The Control of Nonlinear Nonminimum Phase Systems

- VTOL tracking problem

Kwangsoo Kim *
Coordinate Systems Laboratory
Electrical and Computer Engineering
University of Illinois at Urbana-Champaign

December 8, 2000

Abstract

This report deals with the nonlinear nonminimum phase system, espeically with an application of VTOL (Vertical Take-Off and Landing) aircraft. Some element concepts in nonlinear geometric control theory are illustrated for the definition of the nonminimum phase in nonlinear systems. The nonminimum phase characteristics of a planar VTOL aircraft is analyzed with its zero dynamics.

1 Introduction

A few decades ago, there has been a lot of researches on the nonlinear control feedback theory. One of the motivation of this research on nonlinear control of nonlinear systems are to give extensions from the previously developed classical linear control theory. Byrnes and Isidori who have been taking part in research on this problem and achieved some remarkable results [9, 3, 4] suggested such an idea [2].

> .... we describe a research program which we have been pursuing for the past 3 years, whose goal is the development of heuristics for nonlinear control, similar in scope and spirit to classical control, to be used in the analysis and design of nonlinear feedback

* e-mail:kskim@control.csl.uiuc.edu

> control systems. Thus, for example, we wish to develop nonlinear generalizations of some of the concepts familiar from frequency domain theory and to use these, in much the same way as classical control methods, to design and analyze nonlinear versions of PD control, lead-lag compensation, etc, to "shape the response" of nonlinear systems.

One of results from these generalization efforts from linear systems to nonlinear systems is the concept of "nonminimum phase" system. The nonminimum phase characteristics imposes some design difficulties. First, it is impossible to inverse the system due to the instability of the inversed system. Second, in classical control theory, it is well known that the nonminimum phase zeros imposes extra phase lag so that the design of sensitivity functions are more limited [5]. The representative response of a nonminimum phase system is that it response to the reverse direction when the step input is applied. Such a response is also appeared in nonlinear system and this makes the tracking problem of nonminimum phase system difficult.

In this report, I would like to study how to treat nonminimum phase systems with such a structural limitation. The necessary concepts for defining the nonminimum phase system in nonlinear case are briefly presented in section 2. PVTOL aircraft system is well know as a representative example of nonlinear nonminimum phase system. So PVTOL is chosen as an illustrative example in this report and analyzed with the elementary knowledge of section 2 in section 4.

2 Elementary Nonlinear Geometric Control Theory

The definition of nonminimum phase nonlinear systems needs a few concepts of nonlinear geometric control theory. So, in this section, a few basic nonlinear geometric control theory are presented for the later reference, especially for the definition of the nonminimum phase systems in nonlinear case in section 3. The notation and mathematical preliminaries such as Lie derivative and diffeomorphism are referred to the chapter 1 of Isidori's book [8] or chapter 12 of Khalil's text [10] and not shown here. Also, for clarity of exposition and convenience, the presentation in this section is restricted to single-input-single-output systems. So the system form to be considered in this section is as follows:

$$\dot{x} = f(x) + g(x)u \tag{1}$$

$$y = h(x)$$

where the state $x \in \mathbf{R}^n$ and f, g, and h are assumed to be smooth mappings and $f, g, h \in \mathcal{C}^\infty$. The content of this section is mainly adopted from chapter 4 of Isidori's [8] and Appendix A in [1]

2.1 Relative degree

The system described by (1) is said to have **relative degree** r at a point x° if

(i) $L_g L_f^k h(x) = 0$ for all x in a neighborhood of x° and all $k < r - 1$

(ii) $L_g L_f^{r-1} h(x^\circ) \neq 0$.

where $L_g L_f^k h(x)$ denotes the Lie derivative of $L_f^k h(x)$ with respect to g.

It can be known that this definition of nonlinear systems is well compatible of that of linear systems when considering a linear system such as

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx. \end{aligned}$$

In this case, since $f(x) = Ax, g(x) = B, h(x) = Cx$, the integer r is characterized by the conditions

$$\begin{aligned} CA^k B &= 0 \qquad \text{for all } k < r - 1 \\ CA^{r-1} B &\neq 0 \end{aligned}$$

It is well-known that the integer satisfying these conditions is exactly equal to the difference between the degree of the denominator polynomial and the degree of the numerator polynomial of the transfer function

$$H(s) = C(sI - A)^{-1} B$$

of the system.

For the interpretation of the notion of relative degree, assume the system at some time t° is in the state $x(t^\circ) = x^\circ$. The value of the $y(t)$ and its derivative, $y^{(k)}(t)$ for $k = 1, 2, \cdots$, at $t = t^\circ$, can be calculated as the following manner:

$$\begin{aligned} y(t^\circ) &= h(x^\circ) \\ y^{(1)}(t) &= \frac{\partial h}{\partial x}\frac{dx}{dt} = \frac{\partial h}{\partial x}\left(f(x(t)) + g(x(t))u(t)\right) \\ &= L_f h(x(t)) + L_g h(x(t)) u(t) \end{aligned}$$

If the relative degree r is larger than 1, for all t such that $x(t)$ is near x°, we have $L_g h(x(t)) = 0$ and therefore

$$y^{(1)}(t) = L_f h(x(t)).$$

In the similar way,

$$y^{(2)}(t) = L_f^2 h(x(t))$$

if the relative degree is larger than 2. Continuing in this way, one concludes that

$$\begin{aligned} y^{(k)}(t) &= L_f^k h(x(t)) \qquad \text{for all} k < r \text{and all } t \text{ near } t^\circ \\ y^{(r)}(t^\circ) &= L_f^r h(x^\circ) + L_g L_f^{r-1} h(x^\circ) u(t^\circ) \end{aligned}$$

Thus, the relative degree r of a system is equal to the number of times the output must be differentiated to have the input appear explicitly.

2.2 Local Coordinates Transformation

Given a system denoted by (1), let a coordinate transformation be any diffeomorphism denoted by

$$z = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} = \Phi(x) = \begin{bmatrix} \phi_1(x) \\ \phi_2(x) \\ \vdots \\ \phi_n(x) \end{bmatrix} \tag{2}$$

If the relative degree of the system (1) is r, let the first r functions of a coordinate transformation (2) be defined as

$$\begin{bmatrix} \phi_1(x) \\ \phi_2(x) \\ \vdots \\ \phi_r(x) \end{bmatrix} \triangleq \begin{bmatrix} h(x) \\ L_f h(x) \\ \vdots \\ L_f^{r-1} h(x) \end{bmatrix}$$

And it is always possible to find $n - r$ more functions $\phi_{r+1}, \cdots, \phi_n(x)$ such that the mapping $\Phi(x)$ has a jacobian matrix which is nonsingular at x° and therefore qualifies as a **local coordinates transformation** in a neighborhood of x°. Moreover, it is well known that it is always possible to choose $\phi_{r+1}, \cdots, \phi_n(x)$ in such a way that

$$L_g \phi_i(x) = 0 \qquad r + 1 \le i \le n \tag{3}$$

and all x around x°. In summary, the functions of the coordinate transformation (2) are given by

(i) $\phi_i(x) \triangleq L_f^{i-1}h(x) \qquad i = 1, 2, \cdots, r$

(ii) $L_g\phi_j(x) = 0 \qquad j = r+1, \cdots, n.$

2.3 Normal Forms

The description of the system in the new coordinates $z_i = \phi_i(x), 1 \leq i \leq n$, is found with the coordinate transformation given by (2) via the following calculations:

$$\begin{aligned}
\frac{dz_1}{dt} &= \frac{\partial \phi_1}{\partial x}\frac{dx}{dt} = L_f h(x(t)) = \phi_2(x(t)) = z_2(t) \\
&\vdots \\
\frac{dz_{r-1}}{dt} &= \frac{\partial \phi_{r-1}}{\partial x}\frac{dx}{dt} = L_f^{r-1}h(x(t)) = \phi_r(x(t)) = z_r(t) \\
\frac{dz_r}{dt} &= L_f^r h(x(t)) + L_g L_f^{r-1} h(x(t))u(t).
\end{aligned}$$

In the last equation, $x(t)$ must be replaced with its expression as a function of $z(t)$, i.e. $x(t) = \Phi^{-1}(z(t))$. Thus, setting

$$\begin{aligned}
a(z) &= L_g L_f^{r-1} h(\Phi^{-1}(z)) \\
b(z) &= L_f^r h(\Phi^{-1}(z))
\end{aligned}$$

The remaining $n - r$ coordinates, due to the chosen condition (3), are calculated as follows:

$$\frac{dz_i}{dt} = L_f\phi(x(t)) + L_g\phi_i(x(t))u(t) = L_f\phi_i(x(t)).$$

Setting

$$q_i(z) = L_f\phi_i(\Phi^{-1}(z)) \qquad i = r+1, \cdots, n$$

the preceding equation can be rewritten as

$$\frac{dz_i}{dt} = q_i(z(t)).$$

Thus, using the coordinate transformations defined by (2) the nonlinear system will be described in the new coordinates as follows:

$$\begin{aligned}
\dot{z}_1 &= z_2 \\
\dot{z}_2 &= z_3 \\
&\vdots
\end{aligned}$$

$$\begin{aligned}
\dot{z}_{r-1} &= z_r \\
\dot{z}_r &= b(z) + a(z)u \\
\dot{z}_{r+1} &= q_{r+1}(z) \\
&\vdots \\
\dot{z}_n &= q_n(z)
\end{aligned} \tag{4}$$

with the output function in the new coordinate system given by

$$y = z_1.$$

The system given by (4) is said to be in **normal form**.

2.4 Zero dynamics

Partition the state vector in normal form as

$$\xi = \begin{bmatrix} z_1 \\ \vdots \\ z_r \end{bmatrix}, \qquad \eta = \begin{bmatrix} z_{r+1} \\ \vdots \\ z_n \end{bmatrix}$$

Then the normal form equations can be expressed as

$$\begin{aligned}
\dot{z}_1 &= z_2 \\
\dot{z}_2 &= z_3 \\
&\vdots \\
\dot{z}_{r-1} &= z_r \\
\dot{z}_r &= b(\xi, \eta) + a(\xi, \eta)u \\
\dot{\eta} &= q(\xi, \eta).
\end{aligned}$$

Now, consider the *zeroing the output problem* which find, if any except the trivial one, pairs consisting of an initial state x° and of an input function $u^\circ(\cdot)$, defined for all t in a neighborhood of $t = 0$, such that the corresponding output $y(t)$ of the system is identically zero for all t in a neighborhood of $t = 0$. In the normal form, since $y = z_1(t)$, the constraint $y(t) = 0$ for all t implies

$$\dot{z}_1(t) = \dot{z}_2(t) = \cdots = \dot{z}_r(t) = 0$$

that is $\xi(t) = 0\ \forall t$. Hence, when the output of the system is identically the zero function, its state is constrained to evolve in such a way that $\xi(t)$ is exactly zero function. In addition, the input $u(t)$ must necessarily be the unique solution of the equation

$$0 = b(0, \eta(t)) + a(0, \eta(t))u(t)$$

where $\eta(t)$ has a trajectory governed by the differential equation

$$\dot{\eta}(t) = q(0, \eta(t)) \tag{5}$$

with initial condition $\eta(0) = \eta^{\circ}$.

The dynamics of (5) correspond to the dynamics describing the "internal" behavior of the system when input and initial conditions have been chosen in such a way as to constrain the output to remain identically zero. These dynamics are called the **zero dynamics** of the system.

3 The Definition of Nonminimum Phase Systems

The **nonminimum phase system** is defined as a system whose zero dynamics are unstable. In linear case, it is simply referred to a system whose zeros of the system transfer function are located in the right half plane (RHP) in the complex domain. However, it should be noted that the definition of the nonminimum phase system had been defined in the linear case before the definition of the zero dynamics were appeared in the literature. When considering why the terminology "zero" in zero dynamics was used, it is apparent that the definition of zero dynamics defined as (5) is a general definition because it can explain the linear case as well as the nonlinear case. So it will be examined how these dynamics are related to the zeros of the transfer function in a linear system [8].

Let

$$H(s) = K\frac{b_0 + b_1 s + \cdots + b_{n-r-1}s^{n-r-1} + s^{n-r}}{a_0 + a_1 s + \cdots + a_{n-1}s^{n-1} + s^n}$$

denote the transfer function of a linear system. Suppose the numerator and denominator polynomials are coprime polynomials and consider a minimal realization of $H(s)$

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}$$

with

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{pmatrix} \qquad B = \begin{pmatrix} 0 \\ 0 \\ \cdots \\ 0 \\ K \end{pmatrix}$$
$$C = \begin{pmatrix} b_0 & b_1 & \cdots & b_{n-r-1} & 1 & 0 & \cdots & 0 \end{pmatrix}.$$

Let us now calculate the normal form. For the first r new coordinates, it should be taken by

$$\begin{aligned} z_1 &= Cx = b_0x_1 + b_1x_2 + \cdots + b_{n-r-1}x_{n-r} + x_{n-r+1} \\ z_2 &= CAx = b_0x_2 + b_1x_3 + \cdots + b_{n-r-1}x_{n-r+1} + x_{n-r+2} \\ &\vdots \\ z_r &= CA^{r-1}x = b_0x_r + b_1x_{r+1} + \cdots + b_{n-r-1}x_{n-1} + x_n \end{aligned}$$

For the other $n-r$ new coordinates there are some freedom of choice provided that the conditions stated in sec. 2.2. The simplest one is

$$\begin{aligned} z_{r+1} &= x_1 \\ z_{r+2} &= x_2 \\ &\vdots \\ z_n &= x_{n-r}. \end{aligned} \tag{6}$$

This is indeed an admissible choice because the corresponding coordinates transformation $z = \Phi(x)$ is a nonsingular matrix.

In the new coordinates, equations in normal form are obtained, which have the following structure

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_3 \\ &\vdots \\ \dot{z}_{r-1} &= z_r \\ \dot{z}_r &= R\xi + S\eta + Ku \\ \dot{\eta} &= P\xi + Q\eta \end{aligned}$$

where R and S are row vectors and P and Q matrices, of suitable dimensions. The zero dynamics of this system, according to the previous definition, are those of

$$\dot{\eta} = Q\eta$$

The particular choice of the last $n-r$ new coordinates denoted by (6) entails a particularly simple structure for the matrix Q such as

$$Q = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ -b_0 & -b_1 & -b_2 & \cdots & -b_{n-r-1}. \end{pmatrix}$$

From the particular form of this matrix, it is clear that the eigenvalues of Q coincide with the zeros of the numerator polynomial of $H(s)$, i.e. with the zeros of the transfer function. Thus it is concluded that in a linear system the zero dynamics are linear dynamics with eigenvalues coinciding with the zeros of the transfer function of the system.

4 The Analysis of VTOL Aircraft System

There are many nonlinear nonmimimun phase systems such as water level regulator in a nuclear power generator, double inverted pendulum [6], and PVTOL (Planar Vertical Take-Off and Landing) aircraft [7, 11]. In this report, the PVTOL aircraft control problem is considered as an illustrative example.

4.1 Modeling

The complete dynamics of an aircraft, taking into account flexibility of the wings, aeroelastic effects, the (internal) dynamics of the engine and control surface actuators, and the multitude of changing variables, are quite complex and somewhat unmanageable for the purpose of control. For the controller design, it is useful to consider a simple aircraft that has a minimum number of states and inputs but retains many of the features that must be considered when designing control laws [7]. Figure 1 shows a prototype PVTOL aircraft.

The aircraft state is simply the position, x, y, of the aircraft center of mass, the roll anlge, θ, of the aircraft, and the corresponding velocities, $\dot{x}, \dot{y}, \dot{\theta}$. The control inputs, u_1, u_2, are the thrust (directed out the bottom of the aircraft) and the rolling moment. The equation of motion for a protocol PVTOL aircraft are given by

$$\ddot{x} \;=\; -\sin\theta u_1 + \epsilon\cos\theta u_2$$

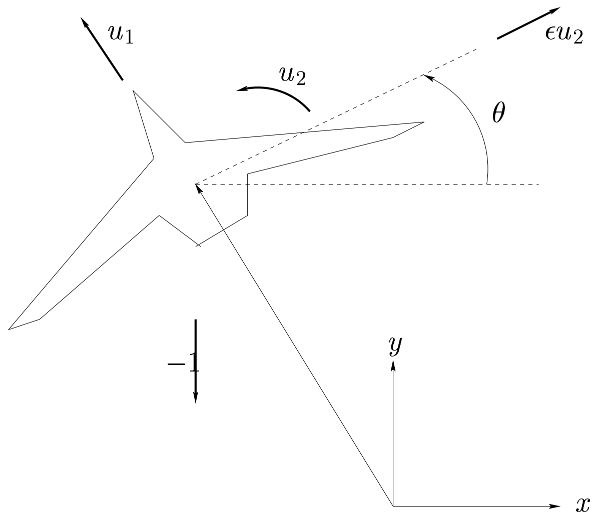

Figure 1: The PVTOL aircraft [7]

$$
\begin{aligned}
\ddot{y} &= \cos\theta u_1 + \epsilon \sin\theta u_2 - 1 \\
\ddot{\theta} &= u_2
\end{aligned}
\tag{7}
$$

where "-1" is the normalized gravitational acceleration and ϵ is the coefficient giving the coupling between the rolling moment and the lateral acceleration of the aircraft.

4.2 Zero dynamics

When the interested outputs are x and y, the PVTOL system denoted by (7) is multi-input-multi-output one in which inputs are u_1 and u_2. The theory of multi-input-multi-output nonlinear geometric control theory is shown in chapter 5 of [8].

By setting states as $x = (\ x \quad \dot{x} \quad y \quad \dot{y} \quad \theta \quad \dot{\theta}\ )^T$, the eq. (7) can be written as follows:

$$
\begin{aligned}
\dot{x} &= \begin{pmatrix} x_2 \\ 0 \\ x_4 \\ -1 \\ x_6 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -\sin x_5 \\ 0 \\ \cos x_5 \\ 0 \\ 0 \end{pmatrix} u_1 + \begin{pmatrix} 0 \\ \epsilon\cos x_5 \\ 0 \\ \epsilon \sin x_5 \\ 0 \\ 1 \end{pmatrix} u_2 \\
y &= \begin{pmatrix} x_1 \\ x_3 \end{pmatrix}
\end{aligned}
\tag{8}
$$

The relative degree of y_1 and y_2 are 2, respectively, because the outputs must be differentiated twice in order to obtain the inputs explicitly. Also the relative degrees are confirmed by using the definition of relative degree in multi-input-multi-output case [8]

Definition 1 *A multivarible nonlinear system has a (vector)* relative degree $\{r_1, \cdots, r_m\}$ *at a point* x° *if*

(i)

$$L_{g_j} L_f^k h_i(x) = 0$$

for all $1 \leq j \leq m$, *for all* $k < r_i - 1$, *for all* $1 \leq i \leq m$, *and for all* x *in a neighborhood of* x°,

(ii) *the* $m \times m$ *matrix*

$$A(x) = \begin{pmatrix} L_{g_1} L_f^{r_1-1} h_1(x) & \cdots & L_{g_m} L_f^{r_1-1} h_1(x) \\ L_{g_1} L_f^{r_2-1} h_2(x) & \cdots & L_{g_m} L_f^{r_2-1} h_2(x) \\ \cdots & \cdots & \cdots \\ L_{g_1} L_f^{r_m-1} h_m(x) & \cdots & L_{g_m} L_f^{r_m-1} h_m(x) \end{pmatrix}$$

is nonsingular at $x = x^\circ$.

As illustrated in sec. 2 for the single-input-single-output case, the multi-input-multi-output nonlinear systems can be transformed into normal forms using the developed methods, e.g. in the Isidori's text [8]. However, in this report, an input-output linearization method is utilized to get a zero dynamics of the aircraft [7]. Rewriting the first two equations of (7) obtains

$$\begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} + \begin{pmatrix} -\sin\theta & \epsilon\cos\theta \\ \cos\theta & \epsilon\sin\theta \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.$$

Since the matrix operating on u is nonsingular, we can linearize the system by choosing the static state feedback law

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} -\sin\theta & \cos\theta \\ \frac{\cos\theta}{\epsilon} & \frac{\sin\theta}{\epsilon} \end{pmatrix} \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right]$$

The resulting system is

$$\begin{aligned} \ddot{x} &= v_1 \\ \ddot{y} &= v_2 \\ \ddot{\theta} &= \frac{1}{\epsilon}(\sin\theta + \cos\theta v_1 + \sin\theta v_2) \end{aligned} \tag{9}$$

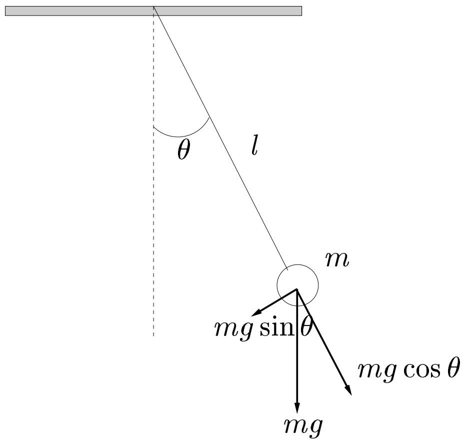

Figure 2: Undamped pendulum

Constraining the outputs to zero by setting $v_1 = v_2 = 0$, the zero dynamics are obtained as

$$\ddot{\theta} = \frac{1}{\epsilon}\sin\theta \tag{10}$$

The eq. (10) is simply the equation of an undamped pendulum shown in Figure. 2 which is modeled by

$$\ddot{\theta} = -\frac{mg}{l}\sin\theta \tag{11}$$

Comparing the (11) with (10), we can know that, if $\epsilon > 0$, the equilibrium point $(\theta, \dot{\theta}) = (0, 0)$ is unstable and the equilibrium point $(\theta, \dot{\theta}) = (\pi, 0)$ is stable but not asymptotically stable and is surrounded by a family of periodic orbits with periods ranging from $2\pi\sqrt{\epsilon}$ to ∞. So the zero dynamics, (10), are unstable and the VTOL aircraft is a nonmimimum phase system by definition.

5 Conclusion

Basic concepts of nonlinear geometric control theory are studied and summarized briefly such as relative degree, local coordinate transformation, normal forms, and zero dynamics. The zero dynamics has an important role in defining the nonminimum phase system. PVTOL aircraft systems was chosen as an example of nonlinear nonminimum phase system in this report and analyzed with its zero dynamics. Future works includes designing controllers which deals with the nonminimum phase characteristics efficiently based on this analysis of the system model.

References

[1] Scott Bezick. Guidance of a homing missile via nonlinear geometric control methods. *Journal of Guidance, Control, and Dynamics*, 18(3):441–448, 1995.

[2] Christopher I. Byrnes and Alberto Isidori. Heuristics for nonlinear control. In *Modeling and Adaptive Control*, pages 48–70. Springer-Verlag, 1988.

[3] Christopher I. Byrnes and Alberto Isidori. Asymptotic stabilization of minimum phase nonlinear systems. *IEEE Transactions on Automatic Control*, 36(10):1122–1137, 1991.

[4] Christopher I. Byrnes, Alberto Isidori, and Jan C. Willems. Passivity, feedback equivalence, and the global stabilization of minimum phase nonlinear systems. *IEEE Transactions on Automatic Control*, 36(11):1228–1240, 1991.

[5] James S. Freudenberg and Douglas P. Looze. *Frequency Domain Properties of Scalar and Multivariable Feedback Systems.* Springer-Verlag, 1988.

[6] Swaminathan Gopalswamy and Karl J. Hedrick. Tracking nonlinear non-minimum phase systems using sliding control. *International Journal of Control*, 57(5):1141–1158, 1993.

[7] John Hauser, Shankar Sastry, and George Meyer. Nonlinear control design for slightly nonminimum phase systems: Application to v/stol aircraft. *Automatica*, 28(4):665–679, 1992.

[8] Alberto Isidori. *Nonlinear Control Systems.* Springer-Verlag, third edition, 1995.

[9] Alberto Isidori and Christopher I. Byrnes. Output regulation of nonlinear systems. *IEEE Transactions on Automatic Control*, 35(2):131–140, 1990.

[10] Hassan K. Khalil. *Nonlinear Systems.* Prentice-Hall Inc., second edition, 1996.

[11] Xiao-Yun Lu, Sarah K. Spurgeon, and Ian Postlethwaite. Robust variable structure control of a pvtol aircraft. *International Journal of Systems Science*, 28(6):547–558, 1997.