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Brief paper

Reduction theorems for stability of closed sets with application to backstepping control design[☆]

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ABSTRACT

We present a solution to the following reduction problem for asymptotic stability of closed sets in nonlinear systems. Given two closed, positively invariant subsets of the state space of a nonlinear system, $\Gamma_1 \subset \Gamma_2$, assuming that Γ_1 is asymptotically stable relative to Γ_2, find conditions under which Γ_1 is asymptotically stable. We also investigate analogous reduction problems for stability and attractivity. We illustrate the implications of our results on the stability of sets for cascade-connected systems and on a hierarchical control design problem. For upper triangular control systems, we present a reduction-based backstepping technique that does not require the knowledge of a Lyapunov function, and mitigates the problem of controller complexity arising in classical backstepping design.

1. Introduction

Consider a dynamical system Σ modelled as

$$\Sigma : \dot{x} = f(x), \tag{1}$$

with state space a domain $\mathcal{X} \subset \mathbb{R}^n$. Assume that f is locally Lipschitz on $\mathcal{X}$, and let $\phi(t, x_0)$ denote the solution of (1) at time t with initial condition $x(0) = x_0$. Suppose that two closed sets $\Gamma_1 \subset \Gamma_2$ are positively invariant for Σ. That is, for all $x_0 \in \Gamma_i$, $i = 1, 2$, and for all $t \geq 0$, $\phi(t, x_0) \in \Gamma_i$. Suppose further that the set Γ_1 is (globally) asymptotically stable *relative to* Γ_2, i.e., it is (globally) asymptotically stable when initial conditions of Σ are restricted to lie on Γ_2. In this paper, we investigate the following

Reduction Problem for Asymptotic Stability (RPAS). Find conditions under which Γ_1 is (globally) asymptotically stable relative to $\mathcal{X}$.

We also investigate reduction problems for stability and attractivity in which Γ_1 is, respectively, stable or (globally) attractive relative to Γ_2, and we seek conditions under which Γ_1 is, respectively, stable or (globally) attractive relative to $\mathcal{X}$.

The above reduction problems were formulated for the first time in Seibert (1969) and Seibert (1970). Seibert and Florio (1995) presented reduction theorems for (global) stability and (global) asymptotic stability of dynamical systems on metric spaces under the restriction that Γ_1 is compact. To date, these are the most general results available for compact Γ_1. See also work by Kalitin (1999) and co-workers (Iggidr, Kalitin, & Outbib, 1996).

In the linear time-invariant setting, the reduction problem takes on a familiar form. For a system $\dot{z} = Az$ with $z \in \mathcal{Z} = \mathbb{R}^n$, if $\mathcal{V} \subset \mathcal{Z}$ is an A-invariant subspace, then a necessary and sufficient condition for $z = 0$ to be asymptotically stable is that $z = 0$ be asymptotically stable relative to $\mathcal{V}$, and that $\mathcal{V}$ be asymptotically stable. Indeed, by A-invariance of $\mathcal{V}$, there exists an isomorphism $T : z \mapsto (x, y)$, such that the system takes on the cascade-connected form

$$\begin{aligned} \dot{x} &= A_{11}x + A_{12}y \\ \dot{y} &= A_{22}y, \end{aligned} \tag{2}$$

where $T(\mathcal{V}) = \{(x, y) : y = 0\}$. The asymptotic stability of $z = 0$ relative to $\mathcal{V}$ is equivalent to the property $\sigma(A_{11}) \subset \mathbb{C}^-$, while the asymptotic stability of $\mathcal{V}$ is equivalent to the property $\sigma(A_{22}) \subset \mathbb{C}^-$. In the context of nonlinear systems, researchers in control theory have focused on another special case of the reduction problem which generalises the linear result above. Consider a cascade-connected system of the form

$$\begin{aligned} \dot{x} &= f(x, y) \\ \dot{y} &= g(y), \end{aligned} \tag{3}$$

where $f(0, 0) = 0$ and $g(0) = 0$. Letting $\Gamma_1 = \{(x, y) : x = y = 0\}$ and $\Gamma_2 = \{(x, y) : y = 0\}$, we have that Γ_1 is (globally) asymptotically stable relative to Γ_2 if and only if $x = 0$ is (globally)

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asymptotically stable for the subsystem $\dot{x} = f(x, 0)$ and the y subsystem does not have finite escape times. Here, the reduction problem seeks conditions under which the equilibrium $(x, y) = (0, 0)$ is (globally) asymptotically stable for (3). Vidyasagar (1980) showed that the required condition is the asymptotic stability of $y = 0$ for $\dot{y} = g(y)$ (this is actually a corollary of Seibert–Florio's results in Seibert and Florio (1995)). Vidyasagar's result was extended by various researchers, see Chaillet (2008), Isidori (1999), Panteley and Loria (1998, 2001), Seibert and Suárez (1990) and Sontag (1990). While the equilibrium stability problem for cascade-connected systems has been researched with vigour, the more general RPAS has received little attention, particularly in the case when the set Γ_1 is not compact. To highlight the distinction between RPAS and the problem of stability of cascade-connected systems, it is worth noting that while in the LTI setting the A-invariance of the subspace $\mathcal{V}$ implies the existence of the upper triangular representation (2), in the nonlinear setting this is not the case. Specifically, the positive invariance of Γ_2 does not guarantee the existence of a coordinate transformation making system Σ take on the cascade-connected form (3) with $\Gamma_2 = \{y = 0\}$.

The main contribution of this paper is the extension, in Section 2, of Seibert–Florio's reduction theorems for the case when Γ_1 is not compact, and a new reduction theorem for attractivity. We also investigate the implications of our reduction theorems on three problems. For cascade-connected systems of the form (3), in Section 3 we derive conditions under which the asymptotic stability of a set $\tilde{\Gamma}_1$ for the system $\dot{x} = f(x, 0)$ implies that $\Gamma_1 := \tilde{\Gamma}_1 \times 0$ is asymptotically stable for (3). For a control system of the form

$$\dot{x} = f(x, u),$$

in Section 4 we investigate a problem of hierarchical control design involving the *simultaneous* asymptotic stabilisation of a chain of nested closed sets $\Gamma_1 \subset \Gamma_2 \subset \cdots \subset \Gamma_l$. Such a problem is encountered in applications in which the designer must simultaneously meet control specifications that can be formulated hierarchically. One such application is illustrated in Example 20. Finally, in Section 5 we specialise the hierarchical control design idea to derive a *reduction-based backstepping* technique to stabilise closed sets for upper triangular systems. This procedure does not require the recursive construction of a Lyapunov function, and it mitigates the problem of controller complexity arising in classical backstepping. Besides the problems discussed in this paper, RPAS arises in other problems of nonlinear control. One of them is the passivity-based stabilisation of closed sets. In El-Hawwary and Maggiore (2010), we used the solution of RPAS presented here[2] to determine conditions for stabilisability of a closed set by passivity-based feedback.

Notation. Given an interval I of the real line and a set $S \in \mathcal{X}$, we denote by $\phi(I, S)$ the set $\phi(I, S) := \{\phi(t, x_0) : t \in I, x_0 \in S\}$. Given a closed nonempty set $S \subset \mathbb{R}^n$, a point $x \in \mathbb{R}^n$, and a vector norm $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}$, the point-to-set distance $\|x\|_S$ is defined as $\|x\|_S := \inf\{\|x - y\| : y \in S\}$. Throughout this paper, we will use the Euclidean norm $\|x\| = (x^\top x)^{1/2}$. Given two subsets S_1 and S_2 of $\mathcal{X}$, the distance of S_1 to S_2, $d(S_1, S_2)$, is defined as $d(S_1, S_2) := \sup\{\|x\|_{S_2} : x \in S_1\}$. For a scalar $\alpha > 0$, a point $x \in \mathcal{X}$, and a set $S \subset \mathcal{X}$, define the open sets $B_\alpha(x) = \{y \in \mathcal{X} : \|y - x\| < \alpha\}$ and $B_\alpha(S) = \{y \in \mathcal{X} : \|y\|_S < \alpha\}$. We denote by $\mathrm{cl}(S)$ the closure of the set S, and by $\mathcal{N}(S)$ an open neighbourhood of S. For $x_0 \in \mathcal{X}$, we will denote by $L^+(x_0)$ the positive limit set of the solution $\phi(t, x_0)$, defined as $L^+(x_0) := \{p \in \mathcal{X} : (\exists \{t_n\} \subset \mathbb{R}^+)\, t_n \to +\infty,\ \phi(t_n, x_0) \to p\}$. The negative limit set of $\phi(t, x_0)$ is denoted $L^-(x_0)$.

[2] In El-Hawwary and Maggiore (2010), the three reduction theorems presented in this paper were reported without proof.

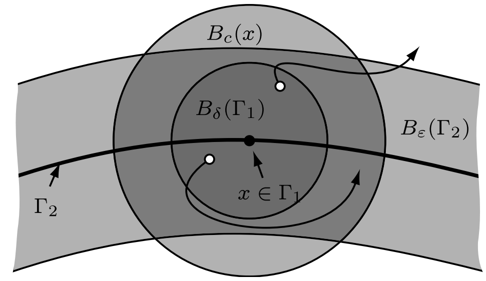

Fig. 1. An illustration of the notion of local stability near Γ_1.

2. Main results

In this section, we present solutions to the reduction problems for stability, attractivity, and asymptotic stability. We begin with stability definitions.

Let $\Gamma \subset \mathcal{X}$ be a closed positively invariant for Σ in (1).

Definition 1 (*Set Stability and Attractivity*)**.**

(i) Γ is *stable* for Σ if for all $\varepsilon > 0$ there exists a neighbourhood $\mathcal{N}(\Gamma)$ such that $\phi(\mathbb{R}^+, \mathcal{N}(\Gamma)) \subset B_\varepsilon(\Gamma)$.
(ii) Γ is an *attractor* for Σ if there exists a neighbourhood $\mathcal{N}(\Gamma)$ such that $\lim_{t\to\infty} \|\phi(t, x_0)\|_\Gamma = 0$ for all $x_0 \in \mathcal{N}(\Gamma)$.
(iii) Γ is a *global attractor* for Σ if it is an attractor with $\mathcal{N}(\Gamma) = \mathcal{X}$.
(iv) Γ is a *uniform semi-attractor* for Σ if for all $x \in \Gamma$, there exists $\lambda > 0$ such that, for all $\varepsilon > 0$, there exists $T > 0$ yielding $\phi([T, +\infty), B_\lambda(x)) \subset B_\varepsilon(\Gamma)$.
(v) Γ is *(globally) asymptotically stable* for Σ if it is stable and attractive (globally attractive) for Σ.

Remark 2. The definitions above, except that of a uniform semi-attractor, are found in Bathia and Szegö (1970). If Γ is not compact, then uniform semi-attractivity is a weaker property than the uniform attractivity notion found in Bathia and Szegö (1970) or Lin, Sontag, and Wang (1996).

Definition 3 (*Local Stability and Attractivity Near* Γ_1)**.** Let Γ_1 and Γ_2, $\Gamma_1 \subset \Gamma_2 \subset \mathcal{X}$, be closed positively invariant sets. The set Γ_2 is *locally stable near* Γ_1 if for all $x \in \Gamma_1$, for all $c > 0$, and all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x_0 \in B_\delta(\Gamma_1)$ and all $t > 0$, whenever $\phi([0, t], x_0) \subset B_c(x)$ one has that $\phi([0, t], x_0) \subset B_\varepsilon(\Gamma_2)$. The set Γ_2 is *locally attractive near* Γ_1 if there exists a neighbourhood $\mathcal{N}(\Gamma_1)$ such that, for all $x_0 \in \mathcal{N}(\Gamma_1)$, $\phi(t, x_0) \to \Gamma_2$ at $t \to +\infty$.

The definition of local stability can be rephrased as follows. Given an arbitrary ball $B_c(x)$ centred at a point x in Γ_1, trajectories originating in $B_c(x)$ sufficiently close to Γ_1 cannot travel far away from Γ_2 before first exiting $B_c(x)$; see Fig. 1. It is immediate to see that if Γ_1 is stable, then Γ_2 is locally stable near Γ_1, and therefore local stability of Γ_2 near Γ_1 is a necessary condition for the stability of Γ_1.

Definition 4 (*Relative Set Stability and Attractivity*)**.** Let Γ_1 and Γ_2, $\Gamma_1 \subset \Gamma_2 \subset \mathcal{X}$, be closed positively invariant sets. We say that Γ_1 is *stable relative to* Γ_2 for Σ if, for any $\varepsilon > 0$, there exists a neighbourhood $\mathcal{N}(\Gamma_1)$ such that $\phi(\mathbb{R}^+, \mathcal{N}(\Gamma_1) \cap \Gamma_2) \subset B_\varepsilon(\Gamma_1)$. Similarly, one modifies all other notions in Definitions 1 and 3 by restricting initial conditions to lie in Γ_2.

Definition 5 (*Local Uniform Boundedness (LUB)*)**.** System Σ is *locally uniformly bounded near* Γ *(LUB)* if for each $x \in \Gamma$ there exist positive scalars λ and m such that $\phi(\mathbb{R}^+, B_\lambda(x)) \subset B_m(x)$.

Now we present the main results, whose proofs are found in the Appendices A and B. When Γ_1 is a compact set, Theorems 6 and 10 below coincide with analogous results in Seibert and Florio (1995). All results below refer to the dynamical system Σ in (1).

Theorem 6 (*Stability*). *Let $\Gamma_1 \subset \Gamma_2$ be two closed positively invariant subsets of $\mathcal{X}$. Then, Γ_1 is stable if the following conditions hold:*

(i) *Γ_1 is asymptotically stable relative to Γ_2.*
(ii) *Γ_2 is locally stable near Γ_1.*
(iii) *If Γ_1 is unbounded, then Σ is LUB near Γ_1.*

By noting that if Γ_2 is stable for Σ, then it is also locally stable near Γ_1, we get the following useful corollary.

Corollary 7. *Let Γ_1 and Γ_2, $\Gamma_1 \subset \Gamma_2 \subset \mathcal{X}$, be two closed positively invariant sets. Then, Γ_1 is stable if conditions* (i) *and* (iii) *in Theorem 6 hold and condition* (ii) *is replaced by the following one:*

(ii)′ *Γ_2 is stable.*

Theorem 8 (*Attractivity*). *Let Γ_1 and Γ_2, $\Gamma_1 \subset \Gamma_2 \subset \mathcal{X}$, be two closed positively invariant sets. Then, Γ_1 is attractive if the following conditions hold:*

(i) *Γ_1 is asymptotically stable relative to Γ_2*
(ii) *Γ_2 is locally attractive near Γ_1,*
(iii) *there exists a neighbourhood $\mathcal{N}(\Gamma_1)$ such that, for all initial conditions in $\mathcal{N}(\Gamma_1)$, the associated solutions are bounded and such that the set $\mathrm{cl}(\phi(\mathbb{R}^+, \mathcal{N}(\Gamma_1))) \cap \Gamma_2$ is contained in the domain of attraction of Γ_1 relative to Γ_2.*

The set Γ_1 is globally attractive if:

(i)′ *Γ_1 is globally asymptotically stable relative to Γ_2,*
(ii)′ *Γ_2 is a global attractor,*
(iii)′ *all trajectories in $\mathcal{X}$ are bounded.*

Conditions (ii) and (ii)′ are also necessary.

Remark 9. This Theorem generalises Theorem 10.5.2 in Isidori (1999). If condition (i) is replaced by the stronger (i)′, then one can replace (iii) by the simpler requirement that trajectories in some neighbourhood of Γ_1 be bounded.

By combining Theorems 8 and 6 we obtain the solution to RPAS.

Theorem 10 (*Asymptotic Stability*). *Let Γ_1 and Γ_2, $\Gamma_1 \subset \Gamma_2 \subset \mathcal{X}$, be two closed positively invariant sets. Then, Γ_1 is (globally) asymptotically stable if the following conditions hold:*

(i) *Γ_1 is (globally) asymptotically stable relative to Γ_2,*
(ii) *Γ_2 is locally stable near Γ_1,*
(iii) *Γ_2 is locally attractive near Γ_1 (Γ_2 is globally attractive),*
(iv) *if Γ_1 is unbounded, then Σ is LUB near Γ_1,*
(v) *(all trajectories of Σ are bounded).*

Conditions (i)–(iii) in the theorem above are necessary. If $\Gamma_1 \subset \Gamma_2 \subset \Gamma_3$ are positively invariant for (1) and conditions (ii)–(v) are relaxed by only assuming that they hold relative to Γ_3, then the conclusions of Theorem 10 hold relative to Γ_3.

By combining Theorem 10 and Corollary 7 we obtain the following corollary.

Corollary 11. *Let Γ_1 and Γ_2, $\Gamma_1 \subset \Gamma_2 \subset \mathcal{X}$, be two closed positively invariant sets. Then, Γ_1 is (globally) asymptotically stable if conditions* (i), (iii), (iv) *(and* (v)*) in Theorem 10 hold, and condition* (ii) *is replaced by the following one:*

(ii)′ *Γ_2 is stable.*

3. Cascade-connected systems

We now return to the cascade-connected system (3). As pointed out in the introduction, when $f(0, 0) = 0$ and $g(0) = 0$, conditions for asymptotic stability and attractivity of the equilibrium $(x, y) = (0, 0)$ are well-known in the control literature (see Vidyasagar (1980, Theorem 3.1), Sontag (1990, Corollary 5.2), Isidori (1999, Corollaries 10.3.2 and 10.3.3)). We now present an application of Theorems 6 and 8, and Corollary 11.

Corollary 12. *Consider system (3) with f and g locally Lipschitz on $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, and let $\tilde{\Gamma}_1 \subset \mathbb{R}^{n_1}$ be a positively invariant set for system $\dot{x} = f(x, 0)$. Denote $\Gamma_1 := \tilde{\Gamma}_1 \times 0$ and suppose that $g(0) = 0$. Then, Γ_1 is an attractor (global attractor) for (3) if*

(i) *$\tilde{\Gamma}_1$ is globally asymptotically stable for $\dot{x} = f(x, 0)$,*
(ii) *$y = 0$ is a (globally) attractive equilibrium for $\dot{y} = g(y)$,*
(iii) *all solutions of (3) originating in some neighbourhood of Γ_1 (originating in $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$) are bounded.*

Moreover, Γ_1 is (globally) asymptotically stable if

(iv) *$\tilde{\Gamma}_1$ is (globally) asymptotically stable for $\dot{x} = f(x, 0)$,*
(v) *$y = 0$ is a (globally) asymptotically stable equilibrium of $\dot{y} = g(y)$,*
(vi) *if Γ_1 is unbounded, then (3) is LUB near Γ_1,*
(vii) *(all trajectories of (3) are bounded).*

The proof is presented in Appendix B. In the special case when Γ_1 is an equilibrium, the part of Corollary 12 concerning attractivity recovers the result in Isidori (1999, Theorem 10.3.1), while the part concerning asymptotic stability recovers well-known results in Vidyasagar (1980, Theorem 3.1) (see also Isidori (1999, Corollaries 10.3.2 and 10.3.3) and Seibert and Suárez (1990)).

Remark 13. The asymptotic stability result of Corollary 12 relies on two boundedness assumptions, (vi) and (vii). Assumption (vi) only needs to be checked when Γ_1 is unbounded, while Assumption (vii) needs to be checked when one wants to infer that Γ_1 is globally asymptotically stable. The requirement, when Γ_1 is unbounded, that system (3) is LUB near Γ_1 may seem surprising, but it can be shown that if trajectories of (3) near Γ_1 are not bounded, then the asymptotic stability of Γ_2, and that of Γ_1 relative to Γ_2 are not sufficient to guarantee asymptotic stability of Γ_1. Instead of assuming that the cascade system (3) is LUB near Γ_1, one could assume that the x subsystem, $\dot{x} = f(x, y(t))$, is LUB near $\tilde{\Gamma}_1$ uniformly with respect to continuous signals $y(t)$ that asymptotically tend to zero. This idea is investigated further in Section 5. In the context of asymptotic stability of equilibria, sufficient conditions guaranteeing that assumption (vii) holds have been widely investigated in the literature. Sontag (1989), used a property of converging input bounded state (CIBS) stability. In the context of time-varying cascades, Panteley and Loria (1998, 2001) proved global uniform stability of equilibria using Lyapunov-type conditions and growth rate conditions. In terms of control design, several results addressed the global stabilisation problem for cascade systems, see Coron, Teel, and Praly (1995). Several of these results present growth rate conditions, see for instance Janković, Sepulchre, and Kokotović (1996), Mazenc and Praly (1996) and Saberi, Kokotović, and Sussmann (1990).

4. Hierarchical control design

Consider a locally Lipschitz control system

$$\dot{x} = f(x, u) \tag{4}$$

with state space a domain $\mathcal{X} \subset \mathbb{R}^n$, and let $\Gamma_1 \subset \Gamma_2 \subset \cdots \subset \Gamma_l$ be a nested sequence of closed subsets of $\mathcal{X}$ encoding hierarchical specifications spec 1, ..., spec l, where spec i is met when $x \in \Gamma_i$. The property that $\Gamma_i \subset \Gamma_{i+1}$ induces a hierarchy of control specifications, where spec i is met only if spec $i+1$ is met, and thus spec $i+1$ has higher priority than spec i. In the next section, we illustrate how such a hierarchical set of specifications arises in a coordination problem for two unicycles. Suppose that one can recursively design a locally Lipschitz feedback $\bar{u}(x)$ which, for each $i \in \{1, \dots, l\}$, asymptotically stabilises Γ_i relative to Γ_{i+1}. The questions we ask in this context are:

(a) Under what conditions does the feedback $\bar{u}(x)$ stabilise the set Γ_1 for (4)?
(b) Additionally, when does it simultaneously stabilise *all* sets $\Gamma_i, i = 1, \dots, l$, for (4)?

The answers to these question are contained in the next proposition. An important special case is the backstepping design technique, discussed in the next section. The problem outlined above bears a vague resemblance to the problem of uniting local and global controllers studied in Prieur (2001), in which the objective is to design a hybrid feedback merging two equilibrium stabilising controllers. However, the design in Prieur (2001) is not hierarchical.

Proposition 14. *Consider system (4), and assume that there exists a locally Lipschitz feedback $\bar{u}(x)$ making the sets $\Gamma_1 \subset \cdots \subset \Gamma_l$, positively invariant for the closed-loop system. Let $\Gamma_{l+1} := \mathcal{X}$, and consider the following conditions for the closed-loop system $\dot{x} = f(x, \bar{u}(x))$:*

(i) *For $i = 1, \dots, l$, Γ_i is asymptotically stable relative to Γ_{i+1} for the closed-loop system.*
(i)′ *For $i = 1, \dots, l$, Γ_i is globally asymptotically stable relative to Γ_{i+1} for the closed-loop system.*
(ii) *For some $k \in \{1, \dots, l\}$, Γ_k is either compact or it is unbounded and the closed-loop system is LUB near Γ_k.*
(iii) *All trajectories of the closed-loop system are bounded.*

Then, the following implications hold:

(a) (i) $\wedge$ *(Γ_1 is compact)* $\Longrightarrow$ *Γ_1 is asymptotically stable for the closed-loop system.*
(b) (i)′ $\wedge$ (iii) $\wedge$ *(Γ_1 is compact)* $\Longrightarrow$ *Γ_1 is globally asymptotically stable for the closed-loop system.*
(c) (i) $\wedge$ (ii) $\Longrightarrow$ *$\Gamma_1, \dots, \Gamma_k$ are asymptotically stable for the closed-loop system.*
(d) (i)′ $\wedge$ (ii) $\wedge$ (iii) $\Longrightarrow$ *$\Gamma_1, \dots, \Gamma_k$ are globally asymptotically stable for the closed-loop system.*

Proof. By assumption (i), Γ_1 is asymptotically stable relative to Γ_2. Moreover, the asymptotic stability of Γ_2 relative to Γ_3 implies that Γ_2 is locally stable and locally attractive near Γ_1 relative to Γ_3. By Theorem 6, if Γ_1 is compact, then it is also asymptotically stable relative to Γ_3 for the closed-loop system. Suppose, by induction, that Γ_1 is asymptotically stable relative to $\Gamma_j, j \in \{3, \dots, l\}$. By assumption (i), Γ_j is locally stable and locally attractive near Γ_1 relative to Γ_{j+1}. By Theorem 6, if Γ_1 is compact, then it is also asymptotically stable relative to Γ_{j+1}. Thus, by induction we conclude that Γ_1 is asymptotically stable for the closed-loop system, proving part (a). The proof of part (b) relies on an analogous argument. Now suppose that assumptions (i), and (ii) hold. If Γ_k is compact, then so too are $\Gamma_1, \dots, \Gamma_{k-1}$. Analogously, if the closed-loop system is LUB near Γ_k, then the same property holds near $\Gamma_1, \dots, \Gamma_{k-1}$. The application of Theorem 10 with an induction argument similar to the one above yields the claim in part (c). The proof of part (d) relies on an analogous argument. □

Remark 15. Assumption (i) in Proposition 14 can be replaced by the weaker requirement that Γ_i be asymptotically stable relative to Γ_{i+1} *provided* that the closed-loop system has no finite escape times. Similarly, assumption (ii) could be made conditional upon the property that the closed-loop system has no finite escape times. This minor relaxation has been implicitly used in proving Corollary 12, and will be used in the backstepping design of Section 5.

5. Reduction-based backstepping

One of the incarnations of the hierarchical control idea explored in the previous section is the backstepping control design technique (Kristić, Kanellakopoulos, & Kokotović, 1995). In this section we will explore the connection with backstepping in the simplest situation when disturbances and uncertainties are ignored. Consider the block-upper triangular control system

$$\begin{aligned}
\dot{x} &= f(x, z_1)\\
\dot{z}_1 &= f_1(x, z_1) + g_1(x, z_1) z_2\\
&\vdots\\
\dot{z}_i &= f_i(x, z^i) + g_i(x, z^i) z_{i+1}\\
&\vdots\\
\dot{z}_l &= f_l(x, z) + g_l(x, z) u
\end{aligned} \tag{5}$$

where $x \in \mathcal{X} \subset \mathbb{R}^n, z_1, \dots, z_l \in \mathbb{R}^m, z^i := \mathrm{col}(z_1, \dots, z_i), i = 1, \dots, l$, and $z := \mathrm{col}(z_1, \dots, z_l)$. All vector fields in (5) are assumed to be smooth. Moreover, the matrix-valued functions $g_i, i = 1, \dots, l$, are assumed to be uniformly bounded and invertible everywhere.

Assumption 16. There exist a smooth function $u_1 : \mathcal{X} \to \mathbb{R}^m$ and a closed set $\Gamma \subset \mathcal{X}$ such that Γ is (globally) asymptotically stable for $\dot{x} = f(x, u_1(x))$.

For notational consistency, we denote $u_1(x, z^0) := u_1(x)$. The control objective is to design a feedback $u(x, z)$ that globally asymptotically stabilises the set

$$\Gamma_0 = \{(x, z) : x \in \Gamma, z_i = u_i(x, z^{i-1}), i = 1, \dots, l\},$$

where $u_i(x, z^{i-1}), i = 2, \dots, l$, are smooth functions to be designed recursively using the backstepping philosophy. In classical backstepping, one begins with a Lyapunov function $V_0(x)$ for the subsystem $\dot{x} = f(x, u_1(x))$, and at step i one defines, recursively, $V_i(x, z^i) = V_{i-1}(x, z^{i-1}) + (1/2) e_i^\top e_i$, where $e_i = z_i - u_i(x, z^{i-1})$. Then, a function $u_{i+1}(x, z^i)$ is chosen to make $\dot{V}_i$ negative definite when $z_{i+1} = u_{i+1}(x, z^i)$. The recursion continues until, at step l, a feedback $u(x, z)$ is found. For large l, this procedure suffers from a well-known explosion in controller complexity. This is due in part to the fact that $u_{i+1}(x, z^i)$ contains the term $\partial V_{i-1}/\partial z_{i-1}$ whose time derivative is needed in the computation of $u_{i+2}(x, z^{i+1})$. As i grows larger, so does the complexity of the time derivative of $\partial V_{i-1}/\partial z_{i-1}$. We now present a reduction-based backstepping design that does not require the computation of the term $\partial V_{i-1}/\partial z_{i-1}$.

For $i = 2, \dots, l$, define

$$u_i(x, z^{i-1}) := g_{i-1}^{-1}(x, z^{i-1}) \Big[-f_{i-1}(x, z^{i-1}) + \dot{u}_{i-1}(x, z^{i-1}) - K_{i-1}(z_{i-1} - u_{i-1}(x, z^{i-2})) \Big], \tag{6}$$

where $K_{i-1} > 0$ and $\dot{u}_{i-1}(x, z^{i-1})$ is the Lie derivative of u_{i-1} along (5). Consider the feedback

$$u(x,z) = g_l^{-1}(x,z)\Big[-f_l(x,z) + \dot{u}_l(x, z^{l-1}) - K_l(z_l - u_l(x, z^{l-1}))\Big], \quad (7)$$

where $K_l > 0$. Letting $e_i := z_i - u_i(x, z^{i-1})$, we have

$$\begin{aligned} \dot{e}_i &= -K_i e_i + g_i(x, z^i) e_{i+1}, \quad i = 1, \ldots, l-1 \\ \dot{e}_l &= -K_l e_l. \end{aligned} \quad (8)$$

Now define closed sets Γ_i, $i = 1, \ldots, l$, as

$$\Gamma_i = \{(x,z) : z_i = u_i(x, z^{i-1}), \ldots, z_l = u_l(x, z^{l-1})\}. \quad (9)$$

We will denote the state space of (5) by Γ_{l+1}. Note that $\Gamma_0 \subset \Gamma_1 \subset \cdots \subset \Gamma_l$. The feedback $u(x,z)$ in (7) makes Γ_i invariant and asymptotically stable relative to Γ_{i+1}, $i = 1, \ldots, l$, provided that there are no escape times. Applying Proposition 14, we get the following result.

Proposition 17. *Consider the upper triangular system (5), and suppose there exist a smooth function $u_1(x)$ and a closed set $\Gamma \subset \mathcal{X}$ satisfying Assumption 16. Consider the following conditions:*

(i) *Γ is asymptotically stable for $\dot{x} = f(x, u_1(x))$.*
(i)′ *Γ is globally asymptotically stable for $\dot{x} = f(x, u_1(x))$.*
(ii) *For all $\bar{x} \in \Gamma$, there exist $\lambda, m > 0$ such that for all $x(0) \in B_\lambda(\bar{x})$, and for any continuous signal $e_1(t)$ with $e_1(t) \to 0$ and $e_1(t) \in B_\lambda(0)$ for all $t \geq 0$, the solution $x(t)$ of $\dot{x} = f(x, u_1(x) + e_1(t))$ satisfies $x(t) \in B_m(\bar{x})$ for all $t \geq 0$.*
(iii) *For any continuous signal $e_1(t)$ such that $e_1(t) \to 0$ as $t \to \infty$, and for any $x(0) \in \mathcal{X}$, the solution $x(t)$ of $\dot{x} = f(x, u_1(x)+e_1(t))$ exists for all $t \geq 0$, and it is bounded.*

Then, the following implications hold for system (5) with feedback $u(x,z)$ in (7):

(a) (i) $\wedge$ *(Γ is compact) $\Longrightarrow$ Γ_0 is asymptotically stable for the closed-loop system.*
(b) (i)′ $\wedge$ (iii) $\wedge$ *(Γ is compact) $\Longrightarrow$ Γ_0 is globally asymptotically stable for the closed-loop system.*
(c) (i) $\wedge$ (ii) $\Longrightarrow$ *Γ_0 is asymptotically stable for the closed-loop system.*
(d) (i)′ $\wedge$ (ii) $\wedge$ (iii) $\Longrightarrow$ *Γ_0 is globally asymptotically stable for the closed-loop system.*

Remark 18. Assumption (ii) in Proposition 17 can be rephrased as follows. The feedback $u_1(x)$ makes system $\dot{x} = f(x, u_1(x))$ LUB near Γ. Moreover, the LUB property persists under small vanishing perturbations of the control input. Assumption (iii) is the familiar *converging input bounded state* property of Sontag in Sontag (1989) applied to the system $\dot{x} = f(x, u_1(x) + e_1)$ with input e_1. If Γ is an equilibrium, then a sufficient condition for assumption (iii) to hold is that system $\dot{x} = f(x, u_1(x) + e_1)$ with input e_1 be input-to-state stable. More generally, one can rewrite the x subsystem as $\dot{x} = f(x, u_1(x)) + {e_1}^\top \tilde{f}(x, z_1)$, and replace assumption (iii) by the requirement that $\|\tilde{f}(x, z_1)\|$ satisfies a suitable growth condition. In Panteley and Loria (2001), the role that such growth conditions play on the boundedness of solutions is discussed in detail.

Proof. We claim that the equilibrium $(e_1, \ldots, e_l) = (0, \ldots, 0)$ is globally asymptotically stable for (8). Indeed, for $i = 1, \ldots, l-1$, the set $\{e_i = 0, \ldots, e_l = 0\}$ is globally asymptotically stable for (8) relative to the set $\{e_{i+1} = 0, \ldots, e_l = 0\}$. Moreover, given an arbitrary initial condition $(e_1(0), \ldots, e_l(0))$, by the second equation in (8) we have that $e_l(t)$ is bounded. By induction, assume that $e_{i+1}(t)$ is bounded. Then, the uniform boundedness of $g_i(x, z^i)$ and the boundedness of $e_{i+1}(t)$ imply that $e_i(t)$ is bounded. Hence, $e_1(t), \ldots, e_l(t)$ are bounded. Applying Proposition 14 to system (8) we get that $(e_1, \ldots, e_l) = (0, \ldots, 0)$ is globally asymptotically stable for (8). Moreover, by Eq. (8), for $i \in \{1, \ldots, l-1\}$, Γ_i is globally asymptotically stable relative to Γ_{i+1} provided there are no finite escape times in the x subsystem. To prove assertion (a), suppose that Γ is compact. Then, by continuity of the functions $u_i(x, z^{i-1})$, Γ_0 is compact as well. The asymptotic stability of $(e_1, \ldots, e_l) = (0, \ldots, 0)$ for system (8) implies that Γ_1 is locally stable near Γ_0. By assumption (i), Γ_0 is asymptotically stable relative to Γ_1. Therefore, by Theorem 6, Γ_0 is stable for the closed-loop system. Since Γ_0 is compact, its stability implies that all solutions of the closed-loop system originating near Γ_0 are bounded, and thus they have no finite escape times. This fact and assumption (i) imply that Γ_1 is locally attractive near Γ_0. Then, Theorem 10 implies that Γ_0 is asymptotically stable for the closed-loop system, proving part (a). If assumption (iii) holds, then all solutions of the closed-loop system are bounded, and Γ_1 is globally asymptotically stable. Thus, by Theorem 10 assumptions (i)′ and (iii) imply that Γ_0 is globally asymptotically stable, proving part (b). Now suppose that Γ, and hence Γ_0, is not compact and assumptions (i), and (ii) hold. For each $\bar{x} \in \Gamma$, let $\lambda(\bar{x})$ be as in assumption (ii). By asymptotic stability of $(e_1, \ldots, e_l) = (0, \ldots, 0)$ for (8), for all $\bar{x} \in \Gamma$ there exists $\delta(\bar{x}) > 0$ such that $\|e_i(0)\| < \delta(\bar{x})$, $i = 1, \ldots, l$, implies $\|e_i(t)\| < \lambda(\bar{x})$, $i = 1, \ldots, l$, for all $t \geq 0$. Define the neighbourhood of Γ_0,

$$\mathcal{N}(\Gamma_0) = \{(x,z) : x \in B_{\lambda(\bar{x})}(\bar{x}), \|z_i - u_i(x, z^{i-1})\| < \delta(\bar{x}), \; i = 1, \ldots, l, \; \bar{x} \in \Gamma\}.$$

By assumption (ii), all solutions of the closed-loop system originating in $\mathcal{N}(\Gamma_0)$ are bounded, and hence they have no finite escape times. The asymptotic stability of $(e_1, \ldots, e_l) = (0, \ldots, 0)$ for (8) and assumption (ii) imply that the closed-loop system is LUB near Γ_0. Thus, by Proposition 14, assumptions (i) and (ii) imply that Γ_0 is asymptotically stable for the closed-loop system, proving implication (c). The global asymptotic stability of $(e_1, \ldots, e_l) = (0, \ldots, 0)$ for (8) and assumption (iii) imply that all solutions of (5) with feedback $u(x,z)$ in (7) are bounded. Since the closed-loop system is LUB near Γ_0, by Proposition 14, assumptions (i)′, (ii), and (iii) imply that Γ_0 is globally asymptotically stable. □

Example 19. In this example we illustrate the difference between classical backstepping and reduction-based backstepping. Consider the control system with state $(x, z_1, z_2) \in \mathbb{R}^3$,

$$\begin{aligned} \dot{x} &= f(x) + z_1 \\ \dot{z}_1 &= z_2 \\ \dot{z}_2 &= u. \end{aligned}$$

If $z_1 = u_1(x) := -f(x) - x$, then Assumption 16 is satisfied with $\Gamma = \{0\}$. In classical backstepping, we would let $V_1(x) = (1/2)x^2$, then define $e_1 := z_1 - u_1(x)$, and let $V_2(x, z_1) = V_1(x) + (1/2)e_1^2$. Since

$$\dot{V}_2 = -x^2 + (\partial V_1)/(\partial x) e_1 + e_1(z_2 - \dot{u}_1),$$

letting $u_2(x, z^2) := -(\partial V_1)/(\partial x) + \dot{u}_1 - e_1$, and $e_2 := z_2 - u_2(x, z^2)$, we get

$$\dot{V}_2 = -x^2 - e_1^2 + e_1 e_2.$$

Finally, letting $V_3(x, z) = V_2(x, z_1) + (1/2)e_2^2$, we get

$$\dot{V}_3 = -x^2 - e_1^2 + e_1 e_2 + e_2(u - \dot{u}_2),$$

from which a globally asymptotically stabilising feedback is $u(x, z) = \dot{u}_2 - e_1 - e_2$. In terms of the problem data, namely $u_1(x)$

and the associated Lyapunov function $V_1(x)$, the feedback $u(x,z)$ is given by

$$u(x,z) = -\frac{d}{dt}[(\partial V_1)/(\partial x)] + \ddot{u}_1 - (z_2 - \dot{u}_1) - (z_1 - u_1) - \Big(z_2 - \big(-(\partial V_1)/(\partial x) + \dot{u}_1 - (z_1 - u_1)\big)\Big). \tag{10}$$

On the other hand, reduction-based backstepping proceeds as follows. From (6), $u_2(x,z_1) = \dot{u}_1 - e_1$, where $e_1 = z_1 - u_1(x)$. The final feedback is $u(x,z) = \dot{u}_2 - e_2$, where $e_2 = z_2 - u_2(x,z_1)$. In terms of the problem data, we have

$$u(x,z) = \ddot{u}_1 - (z_2 - \dot{u}_1) - (z_2 - (\dot{u}_1 - (z_1 - u_1))). \tag{11}$$

Clearly, feedback (11) is simpler than (10). This simplification results from the fact that the control design does not rely on the recursive definition of a Lyapunov function. Rather, at each step the design focuses exclusively on making Γ_i asymptotically stable relative to Γ_{i+1}. Another feature of our design is that it does not rely on the knowledge of a Lyapunov function for the x subsystem.

Example 20. Consider two dynamic unicycles modelled as rolling disks (see Bloch, Bailieul, Crouch, and Marsden (2003)),

$$\begin{aligned}
\dot{x}_1^i &= x_5^i \cos x_3^i \\
\dot{x}_2^i &= x_5^i \sin x_3^i \\
\dot{x}_3^i &= x_4^i \\
\dot{x}_4^i &= \frac{1}{J} w_2^i \\
\dot{x}_5^i &= \frac{R}{(I + mR^2)} w_1^i
\end{aligned} \tag{12}$$

for $i = 1, 2$, where (x_1^i, x_2^i) are the coordinates of the point of contact of the rolling disk with the plane, x_3^i is the heading angle of the unicycle, and x_5^i is the speed of the contact point. The state of unicycle i is $x^i := (x_1^i, x_2^i, x_3^i, x_4^i, x_5^i) \in \mathbb{R}^2 \times S^1 \times \mathbb{R}^2$. The collective state of the two unicycles is $\chi := \mathrm{col}(x^1, x^2)$. The scalars R and m are, respectively, the radius and the mass of each unicycle; I and J are, respectively, the moments of inertia of the unicycle about axes perpendicular to and in the plane of the unicycle, passing through the centre. Finally w_1^i and w_2^i are the torques about those axes. These are the control inputs.

We want to solve the following *coordination problem.* Make the unicycles follow, in the counter-clockwise direction, a common circle of radius $r > 0$ with unspecified centre. On the circle, the unicycles should travel with a constant speed $v > 0$, and keep a constant distance $d \in (0, 2r)$ from each other. In El-Hawwary and Maggiore (2011) we solved the kinematic version of this problem. In this example, we use reduction-based backstepping to generate a solution for the dynamic unicycles in (12). Let $c^i(x^i) = \mathrm{col}(x_1^i - r\sin x_3^i, x_2^i + r\cos x_3^i)$. The point $c^i(x^i)$ is the centre of the circle that unicycle i would follow if the magnitude of its linear velocity were $x_5^i = v$ and its angular velocity where $x_4^i = v/r$. We formulate three hierarchical control specifications (recall that spec $i+1$ has higher priority than spec i).

spec 3. Stabilise a desired "kinematic behaviour", i.e., stabilise $\Gamma_3 = \{\chi_d : x_4^i = u_2^i(\chi), x_5^i = u_1^i(\chi), i = 1, 2\}$, where $u_1^i(\chi), u_2^i(\chi)$ are smooth functions defined later. On Γ_3, the dynamic unicycles become purely kinematic, with new inputs u_1^i, u_2^i.

spec 2. Considering the kinematic motion on Γ_3, make the unicycles follow a common circle, i.e., stabilise $\Gamma_2 = \{\chi_d \in \Gamma_3 : c^1(x^1) = c^2(x^2)\}$.

spec 1. On Γ_2, make the unicycles maintain a distance d from each other. This corresponds to stabilising $\Gamma_1 = \{\chi_d \in \Gamma_2 : |x_3^1 - x_3^2| = 2\sin^{-1}(d/2r) \bmod 2\pi\}$. Note that Γ_1 is not compact, because there is no restriction on the centre of the common circle the unicycles converge to.

Thus, we have a hierarchical control design problem, involving the simultaneous stabilisation of $\Gamma_1 \subset \Gamma_2 \subset \Gamma_3$. Consider the functions

$$\begin{aligned}
u_1^1(\chi) &= v + \varphi_1\left((\cos(x_3^1 - x_3^2) - \cos\alpha)\sin(x_3^1 - x_3^2)\right) \\
u_2^1(\chi) &= \frac{u_1^1}{r} - Kh_1(\chi) \\
u_1^2(\chi) &= v - \varphi_1\left((\cos(x_3^1 - x_3^2) - \cos\alpha)\sin(x_3^1 - x_3^2)\right) \\
u_2^2(\chi) &= \frac{u_1^2}{r} - Kh_2(\chi),
\end{aligned} \tag{13}$$

where $v > 0$ is a design parameter, $\varphi_1 : \mathbb{R} \to \mathbb{R}$ is a smooth odd function which is strictly increasing and such that $\sup_{\mathbb{R}} |\varphi_1(\cdot)| < v$, $\alpha = 2\sin^{-1}(d/2r)$, and $h_i(\chi)$, $i = 1, 2$, are given by

$$h_i(\chi) = (-1)^i r (c^1(x^1) - c^2(x^2))^\top \begin{bmatrix} \cos x_3^i \\ \sin x_3^i \end{bmatrix},$$

In El-Hawwary and Maggiore (2011), we proved that there exists $K^\star > 0$ such that for all $K \in (0, K^\star)$, the functions in (13) make Γ_2 globally asymptotically stable relative to Γ_3, and Γ_1 asymptotically stable relative to Γ_2. In order to extend the kinematic result of El-Hawwary and Maggiore (2011), we should simply design feedbacks that asymptotically stabilise Γ_3. The work in El-Hawwary and Maggiore (2011) does not provide a Lyapunov function, so one cannot use classical backstepping to extend the solution given above to dynamic unicycles. On the other hand, the reduction-based backstepping procedure of Section 5 is readily applicable, and it provides the feedback

$$\begin{aligned}
w_1^i &= \frac{I + mR^2}{R}\left(\frac{\partial u_1^i(\chi)}{\partial \chi}\dot{\chi} - K_1(x_5^i - u_1^i(\chi))\right) \\
w_2^i &= J\left(\frac{\partial u_2^i(\chi)}{\partial \chi}\dot{\chi} - K_2(x_4^i - u_2^i(\chi))\right)
\end{aligned} \tag{14}$$

for $i = 1, 2$, where $K_1, K_2 > 0$ are design constants.

Proposition 21. *Consider system (12) with feedback (14), with $u_j^i(\chi)$, $i, j = 1, 2$, defined in (13), and $K, K_1, K_2 > 0$. Then, there exists $K^\star > 0$ such that for all $K \in (0, K^\star)$, the set Γ_2 is globally asymptotically stable, and Γ_1 is asymptotically stable for the closed-loop system. Thus, for any initial condition the unicycles converge to a common circle of radius r, and for suitable initial conditions the distance between them converges to d.*

Proof. To show that the feedback (14) asymptotically stabilises Γ_1, hence solving the coordination problem, we apply Proposition 17 setting $x = (x_1^1, x_2^1, x_3^1, x_1^2, x_2^2, x_3^2)$, $z_1 = (x_5^1, x_4^1, x_5^2, x_4^2)$, $u_1(x) = (u_1^1(\chi), u_2^1(\chi), u_1^2(\chi), u_2^2(\chi))$, and $\Gamma = \{x : (x^1, x^2) \in \Gamma_1\}$. To this end, we need to show that assumption (ii) in Proposition 17 holds. This is a straightforward adaptation of the analysis presented in El-Hawwary and Maggiore (2011), and it is briefly sketched. The evolution of the centres c^i under the closed-loop system is governed by

$$\begin{bmatrix} \dot{c}^1 \\ \dot{c}^2 \end{bmatrix} = -Kr^2 \begin{bmatrix} S(x_3^1) & -S(x_3^1) \\ -S(x_3^2) & S(x_3^2) \end{bmatrix} \begin{bmatrix} c^1 \\ c^2 \end{bmatrix} + \begin{bmatrix} T(x_3^1)e^1 \\ T(x_3^2)e^2 \end{bmatrix}, \tag{15}$$

where $S(\cdot) = [\cos(\cdot)\ \sin(\cdot)]^\top[\cos(\cdot)\ \sin(\cdot)]$, $T(\cdot) = r[\cos(\cdot)\ \sin(\cdot)]^\top[1\ \ -r]$, and $e^i = [x_5^i - u_1^i(\chi)\ \ x_4^i - u_2^i(\chi)]^\top$. As in Section 5, we let $e_1 = z_1 - u_1(x) = [(e^1)^\top\ \ (e^2)^\top]^\top$. Feedback (14) gives $\dot{e}^i = -K_i e^i$, so (15) can be viewed as a system with asymptotically vanishing input $e_1(t)$. Letting $\tilde{c} = c^1 - c^2$, we have

$$\dot{\tilde{c}} = -Kr^2(S(x_3^1) + S(x_3^2))\tilde{c} + [T(x_3^1) - T(x_3^2)]e_1.$$

Viewing $x_3^i(t)$ as exogenous signals, the above can be viewed as a linear time-varying system with exponentially vanishing input. We now use averaging theory to analyse this system. The averaged system is

$$\dot{\tilde{c}}_{\text{avg}} = -Kr^2(\bar{S}_1 + \bar{S}_2)\tilde{c}, \tag{16}$$

where $\bar{S}_i = \lim_{t\to\infty}(1/t)\int_0^t S(x_3^i(\tau))d\tau$. The signal $e_1(t)$ does not affect the averaged system because it vanishes asymptotically and $T(\cdot)$ is uniformly bounded, and therefore $\lim_{t\to\infty}(1/t)\int_0^t T(x_3^1(\tau)) e^1(\tau) - T(x_3^2(\tau))e^2(\tau)d\tau = 0$. In El-Hawwary and Maggiore (2011) it was shown that the matrix $-Kr^2(\bar{S}_1 + \bar{S}_2)$ is Hurwitz, so that the equilibrium $c_{\text{avg}} = 0$ is globally exponentially stable for the averaged system. By the averaging theorem (Khalil, 2002), there exists $K^\star > 0$ such that for all $K \in (0, K^\star)$ the origin $\tilde{c} = 0$ of (16) is globally exponentially stable. This fact implies that $\|c^i(x^i(t))\| \le M_1\|c^1(x^1(0)) - c^2(x^2(0))\|$ for some $M_1 > 0$. Moreover $\|e^i(t)\| \le M_2\|x_4^i(0) - u_2^i(\chi(0))\|$ for some $M_2 > 0$. The two inequalities above imply that all solutions of the closed-loop system are bounded, and the bound is uniform over neighbourhoods of Γ_2, so that assumption (ii) of Proposition 17 holds near the set $\{(x,z) : (x^1,x^2) \in \Gamma_2\}$, and hence also near $\Gamma = \{(x,z) : (x^1,x^2) \in \Gamma_1\}$. This proves the asymptotic stability of Γ_1. The fact that assumption (ii) holds near $\{(x,z) : (x^1,x^2) \in \Gamma_2\}$ allows us to apply Proposition 17 to prove that Γ_2 is globally asymptotically stable. Letting x, z_1, and $u_1(x)$ be as above, and setting $\Gamma = \{(x,z) : (x^1,x^2) \in \Gamma_2\}$, the global asymptotic stability of Γ_2 relative to Γ_3 implies that assumption (i)′ of Proposition 17 holds. We have already shown that assumption (ii) holds for $x(0)$ near Γ_2. Finally, assumption (iii) holds because the averaging result above shows that for any asymptotically vanishing signal $e_1(t)$, all closed-loop trajectories are bounded. □

6. Conclusion

We presented novel reduction theorems for stability, attractivity, and asymptotic stability of closed invariant sets. We investigated the implications of these theorems on the stability of invariant sets for cascade-connected systems, on a hierarchical control problem, and on backstepping control design.

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Appendix A. Proofs of reduction theorems

A.1. Proof of Theorem 6

The proof of the theorem relies on the next result.

Lemma 22. *Let $\Gamma_1 \subset \mathcal{X}$ be a closed set which is a positively invariant set for Σ. If Γ_1 is unstable, then there exist $\varepsilon > 0$, a bounded sequence $\{x_i\} \subset \mathcal{X}$, and a sequence $\{t_i\} \subset \mathbb{R}^+$, such that $x_i \to \bar{x} \in \Gamma_1$, and $\|\phi(t_i, x_i)\|_{\Gamma_1} = \varepsilon$ for all i.*

Proof. The instability of Γ_1 implies that there exists $\varepsilon > 0$, a sequence $\{x_i\} \subset \mathcal{X}$, and a sequence $\{t_i\} \subset \mathbb{R}^+$, such that $\|x_i\|_{\Gamma_1} \to 0$, and $\|\phi(t_i, x_i)\|_{\Gamma_1} = \varepsilon$. If we show that $\{x_i\}$ above can be chosen to be bounded, then, without loss of generality, there exists $\bar{x} \in \Gamma_1$ such that $x_i \to \bar{x}$ and we are done. Let S be defined as follows

$$S = \{x \in B_\varepsilon(\Gamma_1) : (\exists t > 0)\ \|\phi(x,t)\| = \varepsilon\}.$$

The instability of Γ_1 implies that S is not empty. Moreover, since Γ_1 is positively invariant, $S \cap \Gamma_1 = \emptyset$. Suppose, by way of contradiction, that there does not exist a bounded sequence $\{x_i\}$ and a sequence $\{t_i\}$ such that $\|x_i\|_{\Gamma_1} \to 0$ and $\|\phi(t_i, x_i)\|_{\Gamma_1} = \varepsilon$. This implies that, for any $x \in \Gamma_1$, there exists $\delta(x) > 0$ such that $B_{\delta(x)}(x) \cap S = \emptyset$. For, if this were not true, then there would exist a bounded sequence $\{x_i\} \subset S$, with $x_i \to \Gamma_1$ contradicting the assumption we have made. Let $U = \bigcup_{x\in\Gamma_1} B_{\delta(x)}(x)$. By construction, U is a neighbourhood of Γ_1 such that $U \cap S = \emptyset$. In other words, for all $x \in U$, there does *not* exist $t > 0$ such that $\|\phi(t,x)\|_{\Gamma_1} = \varepsilon$, contradicting the assumption that Γ_1 is unstable. □

Proof of Theorem 6. By way of contradiction, suppose that Γ_1 is unstable. Then, by Lemma 22, there exist $\varepsilon > 0$, a bounded sequence $\{x_i\} \subset \mathcal{X}$, with $x_i \to \bar{x} \in \Gamma_1$, and a sequence $\{t_i\} \subset \mathbb{R}^+$, such that

$$\|\phi(t_i, x_i)\|_{\Gamma_1} = \varepsilon, \quad \text{and} \quad \phi([0,t_i), x_i) \in B_\varepsilon(\Gamma_1).$$

By local uniform boundedness of Σ near Γ_1, there exist $\lambda, m > 0$ such that $\phi(\mathbb{R}^+, B_\lambda(\bar{x})) \subset B_m(\bar{x})$. We can assume $\{x_i\} \subset B_\lambda(\bar{x})$. Take a decreasing sequence $\{\varepsilon_i\} \subset \mathbb{R}^+$, $\varepsilon_i \to 0$. By assumption (ii), Γ_2 is locally stable near Γ_1. Using the definition of local stability with $c = m$ and $\varepsilon = \varepsilon_i$, there exists $\delta_i > 0$ such that for all $x_0 \in B_{\delta_i}(\bar{x})$ and all $t > 0$, if $\phi([0,t], x_0) \subset B_m(\bar{x})$, then $\phi([0,t], x_0) \subset B_{\varepsilon_i}(\Gamma_2)$. By taking $\delta_i \le \lambda$ we have

$$(\forall x_0 \in B_{\delta_i}(\bar{x})) \quad \phi(\mathbb{R}^+, x_0) \subset B_{\varepsilon_i}(\Gamma_2).$$

By passing, if needed, to a subsequence we can assume without loss of generality that, for all i, $x_i \in B_{\delta_i}(\bar{x})$ so that

$$\limsup_{i\to\infty} d(\phi([0,t_i], x_k), \Gamma_2) = 0.$$

Using assumptions (i) and (iii) (if Γ_1 is unbounded), by Lemma 2.5 in El-Hawwary and Maggiore (2009), it follows that Γ_1 is a uniform semi-attractor relative to $\mathcal{O}$. Therefore,

$$\begin{aligned}&(\forall x \in \Gamma_1)(\exists \mu > 0)(\forall \varepsilon' > 0)(\exists T > 0) \quad \text{s.t.}\\ &\phi([T,+\infty), B_\mu(x) \cap \Gamma_2) \subset B_{\varepsilon'}(\Gamma_1).\end{aligned} \tag{A.1}$$

Consider the set $\Gamma_1' = \Gamma_1 \cap \text{cl}(B_{2m}(\bar{x}))$. Since Γ_1' is compact, then the infimum of $\mu(x)$, in (A.1), for all $x \in \Gamma_1'$ exists and is greater than zero. Thus we infer the existence of $\mu > 0$ such that

$$\begin{aligned}&(\forall x \in \Gamma_1')(\forall \varepsilon' > 0)(\exists T > 0)\\ &\phi([T,+\infty), B_\mu(x) \cap \Gamma_2) \subset B_{\varepsilon'}(\Gamma_1).\end{aligned} \tag{A.2}$$

By reducing, if necessary, ε in the instability definition, we may assume that[3] $\varepsilon < \mu$. Now choose $\varepsilon' < \varepsilon/2$. Using again a compactness argument, by (A.2) one infers the following condition

$$(\exists T > 0)(\forall x \in \Gamma_1')\phi([T,+\infty), B_\mu(x) \cap \Gamma_2) \subset B_{\varepsilon'}(\Gamma_1). \tag{A.3}$$

We claim that $B_\mu(\Gamma_1) \cap B_m(\bar{x}) \subset B_\mu(\Gamma_1')$. For, if $\mu \ge m$, then

$$B_\mu(\Gamma_1) \cap B_m(\bar{x}) = B_m(\bar{x}) \subset B_\mu(\bar{x}) \subset B_\mu\Big(\Gamma_1 \cap \text{cl}(B_{2m}(\bar{x}))\Big).$$

[3] In the contradiction assumption that Γ_1 is unstable we employ $\varepsilon > 0$ as in Lemma 22. By instability of Γ_1, any $\epsilon \in (0, \varepsilon]$ works in place of ε. Therefore, it is always possible to find $\varepsilon < \mu$.

If $\mu < m$, then $x \in B_\mu(\Gamma_1) \cap B_m(\bar{x})$ if and only if $\|x\|_{\Gamma_1} < \mu$ and $\|x - \bar{x}\| < m$; in particular, there exists $y \in \Gamma_1$ such that $\|x - y\| < \mu$. Since $\|y - \bar{x}\| \le \|x - y\| + \|x - \bar{x}\| \le \mu + m < 2m$, we have that $y \in \Gamma_1 \cap \mathrm{cl}(B_{2m}(\bar{x}))$, and thus $x \in B_\mu(\Gamma_1 \cap \mathrm{cl}(B_{2m}(\bar{x})))$.

Using (A.3) and the claim we have just proved we obtain

$$(\forall x \in B_\mu(\Gamma_1) \cap B_m(\bar{x}) \cap \Gamma_2)\ \phi([T, +\infty), x) \subset B_{\varepsilon'}(\Gamma_1). \tag{A.4}$$

Now, since $\{t_k\}$ is unbounded there exists $K_1 > 0$ such that $t_k > T$ for all $k \ge K_1$. Since $\phi([0, t_k), x_k) \subset B_\varepsilon(\Gamma_1)$ we have $\phi(t_k - T, x_k) \in B_\varepsilon(\Gamma_1)$ for all $k \ge K_1$. Let

$$y_k = \phi(t_k, x_k), \quad \text{and} \quad z_k = \phi(t_k - T, x_k).$$

Thus, $y_k = \phi(T, z_k)$, $\|y_k\|_{\Gamma_1} = \varepsilon$ and $z_k \in B_\varepsilon(\Gamma_1)$. By local uniform boundedness, it also holds that $z_k \in B_m(\bar{x})$. Pick $\delta \in (0, \mu - \varepsilon)$. Since $z_k \in \phi([0, t_k), x_k) \subset B_m(\bar{x})$, and since

$$\limsup_{k \to \infty} d(\phi([0, t_k], x_k), \Gamma_2) = 0,$$

then there exists $K_2 \ge K_1$ such that, for all $k \ge K_2$, there exists $z_k' \in B_m(\bar{x}) \cap \Gamma_2$ such that $\|z_k - z_k'\| < \delta$. Since $z_k \in B_\varepsilon(\Gamma_1)$, then

$$z_k' \in B_{\varepsilon+\delta}(\Gamma_1) \cap B_m(\bar{x}) \cap \mathcal{O} \subset B_\mu(\Gamma_1) \cap B_m(\bar{x}) \cap \Gamma_2$$

and, by (A.4), $\phi([T, +\infty), z_k') \subset B_{\varepsilon'}(\Gamma_1)$. By continuous dependence on initial conditions, δ can be chosen small enough that

$$(\forall x \in B_m(\bar{x}))(\forall x_0 \in B_\delta(x))\ \|\phi(T, x) - \phi(T, x_0)\| < \varepsilon/2.$$

We have $z_k \in B_m(\bar{x})$ and $\|z_k - z_k'\| < \delta$, hence $\|\phi(T, z_k) - \phi(T, z_k')\| < \varepsilon/2$, which implies

$$y_k \in B_{\varepsilon/2}(\phi(T, z_k')) \subset B_{\varepsilon/2+\varepsilon'}(\Gamma_1) \subset B_\varepsilon(\Gamma_1),$$

contradicting $\|y_k\|_{\Gamma_1} = \varepsilon$. □

A.2. Proof of Theorem 8

Part of the proof was inspired by the stability results using positive semidefinite Lyapunov functions presented in Iggidr et al. (1996) and by the proof of Lemma 1 in de Leenheer and Aeyels (2002).

By assumption (ii), there exists a neighbourhood $\mathcal{N}_1(\Gamma_1)$ of Γ_1 such that all trajectories originating there asymptotically approach Γ_2 in positive time. Let $\mathcal{N}_2(\Gamma_1)$ be the neighbourhood in assumption (iii), and define $\mathcal{N}_3(\Gamma_1) = \mathcal{N}_1(\Gamma_1) \cap \mathcal{N}_2(\Gamma_1)$. Clearly, $\mathcal{N}_3(\Gamma_1)$ is a neighbourhood of Γ_1. By construction, for all $x_0 \in \mathcal{N}_3(\Gamma_1)$, the solution is bounded and approaches Γ_2. Therefore, the positive limit set $L^+(x_0)$ is non-empty, compact, invariant, and $L^+(x_0) \subset \Gamma_2$. Moreover, by definition of positive limit set, and by assumption (iii) we have the following inclusion,

$$\begin{aligned} L^+(x_0) &\subset \mathrm{cl}(\phi(\mathbb{R}^+, x_0)) \cap \Gamma_2 \\ &\subset \{\text{domain of attraction of } \Gamma_1 \text{ rel. to } \Gamma_2\}. \end{aligned} \tag{A.5}$$

We need to show that $L^+(x_0) \subset \Gamma_1$. Assume, by way of contradiction, that there exists $\omega \in L^+(x_0)$ and $\omega \notin \Gamma_1$. By the invariance of $L^+(x_0)$, $\phi(\mathbb{R}, \omega) \subset L^+(x_0)$, and therefore $L^-(\omega) \subset L^+(x_0)$. By the inclusion in (A.5), all trajectories in $L^-(\omega)$ asymptotically approach Γ_1 in positive time, and so since $L^-(\omega)$ is closed, $L^-(\omega) \cap \Gamma_1 \ne \varnothing$. Let $p \in L^-(\omega) \cap \Gamma_1$. Pick $\varepsilon > 0$ such that $\|\omega\|_{\Gamma_1} > \varepsilon$. By the stability of Γ_1 relative to Γ_2, there exists a neighbourhood $\mathcal{N}_4(\Gamma_1)$ of Γ_1 such that $\phi(\mathbb{R}^+, \mathcal{N}_4(\Gamma_1) \cap \Gamma_2) \subset B_\varepsilon(\Gamma_1)$. Since $p \in L^-(\omega)$, there exists a sequence $\{t_k\} \subset \mathbb{R}^+$, with $t_k \to +\infty$, such that $\phi(-t_k, \omega) \to p$ at $k \to +\infty$. Since $p \in \Gamma_1$, we can pick $k^\star$ large enough that $\phi(-t_{k^\star}, \omega) \in \mathcal{N}_4(\Gamma_1)$. Let $T = t_{k^\star}$ and $z = \phi(-t_{k^\star}, \omega)$. We have thus obtained that $z \in \mathcal{N}_4(\Gamma_1)$, but $\phi(T, z) = \omega$ is not in $B_\varepsilon(\Gamma_1)$. This contradicts the stability of Γ_1, and therefore, for all $x_0 \in \mathcal{N}_3(\Gamma_1)$, $L^+(x_0) \subset \Gamma_1$, proving that Γ_1 is an attractor for Σ. To prove global attractivity of Γ_1 it is sufficient to notice that by assumptions (ii)′ and (iii)′, for all $x_0 \in \mathcal{X}$, $L^+(x_0)$ is non-empty and $L^+(x_0) \subset \Gamma_2$. On Γ_2, by assumption (i)′ all trajectories approach Γ_1, so by the contradiction argument above we conclude that $L^+(x_0) \subset \Gamma_1$. □

Appendix B. Proof of Corollary 12

The attractivity part of the Corollary follows directly from Theorem 8 and Remark 9. Now let $\Gamma_2 = \{(x, y) : y = 0\}$. Assumption (iv) implies that Γ_1 is (globally) asymptotically stable relative to Γ_2. In light of Theorem 10, to prove asymptotic stability of Γ_1 we need to show that Γ_2 is locally stable and locally attractive near Γ_1. These properties are implied by the asymptotic stability of $y = 0$ for $\dot{y} = g(y)$ *provided* that system (3) has no finite escape times in a neighbourhood of Γ_1. If Γ_1 is unbounded, the LUB assumption (vi) rules out finite escape times near Γ_1. Now suppose that Γ_1 is compact. The stability of $y = 0$ for $\dot{y} = g(y)$ implies that Γ_2 is locally stable near Γ_1. Thus, by Theorem 6, assumptions (iv) and (v) imply that Γ_1 is stable. The compactness of Γ_1 and its stability imply that solutions in a neighbourhood of Γ_1 have no finite escape times. This concludes the proof of asymptotic stability of Γ_1. For the global version, assumption (vii) implies that the cascade system (3) has no finite escape times, and so assumption (v) guarantees that Γ_2 is globally asymptotically stable. Global asymptotic stability of Γ_1 then follows directly from Theorem 10. □

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