_{i,k})\,dx + f(X_k, y(X_k), Y'_k).$$

By (13) and (18) this is transformed into

$$(55) \quad \lambda_0 \partial I(b, e)/\partial e_k = -\lambda_0 f(X_k, y(X_k), y'(X_k)) - c_i \eta_{i,k}(x_1) + \eta_{i,k}(X_k) F_{y'_i}(x_k, y(X_k), y'(X_k), \lambda) + F(X_k, y(X_k), Y'_k, \lambda),$$

not summed on k; the left member is understood to be evaluated at $b = e = 0$. If we differentiate both members of (51) with respect to e and set $b = e = 0$, we obtain (interpreting e as any one e_k)

$$-\bar{y}'_i(X_k, 0, 0) + \eta_{ik}(X_k) = -Y'_i(X_k, 0).$$

Since $\bar{y}_i(x, 0, 0) \equiv y_i(x)$ and $Y'_i(X_k, 0) = Y'_{ik}$, this becomes

$$\eta_{ik}(X_k) = -(Y'_{ik} - y'_i(X_k)).$$

Substituting this in equation (55) yields

$$(56) \quad \lambda_0 \partial I(b, e)/\partial e_k = -c_i \eta_{i,k}(x_1) + \mathcal{E}(X_k, y(X_k), y'(X_k), Y'_k, \lambda),$$

where as usual we have written

$$\mathcal{E}(x, y, y', Y', \lambda) = F(x, y, Y', \lambda) - F(x, y, y', \lambda) - (Y'_i - y'_i) F_{y'_i}(x, y, y', \lambda).$$

Observe that the right member of (56) is a continuous function of X_k.

2. A convex set defined by the variations. Next we change notation slightly; we replace the symbol e_k by b_k, $k = l+1, \cdots, s$. We define

$$(57) \quad \begin{aligned} \rho_0(b) &= I(b) - I(0), \\ \rho_\mu(b) &= \psi_\mu(x_1(b), y(x_1(b), b), x_2(b), y(x_2(b), b)), \\ &\qquad (\mu = 1, \cdots, p). \end{aligned}$$

As remarked just after equation (53), these can be regarded as defined, continuous and continuously differentiable for all b near zero, although we must recall that in order that the parameter b shall define a curve of the family $y(x, b)$, all the parameters $b_{l+1}, \cdots, b_s$ must be non-negative.

Let K be the set of points $(u_0, \cdots, u_p)$ in $p+1$-dimensional space defined by the equations

$$(58) \quad u_j = \rho_{ja} b_a, \qquad b_a \geqq 0, \qquad (\alpha = 1, \cdots, s),$$

wherein

$$\rho_{ja} = \left.\frac{\partial \rho_j(b)}{\partial b_a}\right|_{b=0}.$$

It is easily seen that K is a convex point set; in fact, it is the linear image of the convex set $b_1 \geqq 0, \cdots, b_s \geqq 0$, by (58). We shall need the following lemma.

No point of the negative u_0-axis is interior to the set K.

Suppose there were such a point $\bar{u} : \bar{u}_0 < 0, \bar{u}_1 = \cdots = \bar{u}_p = 0$. Since $\bar{u}$ is in K it can be represented by equations

$$\bar{u}_i = \rho_{i\alpha}\bar{b}_\alpha, \qquad \bar{b}_\alpha \geqq 0, \qquad (i = 0, \cdots, p, \ \alpha = 1, \cdots, s). \tag{59}$$

We first show that the numbers $\bar{b}_\alpha$ in (59) may be supposed actually positive. The point $\bar{u}$ is interior to K; so if ϑ is a sufficiently small positive number, the point

$$u'_i = \rho_{i\alpha}b'_\alpha, \qquad b'_\alpha = \bar{b}_\alpha + \vartheta > 0$$

is also interior to K. So is $u'' = 2\bar{u} - u'$; hence

$$u_i'' = \rho_{i\alpha}b_\alpha'', \qquad b_\alpha'' \geqq 0.$$

Thus

$$\bar{u}_i = \tfrac{1}{2}(u_i' + u_i'') = \rho_{i\alpha}[\tfrac{1}{2}(b_\alpha' + b_\alpha'')],$$

and here the coefficients of the $\rho_{i\alpha}$ are positive. If the rows of the matrix $\rho_{i\alpha}$ were linearly dependent, the coördinates u_i of the points of K would satisfy a linear relationship, and K could have no interior points. Hence, as we are supposing $\bar{u}$ interior to K, the rows are linearly independent. Consequently we can adjoin $s - (p+1)$ linear functions $\rho_h(b) = \rho_{h\alpha}b_\alpha$, $h = p+1, \cdots, s-1$, in such a way that the square matrix $\| \rho_{i\alpha} \|$ is non-singular. Consider now the equations

$$u_i - \rho_i(b) = 0, \qquad (i = 0, \cdots, s-1). \tag{60}$$

These are satisfied if $u = b = 0$, and the jacobian with respect to the b_i is non-vanishing. Hence they have solutions $b_i = b_i(u)$, $i = 1, \cdots, s$, which are defined and continuous and possess continuous first partial derivatives near $u = 0$, and are such that $b_i(0) = 0$.

The numbers $\bar{u}_i$, $i = 0, \cdots, p$ are defined by equation (59). We augment this set, defining

$$\bar{u}_i = \rho_{i\alpha}\bar{b}_\alpha, \qquad (i = 0, \cdots, s-1). \tag{61}$$

The equation

$$t\bar{u}_i = \rho_i(b(t\bar{u})) \tag{62}$$

is an identity by definition of the function $b(u)$. If we differentiate with respect to t and set $t = 0$, we obtain

$$\bar{u}_i = \rho_{i\alpha}\frac{db_\alpha(t\bar{u})}{dt}.$$

The matrix $\| \rho_{i\alpha} \|$ is non-singular, so this and (61) imply

$$\frac{db_\alpha(t\bar{u})}{dt} = \bar{b}_\alpha > 0.$$

Hence for all small positive t the functions $b_\alpha(t\bar{u})$ are all positive-valued. Also, if t is small the curve $y = y(x, b)$ is in an arbitrarily small neighborhood of E_{12}. Equation (62), with (57) and the fact that $(\bar{u}_0, \cdot \cdot \cdot, \bar{u}_p)$ is on the negative u_0-axis, yields

$$I(b(t\bar{u})) < I(0),$$
$$\psi_\mu(X_1(b(t(\bar{u})), y(x_1(b(t\bar{u})), b(t\bar{u})), x_2(b(t\bar{u}))),$$
$$y(x_2(b(t\bar{u})), b(t\bar{u})) = 0.$$

The curves $y = y(x, b)$ were constructed so as to satisfy the differential equations $\phi_\alpha = 0$. Now we see that if $b = b(t\bar{u})$, t small and positive, they also satisfy the end conditions $\psi_\mu = 0$ and give I a smaller value than $I(0)$. This contradicts the hypothesis that E_{12} is a minimizing curve, and our lemma is established.

The set K depends for its definition on the sets $[X_k, Y'_k]$ and $[\xi_{i1}, \xi_{i2}, \eta_{ij}(x)]$ used in defining the functions $y(x, b)$. Let K^* be the closure of the sum of all sets K, for all possible choices of sets $[X_k, Y'_k]$ and $[\xi_i, \eta_i]$. That is, u is in K^* if every neighborhood of u contains points belonging to some set K, as defined by (58). We first prove

The set K^* is convex.

Let v, $\tilde{v}$ be two points in K^*. For every positive number ϵ there are points u, $\tilde{u}$ having distances less than ϵ from v, $\tilde{v}$ respectively and belonging to sets K, $\tilde{K}$. Let $[X_k, Y'_k]$ and $[\xi_i, \eta_i]$ be the sets defining K, and $[\tilde{X}_h, \tilde{Y}'_h]$ and $[\tilde{\xi}_j, \tilde{\eta}_j]$ the sets defining $\tilde{K}$. Then u is defined by (58), and $\tilde{u}$ analogously. We may suppose that the points $\tilde{X}_h$ and X_k are all distinct, since by the remark after equation (56) the X_k can be moved slightly so as to produce an arbitrarily small change in the $\rho_{i\alpha}$. If we now use all the sets $[X_k, Y'_k]$, $[\tilde{X}_h, \tilde{Y}'_h]$, $[\xi_i, \eta_i]$, $[\tilde{\xi}_j, \tilde{\eta}_j]$ to define a new set K_1 as in (58), the set K_1 contains both K and $\tilde{K}$, hence contains u and $\tilde{u}$. Being convex, it contains the line segment joining u and $\tilde{u}$. Every point of the line segment joining v and $\tilde{v}$ has distance less than ϵ from some point of the segment joining u and $\tilde{u}$. Since ϵ is arbitrary, every point of the line segment joining v and $\tilde{v}$ is in K^*, and K^* is convex.

No point of the negative u_0-axis is interior to K^*.

Suppose there is such a point $\bar{u}$. It is then possible to find $p + 2$ points $v^{(1)}, \cdot \cdot \cdot, v^{(p+2)}$ in K^* such that $\bar{u}$ is interior to the simplex with vertices

$v^{(1)}, \cdots, v^{(p+2)}$. Arbitrarily near each $v^{(i)}$ there is a point $u^{(i)}$ belonging to some set $K^{(i)}$ defined by sets $[X_k^{(i)}, Y_k^{(i)\prime}]$ and $[\xi_j^{(i)}, \eta_j^{(i)}]$. As before, we may suppose that all the numbers $X_k^{(i)}$ are distinct, and combine all the sets into a single set. This aggregate of sets then defines a convex point set K_1 containing each of the $K^{(i)}$. In particular, it contains the entire simplex with vertices $u^{(i)}$; and if the $u^{(i)}$ are near enough to the $v^{(i)}$, the point $\bar{u}$ is interior to this simplex. But now $\bar{u}$ is interior to K_1, contradicting a preceding lemma.

As K^* does not consist of the entire u-space, it has frontier points. Let $\tilde{u}$ be a frontier point of K^*. Since K^* is convex, there is a hyperplane of support[4] of K^* passing through $\tilde{u}$. That is, there are numbers $\lambda_0, d_1, \cdots, d_p, q$ such that

$$(63) \qquad \begin{aligned} &\lambda_0 u_0 + d_\mu u_\mu + q \geqq 0 \text{ for all } u \text{ in } K^*, \\ &\lambda_0 \tilde{u}_0 + d_\mu \tilde{u}_\mu + q = 0. \end{aligned}$$

It is easy to see that q must vanish. If $\tilde{u} = (0, \cdots, 0)$ this is obvious from the second of conditions (63). Otherwise, we observe that the points $u = (0, \cdots, 0)$ and $u = 2\tilde{u}$ both are in K^*. Substituting these in the first of conditions (63) and using the second yields $q = 0$.

We now wish to show that it is possible to choose λ_0 and the d_i in such a way that λ_0 is non-negative. If $\tilde{u} = (-1, 0, \cdots, 0)$ is a frontier point of K^*, we choose λ_0, d_μ so as to define a hyperplane of support at $\tilde{u}$. By the second of conditions (63) we obtain $\lambda_0 = 0$. If $(-1, 0, \cdots, 0)$ is not a frontier point of K^*, it is exterior to K^*. There is then a hyperplane separating[5] $(-1, 0, \cdots, 0)$ from K^*. We choose the notation so that

$$(64) \qquad \begin{aligned} &\lambda_0 u_0 + d_\mu u_\mu + q > 0 \text{ for all } u \text{ in } K^*, \\ &\lambda_0(-1) + d_\mu 0 + q < 0. \end{aligned}$$

The first of these, with $u = (0, \cdots, 0)$, implies $q > 0$; the second then implies $\lambda_0 > 0$. For this λ_0 and d_μ the first of conditions (63) holds with q replaced by zero; if it failed for some $\bar{u}$, then for a sufficiently large number N the numbers $u_i = N\bar{u}_i$ would violate (64). Hence $\lambda_0 u_0 + d_\mu u_\mu = 0$ is a hyperplane supporting K^* at the origin.

Summing up our lemmas, we have the following statement.

There are numbers $\lambda_0 \geqq 0, d_1, \cdots, d_p$, not all zero, such that for every finite collection of sets $[X_k, Y'_k]$ with $(X_k, y(X_k), Y'_k)$ admissible[6] and every finite collection of sets of admissible variations $[\xi_i, \eta_i]$ the inequalities

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[4] C. Carathéodory, "Über den Variabilitätsbereich der Fourier'schen Konstanten von positiven harmonischen Funktionen," *Rend. Cir. Math. Palermo*, vol. 32 (1911), pp. 197, 198.

[5] C. Carathéodory, *loc. cit.*

[6] We still suppose that $(X_k, y(X_k))$ is not an end or corner of E_{12}.

$$(65) \qquad [\lambda_0\rho_{0\beta} + d_\mu\rho_{\mu\beta}]b_\beta \geqq 0$$

hold whenever the numbers b_β are all non-negative.

3. The multiplier rule and the Weierstrass condition. Now let us suppose that there are no sets $[X_k, Y'_k]$ and just one set of variations $[\xi, \eta]$. The matrix $\| \rho_{i\alpha} \|$ has then a single column $\alpha = 1$. Inequality (65) takes the form

$$(66) \qquad \lambda_0\rho_{0,1} + d_\mu\rho_{\mu,1} \geqq 0.$$

However, $[-\xi, -\eta]$ is also an admissible set of variations, and for these the matrix $\| \rho_{i\alpha} \|$ consists of a single column $(-\rho_{0,1}, \cdots, -\rho_{p,1})$; hence by (65) we have

$$\lambda_0(-\rho_{0,1}) + d_\mu(-\rho_{\mu,1}) \geqq 0.$$

This, with (66), implies

$$\lambda_0\rho_{0,1} + d_\mu\rho_{\mu,1} = 0.$$

In this equation we substitute for the derivatives $\rho_{i,1}$ their values as computed from (57); we obtain

$$\lambda_0 I_1 + d_\mu\Psi_\mu(\xi, \eta) = 0.$$

With the help of equations (44) and (45) this becomes

$$(67) \qquad \begin{aligned} -\int_{x_1}^{x_2} \lambda_r\zeta_r dx &+ [-\lambda_0 f(x_1) + d_\mu(\psi_{\mu x_1} + \psi_{\mu y'_i} y'_{i1})]\xi_1 \\ &+ [\lambda_0 f(x_2) + d_\mu(\psi_{\mu x_2} + \psi_{\mu y_{i2}} y'_{i2})]\xi_2 \\ &+ [-c_i + d_\mu\psi_{\mu y_{i1}}]\eta_i(x_1) \\ &+ [F_{y'_i}(x_2) + d_\mu\psi_{\mu y_{i2}}]\eta_i(x_2) = 0. \end{aligned}$$

The constants λ_0, c_i have so far been arbitrary, save that the λ_0 has recently been chosen in the lemma at the end of § 2. We now choose the c_i so that the coefficients of the $\eta_i(x_1)$ vanish:

$$(68) \qquad c_i = d_\mu\psi_{y_{i1}}.$$

The ξ_1 and ξ_2 are arbitrary, so if (67) is to hold their coefficients must vanish. Likewise, for each set ζ_r the solution $\eta_i(x)$ of equation (11) can be chosen so as to have arbitrary values at x_2; hence the coefficient of $\eta_i(x_2)$ must vanish. Finally, the ζ_r are arbitrary continuous functions, so $\lambda_r(x)$ must vanish identically, $r = m + 1, \cdots, n$.

The conditions on the various coefficients imply that the rank of the matrix

$$(69) \qquad \left\| \begin{matrix} -\lambda_0 f(x_1) & -c_i & \lambda_0 f(x_2) & F_{y'_i}(x_2) \\ \psi_{\mu x_1} + \psi_{\mu y_{i1}} y'_{i1} & \psi_{\mu y_{i1}} & {}_{\mu x_2} + {}_{\mu y_{i2}} y'_{i2} & \psi_{\mu y_{i2}} \end{matrix} \right\|$$

is at most p, for if the rows be multiplied by $1, d_1, \cdots, d_p$ respectively and added, the sum is a row of zeros. By (15) we have $c_i = F_{y_i'}(x_1)$. We substitute this in the matrix. Adding a multiple of one column of a matrix to another column leaves the rank unchanged, so the matrix

$$(70) \quad \left\| \begin{matrix} -F(x_1) + y'_{i1}F_{y'_i}(x_1) & -F_{y'_i}(x_1) & F(x_2) - y'_{i2}F_{y'_i}(x_2) & F_{y'_i}(x_2) \\ \psi_{\mu x_1} & \psi_{\mu y_{i1}} & \psi_{\mu x_2} & \psi_{\mu y_{i2}} \end{matrix} \right\|$$

has rank less than $p + 1$.

Next, let us use a single set $[X_1, Y'_i]$ and no sets $[\xi, \eta]$. Inequality (65) now takes the form (by (56) and (57))

$$\mathcal{E}(X_1, y(X_1), y'(X_1), Y'_1, \lambda) + c_i\eta_i(x_1) \\ + d_\mu\Psi_\mu(0, 0, \eta(x_1), 0) \geqq 0.$$

If we use (68) and (44), this yields

$$\mathcal{E}(X_1, y(X_1), y(X_1), Y', \lambda) \geqq 0.$$

Collecting the various statements established above and adding two minor ones yet to be proved, we have the following theorem.

For every minimizing arc for the problem of Lagrange with variable end points there exists a non-negative constant λ_0 *and a set of functions* $\lambda_1(x), \cdots, \lambda_m(x)$ *such that for the function*

$$F(x, y, y', \lambda) \equiv \lambda_0 f + \lambda_1\phi_1 + \cdots + \lambda_m\phi_m$$

the following statements hold.

(i) (Du Bois-Reymond relation).

There are constants $c_1, \cdots, c_n$ *such that the equations*

$$F_{y_i'}(x, y(x), y'(x), \lambda) = \int_{x_1}^{x} F_{y_i} dx + c_i$$

hold on the entire interval $[x_1, x_2]$.

(ii) (Transversality).

The rank of the matrix (70) *is less than* $p + 1$.

(iii) (Weierstrass Condition).

For all x *in the interval* $[x_1, x_2]$ *and all* Y' *such that* $(x, y(x), Y')$ *is admissible, the inequality*

$$\mathcal{E}(x, y(x), y'(x), Y', \lambda) \geqq 0$$

is satisfied.

(iv) (Clebsch Condition).

For all x *in the interval* $[x_1, x_2]$ *and all sets of numbers* $\pi_1, \cdots, \pi_n$ *satisfying the equations*

$$\pi_i \phi_{\alpha y'_i} = 0, \qquad (\alpha = 1, \cdots, m)$$

the inequality

$$\pi_i F_{y'_i y'_j} \pi_j \geqq 0$$

is satisfied.

Moreover, the constant λ_0 *and the functions* $\lambda_\alpha(x)$ *are not all identically zero on* $[x_1, x_2]$, *and are continuous except possibly at values of* x *defining corners of* E_{12}.

The continuity properties of the $\lambda_\alpha(x)$ have already been established. If λ_0 and the $\lambda_\alpha(x)$ all were identically zero, the first row of matrix (69) would be identically zero. The numbers λ_0, d_μ are not all zero; since $\lambda_0 = 0$, not all the d are zero. But we have seen that if the rows of (69) are multiplied by $1, d_1, \cdots, d_p$ respectively and added, the sum is a row of zeros. If the first row vanished, we would have a linear dependency (with coefficients d_μ) between the remaining rows, and the rows of matrix (42) would be linearly dependent, contrary to hypothesis.

We have established the Weierstrass condition only under the assumption that $(x, y(x))$ is neither a corner nor an end of E_{12}. It extends to such points readily, by simple continuity considerations; at a corner, we can understand $y'(x)$ to mean either the right or the left derivative.

The condition of Clebsch is a consequence of the condition of Weierstrass, as shown on page 718.

The conclusion that $\lambda_0, \lambda_\alpha(x)$ are not all identically zero can be sharpened to the form that for no x in the interval (x_1, x_2) is $(\lambda_0, \lambda_1(x), \cdots, \lambda_m(x))$ equal to $(0, 0, \cdots, 0)$; cf. [II], p. 27. Furthermore [II, p. 31] we can add one more equation analogous to the Du Bois-Reymond equations:

$$F - y'_i F_{y'_i} = c_0 + \int_{x_1}^{x} F_x dx. \tag{71}$$

4. Remarks on the determination of the multipliers. If the minimizing curve E_{12} happens to be normal, we can take $\lambda_0 = 1$, and then there exists a unique set of functions $\lambda_\alpha(x)$ for which the Du Bois-Reymond equations and the transversality conditions hold. By the theorem just stated, for these same multipliers the Weierstrass and Clebsch conditions must hold.

If E_{12} is not normal, there may be many determinations of λ_0 and the $\lambda_\alpha(x)$ for which the Du Bois-Reymond equations and the transversality conditions hold. For at least one, and possibly for several, of these determinations, the Weierstrass and Clebsch conditions will also hold. However, these latter conditions may impose additional restrictions on the choice of the λ_0 and $\lambda_\alpha(x)$. For example, let $x_1 = 0, x_2 = 1, f(x, y, y') = (y'_i y'_i)^2$. We suppose the end-values fixed at zero, and take a single side-equation $\phi(x, y, y') = 0$. If we choose $\phi = y'_1 + (y'_1)^3$, then for any λ_0 and any constant $\lambda(x)$ the Du Bois-Reymond relations hold along the arc $E_{12}: y = 0$. But unless $\lambda(x)$ is identically zero the Weierstrass condition is not satisfied. Again, if we choose $\phi = y'_1 + y'_i y'_i$, for any λ_0 and any constant $\lambda(x)$ the Du Bois-Reymond relations hold. But for the Weierstrass condition to hold we must have $\lambda(x) \geqq 0$.

It is possible to establish the Jacobi condition by considerations similar to the preceding, but not without assumptions of normality; the method seems at present to be useful only for normal problems and certain problems with order of anormality 1. The inclusion of the proof of the Jacobi condition for such cases would about double the length of this paper, and the theorem does not seem to be interesting enough to justify the use of the extra pages.

If the necessary condition of Jacobi could be established without assumptions of normality, we would be in possession of a complete set of necessary conditions and sufficient conditions for a minimum without hypotheses of normality. It would be rash to conclude that the concept of normality would be rendered useless. For within the class of anormal problems there are problems of definitely unruly behavior, such as those in which there is but a single curve satisfying the differential equation and end conditions. The distinction between such problems and those of more placid aspect is intrinsic, and not erased by any amount of analytic ingenuity. Nevertheless, it seems desirable to have proofs constructed so as to apply simultaneously both to normal and to anormal problems, if for no other reason than to avoid having to verify the presence of normality in individual cases where such verification may be difficult.

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