

Model-Free Adaptive Switching Control of Time-Varying Plants

Giorgio Battistelli, João P. Hespanha, *Fellow, IEEE*, Edoardo Mosca, *Life Fellow, IEEE*, and Pietro Tesi

Abstract—This paper addresses the problem of controlling an uncertain time-varying plant by means of a finite family of candidate controllers supervised by an appropriate switching logic. It is assumed that, at every time, the plant consists of an uncertain single-input/single output linear system. It is shown that stability of the switched closed-loop system can be ensured provided that 1) at every time there is at least one candidate controller capable of potentially stabilizing the current time-invariant “frozen” plant model, and 2) the plant changes are infrequent or satisfy a slow drift condition.

Index Terms—Adaptive control, control of uncertain plants, switching control, time-varying systems.

I. INTRODUCTION

ONE of the approaches for controlling uncertain or time-varying plants relies on the introduction of adaptation in the feedback loop. In recent years, *adaptive switching control* has emerged as an alternative to conventional continuous adaptation. In switching control, a so-called supervisor selects a specific feedback controller among a family of admissible ones, based on the measured data. The latter are processed so as to enable the supervisor to determine whether or not the current controller is adequate, and, in the negative, to replace it by a different candidate controller. For an early overview of the topic, the reader is referred to [1].

To date, most of the contributions on switching control have been basically of a two-fold nature. On one side the major emphasis in [2]–[9] has been on robust adaptive stabilization of time-invariant systems. Within these contributions, the *Unfalsified Control* approach developed by M.G. Safonov and co-workers [5], [6] provides guarantees of stability for plants subject to large parametric uncertainties, unmodeled dynamics and disturbances. Unfortunately, as will be discussed in detail later, this methodology is tailored to control time-invariant systems and has no direct generalization to time-varying systems.

Manuscript received June 24, 2011; revised June 04, 2012; accepted November 29, 2012. Date of publication January 30, 2013; date of current version April 18, 2013. Recommended by Associate Editor A. Astolfi.

G. Battistelli and E. Mosca are with Dipartimento di Ingegneria dell’Informazione, DINFO—Università di Firenze, 50139 Florence, Italy (e-mail: giorgio.battistelli@unifi.it; mosca@dsi.unifi.it).

J. P. Hespanha is with the Department of Electrical and Computer Engineering, University of California, Santa Barbara, CA 93106-9560 USA (e-mail: hespanha@ece.ucsb.edu).

P. Tesi is with the Department of Informatics, Bioengineering, Robotics, and Systems Engineering, University of Genoa, 16145 Genoa, Italy (e-mail: pietro.tesi@unige.it).

Digital Object Identifier 10.1109/TAC.2013.2243974

On the other side the main focus in [1], [10]–[15], has been on switching control schemes capable, at least in principle, of dealing with plant variations. However, these techniques can become ineffective if the available knowledge about the plant is limited so that the family of candidate plant models does not tightly approximate the process dynamics over the whole uncertainty set.

The main goal of this paper is to consider a novel supervisory control scheme by which previous theoretical results on Unfalsified Control can be extended to time-varying plants. The proposed scheme embeds a family of pre-designed candidate controllers and a supervisory unit that selects the controller minimizing a certain cost function. Unlike previous works on Unfalsified Control [4]–[6], [8], the proposed scheme follows a *fading memory* paradigm in that the cost functions used for controller switching are evaluated over a suitable time-window. The key feature of the scheme is the use of a dynamic window, *viz.* a window whose length can vary with time. Specifically, we propose a resetting logic by which the length of the time-window is adjusted online, according to the rule that past records of the cost functions are discarded or retained depending on whether they contain information relevant to stability. As will be seen, this supervisory scheme not only retains the desirable robustness features of Unfalsified Control, but, being based on a fading memory paradigm, can also provide stability guarantees for plants whose parameters are either slowly time-varying or subject to infrequent jumps.

A final point is worth mentioning. Various schemes for switching adaptive control have been proposed in the literature, which differ on how they choose *when* to switch controllers, and *how* to select a new controller. A first class of algorithms, usually referred to as pre-routed [3], [11], [16], essentially try to determine whether the online controller meets a desired level of performance that is consistent with a stable closed-loop. In the negative, the current controller is replaced in the feedback loop by a different one. In this context, a recent important contribution is given by [17], where suitable logics are derived which can ensure stability even for time-varying plants. The supervisory scheme proposed hereafter belongs to a different class of algorithms in that controller switching occurs whenever the inferred virtual performance of an offline controller turns out to be better than the one achieved by the active controller. This obviates the need for a pre-routed search and permits the logic to pick those controllers with the potential of yielding the highest performance. Specifically, we use a hysteresis logic which switches to a new controller when a “significant” difference between performance levels is detected. This type of supervisor does not directly enforce a dwell-time [10], [17],

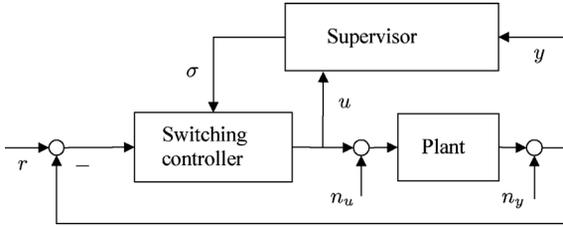


Fig. 1. Adaptive switching control scheme.

[18] between two consecutive switching times. For a discussion on the potential advantages of hysteresis-based switching algorithms the reader is referred to [13].

Notations

Throughout the paper, the prime denotes transpose, $\|\cdot\|$ Euclidean norm, and \mathcal{S} the space of all real-valued vector sequences on the set \mathbb{Z}_+ of nonnegative integers. For any $s \in \mathcal{S}$, and $t_0, t \in \mathbb{Z}_+$, $t_0 \leq t$, we define $s|_{t_0}^t := \{s(t_0), \dots, s(t)\}$. For simplicity, if $t_0 = 0$, $s|_{t_0}^t = s^t$. Given λ , $0 < \lambda \leq 1$, we denote the λ -exponentially weighted ℓ_2 -norm of $s|_{t_0}^t$ by $\|s|_{t_0}^t\|_\lambda := \sqrt{\sum_{\tau=t_0}^t \lambda^{2(t-\tau)} |s(\tau)|^2}$ whenever $t \geq t_0$, or the zero number otherwise. If $\lambda = 1$, we let $\|s|_{t_0}^t\|$ denote the ℓ_2 -norm of $s|_{t_0}^t$. The ℓ_∞ -norm of $s|_{t_0}^t$ is defined as $\|s|_{t_0}^t\|_\infty := \max_{\tau \in \{t_0, \dots, t\}} \max_i |s_i(\tau)|$ where s_i denotes the i -th component of s . The sequence $s \in \mathcal{S}$ is said to be *bounded* if its ℓ_∞ -norm is finite.

II. PROBLEM FORMULATION AND PAPER OVERVIEW

A. Plant and Control System

We consider the switched system depicted in Fig. 1. The plant P consists of a discrete-time strictly causal SISO linear time-varying dynamic system described by

$$A_t(d)y(t) = B_t(d)(u(t) + n_u(t)) + A_t(d)n_y(t), \quad t \in \mathbb{Z}_+. \quad (1)$$

In (1), u is the input, y is the output, n_u is the input disturbance and n_y is the output disturbance. $A_t(d)$ and $B_t(d)$ denote time-varying polynomials in the unit backward shift operator d . It is assumed that at any time t the time-invariant system obtained by freezing the parameters of (1) at their values at time t belongs to a set P , which represents the set of all possible plant configurations. Further, let P_* denote a generic member of \mathcal{P} , and $B_*(d)/A_*(d)$ its corresponding transfer function. The set P will also account for both the range of parametric uncertainty and the unmodelled dynamics. Indeed, as we will see later, no assumption on P will be required other than it is a compact set, i.e., for any P_* the polynomials A_* and B_* have (possibly unknown) bounded degrees and their coefficients belong to a (possibly unknown) compact set.

The switching controller C_σ is assumed to have the one-degree-of-freedom form $u(t) = C_{\sigma(t)}(r(t) - y(t))$, where r is the output reference, while the subscript $\sigma(t)$ identifies the specific candidate controller connected in feedback with the plant at time t . We shall assume that σ takes values in the (finite) set $\underline{N} := \{1, \dots, N\}$. Then, $\mathcal{C} := \{C_i, i \in \underline{N}\}$ will denote the finite family of candidate controllers. In particular, each member

of \mathcal{C} is taken as a linear time-invariant (LTI) controller with transfer function $S_i(d)/R_i(d)$. Accordingly, the plant input u is given by

$$R_{\sigma(t)}(d)u(t) = S_{\sigma(t)}(d)(r(t) - y(t)) \quad (2)$$

so that the switching controller reduces to a single controller with switched parameters. Given a finite family \mathcal{C} of candidate controllers, $\mathcal{E}_{\mathcal{S}}(P_*)$ will denote the subset of \mathcal{C} composed by all controllers which (internally) stabilize P_* .

Definition 1: The switched system (1)–(2) is said to be *stable* if, for all initial conditions, any bounded exogenous input (r, n_u, n_y) produces a bounded output (u, y) . The problem is said to be *feasible* if $\mathcal{E}_{\mathcal{S}}(P_*) \neq \emptyset, \forall P_* \in \mathcal{P}$ \blacktriangle

B. Supervisory Control Architecture

The supervisor processes the recorded plant I/O data to generate the sequence σ that specifies the switching controller C_σ . In particular, to decide when and how to change the controller, the supervisor embodies a family $\Pi := \{\Pi_i; i \in \underline{N}\}$ of test functionals which quantify the suitability of each candidate controller to control P . Given Π , the hysteresis switching logic considered hereafter is as follows: At each step, one computes the least index $i_*(t)$ in \underline{N} such that $\Pi_{i_*(t)}(t) \leq \Pi_i(t), \forall i \in \underline{N}$. Then, the switching index sequence σ is given by

$$\left. \begin{aligned} \sigma(t+1) &= l(\sigma(t), \Pi(t)), & \sigma(0) &= i_0 \in \underline{N} \\ l(i, \Pi(t)) &= \begin{cases} i, & \text{if } \Pi_i(t) < \Pi_{i_*(t)}(t) + h \\ i_*(t), & \text{otherwise} \end{cases} \end{aligned} \right\} \quad (3)$$

where $h > 0$ is the hysteresis constant.

Many adaptive control schemes based on hysteresis switching have been considered for time-invariant plants, and shown to provide robustness against large uncertainties, unmodeled dynamics and disturbances. However, no similar results are available for plants subject to time variations.

To handle time variations in the plant dynamics, one needs to select Π with a finite or fading memory. Unfortunately, this involves the potential risk that the switched system become unstable due to persistent switching. To the best of the authors' knowledge, efforts to ensure stability without relying on a finite switching stopping time are restricted to [1], [10]–[15]. However, these techniques can become ineffective if the plant models do not tightly approximate the admissible plants. To prevent the type of instability induced by persistent switching and to provide robustness of the control scheme several techniques have been proposed [2], [5]–[8], [19]. These techniques combine (3) with test functionals of the form

$$\Pi_i(t) := \|\pi_i^t\|_\infty \quad (4)$$

where $\pi := \{\pi_i; i \in \underline{N}\}$ is an underlying family of test functionals. This type of supervisory schemes, including unfalsified control, provides a simple means for preventing the risk of instability caused by persistent switching. Indeed, the maximum norm in (4) ensures that all the test functionals admit a limit as t tends to infinity. Then, by the presence of the hysteresis in the logic (3), the switching stops whenever at least one of the π_i 's is bounded ([20, Lemma 1]). However, such a maximum norm may yield unbounded Π_i 's in a time-varying plant case.

This paper aims at overcoming such limitations by considering novel hysteresis switching algorithms in which the test functionals have an *adaptive* memory. In particular, the scheme here proposed embeds in (3) a *resetting logic*, *viz.* a mechanism according to which the supervisor resets all the Π_i 's to zero whenever suitable events (resetting conditions) occur. Specifically, we consider test functionals of the form

$$\Pi_i(t) := \|\pi_i|_{t_k}^t\|_\infty, \quad t \in \mathbb{T}_k := \{t_k, \dots, t_{k+1} - 1\} \quad (5)$$

where $\{t_k\}_{k \in \mathbb{Z}_+}$ denotes a sequence of resetting instants to be specified. For clarity, we shall denote by HSL- ∞ the hysteresis switching logic defined by (3) and (4), and by hysteresis switching logic with resetting (HSL-R) the new switching logic defined by (3) and (5).

The remainder of the paper is as follows. In Section III, we recall basic concepts underlying unfalsified control, and derive certain key properties of HSL-R. The adaptive mechanism used by the supervisor to generate the resetting instants is analyzed in Sections IV and V. It is shown that, for time-invariant systems, the proposed supervisory scheme can ensure stability without relying on a finite switching stopping time, thus extending the theoretical properties of unfalsified control to logics other than HSL- ∞ . In Section VI, we show that without any further modification, the same supervisory scheme can also ensure stability in the presence of time variations of the plant parameters. Section VII ends the paper with some concluding remarks.

III. MODEL-FREE ADAPTIVE CONTROL

In unfalsified control, the feedback adaptation task is replaced by the so-called controller falsification. The basic concept of this approach is as follows. At each time and for each $i \in \underline{N}$ one computes in real-time the solution v_i to the difference equation (*cf.* Remark 1)

$$S_i(d) (v_i(t) - y(t)) = R_i(d) u(t). \quad (6)$$

As shown in Fig. 2, v_i equals the *virtual* reference sequence [4] which would reproduce the recorded I/O sequence (u, y) should the plant P be fed-back by the controller C_i , irrespective of the way the plant input u is generated. This makes it possible to evaluate what would have been achieved by the feedback connection of the controller C_i with the process (denoted by (P/C_i)) if the reference have been equal to v_i . In this respect, consider the time-varying feedback system (P/C_i) mapping the “input” $w_i := [v_i \ n_u \ n_y]'$ to the “output” $\zeta_i := [u \ (v_i - y)]'$. Accordingly, a possible related performance measure can be constructed as follows:

$$\pi_i(t) := \frac{\|\zeta_i^t\|_\lambda}{\mu + \|v_i^t\|_\lambda}, \quad t \in \mathbb{Z}_+ \quad (7)$$

where μ is a positive constant. In case of noise-free LTI plant, π_i provides an estimate from below of the λ -weighted \mathcal{H}_∞ mixed-sensitivity norm [21] of the loop (P/C_i) with virtual reference input v_i and output ζ_i containing the control input u and the virtual tracking error $v_i - y$, which we would like to minimize. The test functional (7) can be viewed as a variant

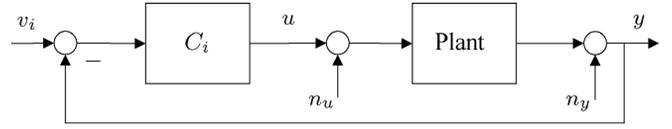


Fig. 2. The i th virtual candidate loop.

of the (unweighted) \mathcal{H}_∞ mixed-sensitivity performance criterion often considered in unfalsified control [5]. The performance measure given by (7) is obtained with no *plant model* identification effort, and this motivates the adoption of the term “model-free.”

Remark 1: Online computation of (6) requires that all the candidate controllers be stably causally invertible (SCI), but suitable arrangements which remove this design constraint have been reported in the literature [22]. ▼

Assumptions

We now introduce the assumptions needed to provide stability of the switched system. For ease of reference, some shorthand notations are defined.

Definition 2: A polynomial $p(d)$ is said to be a λ -Hurwitz polynomial (in the indeterminate d) if it has no root in the closed disk of radius λ^{-1} of the complex plane. ▲

Definition 3: The feedback loop (P_*/C_i) composed by the time-invariant plant P_* and the controller C_i is said to be λ -stable if its characteristic polynomial $\chi_{*/i}(d) := A_*(d)R_i(d) + B_*(d)S_i(d)$ is a λ -Hurwitz polynomial. ▲

We make the following assumptions.

- A1) The plant uncertainty set \mathcal{S} is compact, in the sense that there exists an integer n^* (possibly unknown) such that for every $P_* \in \mathcal{S}$ the polynomials A_* and B_* have order smaller than n^* and their coefficients belong to a compact subset of \mathbb{R}^{n^*+1} . •
- A2) For any $P_* \in \mathcal{S}$, there exists a candidate controller $C_i \in \mathcal{C}$ such that (P_*/C_i) is λ -stable, λ as in (7). •
- A3) For each candidate controller $C_i \in \mathcal{C}$, S_i is a λ -Hurwitz polynomial, λ as in (7). •
- A4) The exogenous inputs r , n_u , and n_y are bounded. •

Remark 2: A2 implies feasibility, *viz.* $\mathcal{C}_S(P_*) \neq \emptyset, \forall P_* \in \mathcal{S}$. Specifically, it guarantees that for any $P_* \in \mathcal{S}$ there exists a controller $C_i \in \mathcal{C}$ such that the closed-loop eigenvalues of (P_*/C_i) are strictly less than λ . A2 is related to the choice of the cost function π_i in (7). As will be seen in next Lemma 3, in fact, λ -stability of a feedback loop (P_*/C_i) implies boundedness of the corresponding cost function π_i . From a practical point of view, however, A2 is no stronger than feasibility in the sense that under feasibility we can always satisfy A2 by selecting λ close enough to one in (7). A similar remark applies to A3. In fact, one can always increase λ in (7) so that A3 is no stronger than requiring all controllers to be SCI (see Remark 1 above). ▼

A. Key Lemmas

In this subsection, we introduce certain key properties upon which the stability analysis depends. To this end, some preliminary observations are needed.

To avoid needless complications, we assume that the switching controller (2) as well as (6) are both initialized at time zero from zero initial conditions: Let

$m := \max \{\deg S_i, \deg R_i; i \in \underline{N}\}$ and assume $\sigma(t) = i$. The control input is then given by

$$\begin{aligned} u(t) &= \sum_{k=0}^m s_{ik} (r(t-k) - y(t-k)) - \sum_{k=1}^m r_{ik} u(t-k) \\ r(k) &= u(k) = y(k) = 0, \quad k = -1, \dots, -m \end{aligned} \quad (8)$$

where s_{ik} and r_{ik} denote the coefficients of the polynomials $S_i(d)$ and $R_i(d)$, respectively. An analogous initialization is made for (6). Regarding (1), consider that in the difference equation

$$\begin{aligned} y(t) &= n_y(t) + \sum_{k=1}^{n^*} b_{k_t} (u(t-k) + n_u(t-k)) \\ &\quad - \sum_{k=1}^{n^*} a_{k_t} (y(t-k) - n_y(t-k)), \quad t \in \mathbb{Z}_+ \end{aligned}$$

where b_{k_t} and a_{k_t} denote the coefficients of the polynomials $B_t(d)$ and $A_t(d)$, the values taken on by $u(k)$ and $y(k)$ for $k = -1, \dots, -n^*$ need not be consistent with those specified in (8). In fact, the plant is supposed to be controlled with the architecture of Fig. 1 only from time $t = 0$, whereas no assumption is made on how the plant inputs were generated for $t < 0$. For notational simplicity we set $u(k)$ and $y(k)$ equal to zero for negative times and denote by $\xi_P := [n_u(-1) \dots n_u(-n^*) \ n_y(-1) \dots n_y(-n^*)]'$ the vector composed by the plant initial conditions. This entails no loss of generality in that any nonzero plant initial state ξ_P can be thought of as generated by a suitable plant input/output disturbance sequence.

Let now

$$\Pi^k := \min_{i \in \underline{N}} \left\{ \max_{t \in \mathbb{T}_k} \pi_i(t) \right\} + h, \quad k \in \mathbb{Z}_+. \quad (9)$$

The following lemmas are the main results of this section and fundamental for the developments of the paper.

Lemma 1: Consider the HSL-R. Let \mathfrak{N}_k denote the number of switchings over the interval \mathbb{T}_k , and $\lceil \alpha \rceil$ denote the smallest positive integer greater than or equal to $\alpha \in \mathbb{R}_+$. Then,

$$\pi_{\sigma(t+1)}(t) < \Pi^k, \quad \forall t \in \mathbb{T}_k \quad (10)$$

$$\mathfrak{N}_k \leq N \left\lceil \frac{\Pi^k}{h} \right\rceil, \quad \forall k \in \mathbb{Z}_+ \quad (11)$$

hold for any resetting sequence $\{t_k\}_{k \in \mathbb{Z}_+}$. ■

The proof of Lemma 1 can be omitted since (10) follows directly from the hysteresis logic and (11) is a straightforward consequence of the fact the test functionals Π_i are monotone nondecreasing over each time interval \mathbb{T}_k .

Let Σ_σ denote the switched system (1)–(2) mapping the “input” $w := [r \ n_u \ n_y]'$ to the “output” $\zeta := [u \ (r-y)]'$ under a given switching sequence σ .

Lemma 2: Consider a HSL-R switched system Σ_σ based on the test functionals (7). Let A1, A3, and A4 hold. Then, there exists a finite-valued function \mathfrak{g} such that

$$\|\zeta^t\|_\lambda \leq \mathfrak{g}(\Pi^k) (\mu + |\xi_P| \lambda^{t+1} + \|w^t\|_\lambda + \|\zeta^{t_k-1}\|_\lambda \lambda^{t-t_k+1}), \quad \forall t \in \mathbb{T}_k \quad (12)$$

holds for any resetting sequence $\{t_k\}_{k \in \mathbb{Z}_+}$.

Proof: See the Appendix. ■

Before concluding this section it is important to observe that in Lemma 2 the time variations of the plant parameters can be arbitrary and, at this time, we do not yet need assumption A2.

IV. STABILITY UNDER ADMISSIBLE RESETTING

As seen from Lemma 2, the bound in (12) depends on both the sequences $\{\Pi^k\}_{k \in \mathbb{Z}_+}$ and $\{\zeta^{t_k-1}\}_{k \in \mathbb{Z}_+}$. In unfalsified control based on HSL- ∞ , the term $\|\zeta^{t_k-1}\|_\lambda$ is absent since there is only one reset time, $t_0 := 0$ at startup, and $\mathbb{T}_0 = \mathbb{Z}_+$. In this case, stability depends only on boundedness of Π^0 , and, hence, it can be guaranteed simply by ensuring boundedness of at least one of the test functionals (this nice property is precisely the motivation which led [5] to introduce the notion of *cost detectability* of the test functionals). Under resetting, the analysis becomes more complicated. Indeed, even if the plant is time-invariant but there are infinitely many resets, $t - t_k$ does not grow unbounded as $t \rightarrow \infty$, and, hence, the term $\|\zeta^{t_k-1}\|_\lambda \lambda^{t-t_k+1}$ need not vanish. This is consistent with the intuition that resetting destroys the monotonicity of the test functionals and therefore that boundedness of $\{\Pi^k\}_{k \in \mathbb{Z}_+}$ alone may not prevent instability due to persistent switching.

The remainder of this section is devoted to show that, nonetheless, adaptive resetting mechanisms do exist which preserve stability of the switched system.

A. Admissible Resetting Times

The notation for this subsection is as in Section III-C. Consider the switched system Σ_σ and define the following performance measure for the closed-loop:

$$\pi_*(t) := \frac{\|\zeta^t\|_\lambda}{\mu + \|r^t\|_\lambda}, \quad t \in \mathbb{Z}_+. \quad (13)$$

Consider now the following definition.

Definition 4: (Admissible Resetting Times). A sequence of reset times $\{t_k\}_{k \in \mathbb{Z}_+}$ is called *admissible* if, for every $k \in \mathbb{Z}_+$, we have that

$$\pi_*(t_k - 1) \leq \pi_{\sigma(t_k)}(t_k - 1) + \epsilon, \quad \epsilon > 0. \quad (14)$$

In essence, (14) only allows the k th reset to occur at the time t_k if (14) holds. ▲

To understand the rationale for (14), observe that π_* acts as an estimate of the *actual* reference-to-data induced gain. In particular, to obtain stability it is required that π_* remains bounded. The test functional π_σ , related to the switched-on controller, however, does only provide an estimate of the *virtual* reference-to-data induced gain. Depending on the values taken on by the virtual reference v_σ , π_σ can be greater or smaller than π_* . The inequality (14) imposes that a reset can occur only if π_* is not much larger than π_σ . As shown hereafter, for time-invariant plants, the selection of σ through the switching logic makes sure that π_σ remains bounded. Condition (14) therefore ensures that at the times of resetting π_* satisfies a boundedness constraint. We will also see that this is sufficient for closed-loop stability to hold.

Consider an admissible resetting sequence. Combining (13) with (14) and (10) we obtain $\|\zeta^{t_k-1}\|_\lambda \leq (\Pi^{k-1} + \epsilon)(\mu + \|r^{t_k-1}\|_\lambda)$. Notice that the above inequality is well-defined for $k = 0$ since $\|\zeta^{-1}\|_\lambda = 0$, which leads to $\Pi^{-1} = 0$. Further, $\|r^{t_k-1}\|_\lambda \lambda^{t-t_k+1} \leq \|r^t\|_\lambda \leq \|w^t\|_\lambda$. Then, under an admissible resetting sequence, Lemma 2 implies that

$$\|\zeta^t\|_\lambda \leq \mathbf{g}(\Pi^k)(|\xi_P| \lambda^{t+1} + (\Pi^{k-1} + \epsilon + 1)(\mu + \|w^t\|_\lambda)), \quad \forall t \in \mathbb{T}_k. \quad (15)$$

From (15) one sees that a sufficient condition for stability is that $\{\Pi^k\}_{k \in \mathbb{Z}_+}$ is bounded. This is formalized in next theorem.

Theorem 1: Consider the same assumptions as in Lemma 2 and further assume that $\Pi^k \leq \Pi^*$, $\forall k \in \mathbb{Z}_+$, for some finite constant Π^* . Then, the HSL-R switched system Σ_σ is stable for any admissible resetting sequence $\{t_k\}_{k \in \mathbb{Z}_+}$ and

$$\|\zeta^t\|_\lambda \leq \mathbf{g}(\Pi^*)|\xi_P| \lambda^{t+1} + \mathbf{h}(\Pi^*)(\mu + \|w^t\|_\lambda), \quad \forall t \in \mathbb{Z}_+ \quad (16)$$

where $\mathbf{h}(\Pi^*) := \mathbf{g}(\Pi^*)(\Pi^* + \epsilon + 1)$. ■

In essence, Theorem 1 indicates that, under the admissibility condition (14), stability of the switched system depends only on boundedness of $\{\Pi^k\}_{k \in \mathbb{Z}_+}$. This is precisely the point where assumption A2 becomes important. In particular, as discussed in next subsection, for time-invariant plants, A2 is sufficient to prove that, like HSL- ∞ , HSL-R leads to stability, as long as the resetting sequence is admissible in the sense of Definition 4.

B. Stability in the Time-Invariant Case

To prove boundedness of $\{\Pi^k\}_{k \in \mathbb{Z}_+}$ for time-invariant plants, we use the following result.

Lemma 3: Let the HSL-R switched system Σ_σ be based on the test functionals (7). Let A1–A4 hold and further assume that on a given interval $\{\tau, \tau+1, \dots, T\}$ the coefficients of the polynomials A_t and B_t in (1) remain constant. Then, there exist positive constants g_0, g_1, g_2 and g_3 such that, for any $P_* \in \mathcal{S}$

$$\pi_s(t) \leq g_0 + g_1 |\xi_P| \lambda^{t+1} + g_2 \|w^t\|_\lambda + g_3 \|\zeta^{\tau-1}\|_\lambda \lambda^{t-\tau+1}, \quad \forall t \in \{\tau, \tau+1, \dots, T\} \quad (17)$$

holds true for some $s \in \underline{N}$.

Proof: See the Appendix. ■

From Lemma 3 one sees that when the plant is a time-invariant system ($\tau = 0$ and $T = \infty$), A2 is sufficient to ensure boundedness of $\{\Pi^k\}_{k \in \mathbb{Z}_+}$. Indeed, by letting $\kappa_0 := g_0$, $\kappa_1 := g_1$ and $\kappa_2 := (1 - \lambda^2)^{-1/2} g_2$ and recalling that $\|\zeta^{-1}\|_\lambda = 0$, we have from (17) that

$$\pi_s(t) + h \leq \kappa_0 + \kappa_1 |\xi_P| + \kappa_2 \|w\|_\infty + h =: \Pi_{TI}^*, \quad \forall t \in \mathbb{Z}_+ \quad (18)$$

holds true for some $s \in \underline{N}$. Hence, from definition of Π^k in (9), we obtain that $\Pi^k \leq \Pi_{TI}^*$ for all $k \in \mathbb{Z}_+$. From boundedness of $\{\Pi^k\}_{k \in \mathbb{Z}_+}$, next result follows directly from Theorem 1.

Theorem 2: Let the HSL-R switched system Σ_σ be based on the test functionals (7). Then, if the plant is time-invariant,

under A1–A3, Σ_σ is stable for any admissible resetting sequence $\{t_k\}_{k \in \mathbb{Z}_+}$. ■

Remark 3: For time-invariant systems, stability of switched systems based on unfalsified control can be proven using analysis tools quite simple compared to the ones given here (cf. [5]–[7]). On the other hand, the present analysis tools do not rely on switching stopping, a property that is crucial for the results in [5], [6] and [7]. ▼

V. FINITE-TIME RESETTING

As described in the previous subsection, for time-invariant plants, HSL-R allows one to prove stability results similar to those available for HSL- ∞ . In this subsection, we show that the reset admissibility condition (14) is always attained in finite time, which ensures that past data records are periodically discarded whenever the plant dynamics remain constant over a large enough horizon, and it will become crucial in the presence of plant variations.

Taking (14) and Theorem 1 into account, consider the following *resetting rule* ($t_0 := 0$):

$$t_{k+1} := 1 + \min \{t : t \geq t_k; \pi_*(t) \leq \pi_{\sigma(t+1)}(t) + \epsilon\}, \quad k \in \mathbb{Z}_+ \quad (19)$$

which, by construction, always generates an admissible resetting sequence satisfying (14). Notice now that, under the same assumptions as in Theorem 1, the following upper bound on the plant data can be derived

$$\begin{aligned} \max_{t \in \mathbb{Z}_+} \|\zeta^t\|_\lambda &\leq \mathbf{g}(\Pi^*)|\xi_P| + (1 - \lambda^2)^{-1/2} \mathbf{h}(\Pi^*)(\mu + \|w\|_\infty) \\ &=: Z(\Pi^*) \end{aligned} \quad (20)$$

where $\|w\|_\infty$ is finite in view of assumption A4. Then, by letting

$$\begin{aligned} \mathfrak{N}_* &:= N \left\lceil \frac{\Pi^*}{h} \right\rceil \\ \Delta(\Pi^*) &:= (\mathfrak{N}_* + 1) \left\lceil \log_\lambda \frac{\epsilon \mu}{Z(\Pi^*)} \right\rceil \end{aligned} \quad (21)$$

the following result states that when $\{\Pi^k\}_{k \in \mathbb{Z}_+}$ is bounded, the HSL-R based on (19) always experiences at least one resetting.

Lemma 4: Consider the HSL-R based on (19). Then, under the same assumptions as in Theorem 1, one has

$$t_{k+1} - t_k \leq \Delta(\Pi^*), \quad \forall k \in \mathbb{Z}_+. \quad (22)$$

Proof: See the Appendix. ■

From Lemma 4 it is also immediate to conclude that, when the plant is time-invariant, under A1–A4, a reset always occurs after at most $\Delta(\Pi_{TI}^*)$ time steps, where Π_{TI}^* is as in (18). Lemma 4, along with Theorem 2, completes the analysis for LTI plants. It is important to emphasize that the complexity of the control scheme proposed here does not depend on the “complexity” of the set \mathcal{S} of frozen plant models, which could contain plants with very high order or be very non-convex. In fact, the complexity of the proposed control scheme is comparable to that of standard Unfalsified Control. The only additional requirements are the computation of the actual reference-to-data induced gain π_* and the application of the reset rule (19).

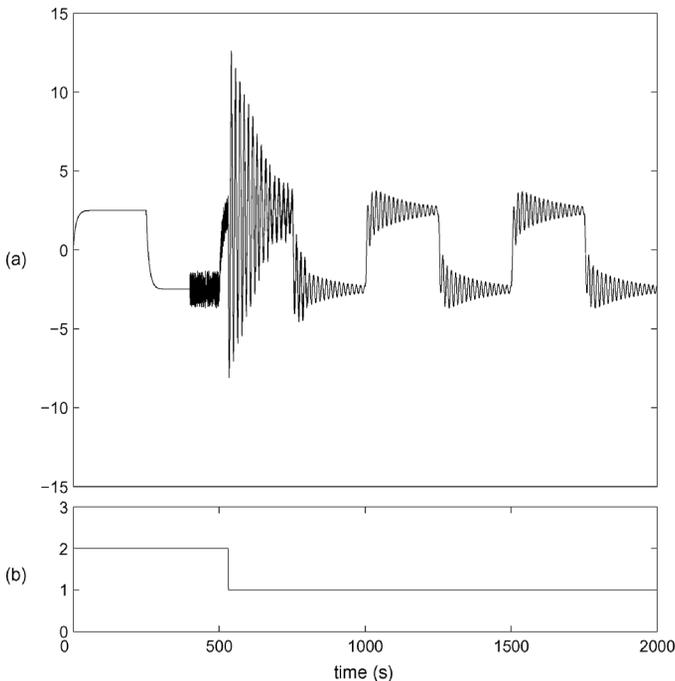


Fig. 3. Supervision based on HSL- ∞ . (a) Plant Output. (b) Switching sequence.

A. Example 1

Although the introduction of HSL-R is mainly motivated by the goal of handling possible plant variations, there are good reasons to use it also in the time-invariant case. In fact, when the plant is time-invariant, (17) implies that

$$\pi_s(t) \leq g_0 + g_1 |\xi_P| \lambda^{t+1} + g_2 \|w|_{t_k}^t\|_\lambda + g_2 \|w^{t_k-1}\|_\lambda \quad \forall t \in \mathbb{T}_k, k \in \mathbb{Z}_+ \quad (23)$$

holds for some $s \in \underline{N}$. This equation indicates that the effect of plant initial conditions and past disturbances on Π^k and, hence, on the controller selection, vanishes as k increases. In view of Lemma 4, this property, which cannot be exploited in HSL- ∞ because of infinite memory, can provide definite improvements in performance for HSL-R.

As a simple illustration, consider the continuous-time LTI unstable plant with transfer function $P(s) = K/(s - 0.4)$, $K \in [0.1, 1]$, controlled by feeding its input via a zero-order holder and sampling its output every $0.2s$. Two proportional-integral (PI) controllers, $C_1(s) = (3.485 - 3.265d)/(1 - d)$ and $C_2(s) = (27.36 - 25.63d)/(1 - d)$, have been designed so as to provide good performance over the uncertainty set. We compare HSL- ∞ and HSL-R both with hysteresis $h = 1$ and with $\lambda := 0.99$ and $\mu = 1$ in the test functionals (7). In particular, λ has been selected so as to satisfy A2 and A3 for the uncertain discrete-time plant and the candidate controllers. For HSL-R, we have adopted the reset rule (19) with $\epsilon := 0.01$. Let $K = 0.16$ so that both the controllers stabilize the plant but only C_2 performs satisfactorily and further assume that n_y and n_u are zero everywhere except on the interval $[400, 800]$ where a burst in noise is modeled by taking n_y to be uniformly distributed on $[-0.5, 0.5]$. Figs. 3–4 show the plant output response for zero plant initial conditions and a square-wave reference of period $500s$ and amplitude 2.5 . In both HSL- ∞ and HSL-R, the output

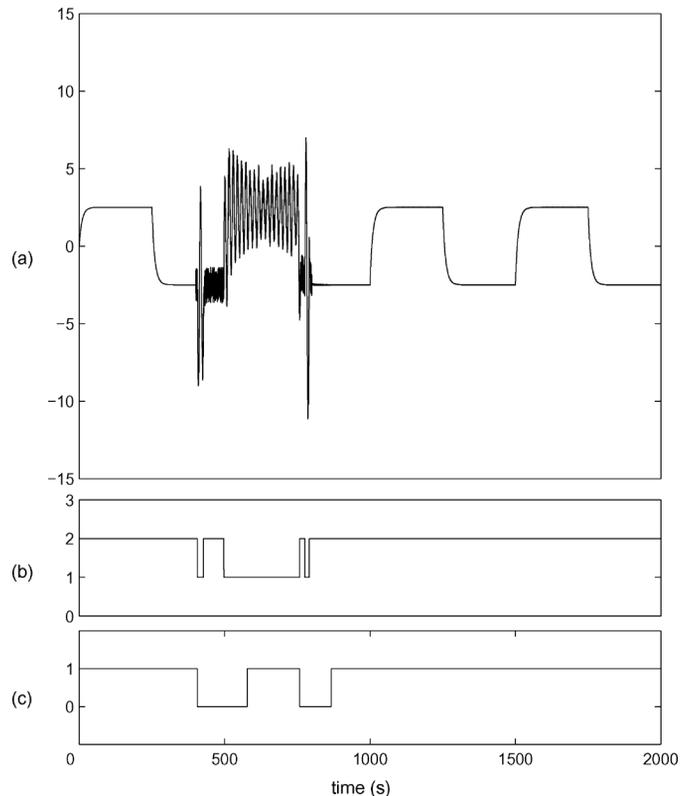


Fig. 4. Supervision based on HSL-R. (a) Plant Output. (b) Switching sequence. (c) Resetting sequence (1 stands for resetting).

noise causes C_2 to be switched-off. However, due to its memory feature, the HSL- ∞ does not allow C_2 to be switched-on again. On the contrary, with HSL-R, C_2 is promptly re-selected right after resetting, as shown in Fig. 5.

Remark 4: In contrast with HSL- ∞ , HSL-R need not ensure finite-time convergence for switching even when the plant is time-invariant. This is the price paid for considering time-windowed test functionals. Hence, for time-invariant plants, HSL- ∞ still maintains an advantage in this respect. To prevent possible spurious switching caused by resetting (we discuss qualitatively the situation), the basic scheme of HSL-R can be suitably modified in many ways, e.g., one can replace Π in the logic (5) with

$$\Pi_i(t) := \left\| \pi_i|_{t_k-T+1}^t \right\|_\infty, \quad t \in \mathbb{T}_k \quad (24)$$

where $T, T \geq 1$, indicates the minimum memory-length of the Π_i 's, i.e., the number of past performance records which are not discarded upon resetting. Clearly, T has to be chosen by trading off readiness of the algorithm in discarding past information versus false-alarm rate. ▼

VI. STABILITY UNDER TIME VARIATIONS OF THE PLANT PARAMETERS

This section shows how, even in the presence of plant variations, HSL-R makes it possible to achieve stability properties similar to those derived for LTI plants. Recall from Theorem 1 that a sufficient condition for stability is the existence of a finite constant Π^* such that $\Pi^k \leq \Pi^*$ for every $k \in \mathbb{Z}_+$. In the presence of plant variations, it is not immediate to conclude that such a property still holds. This is because a single controller able to

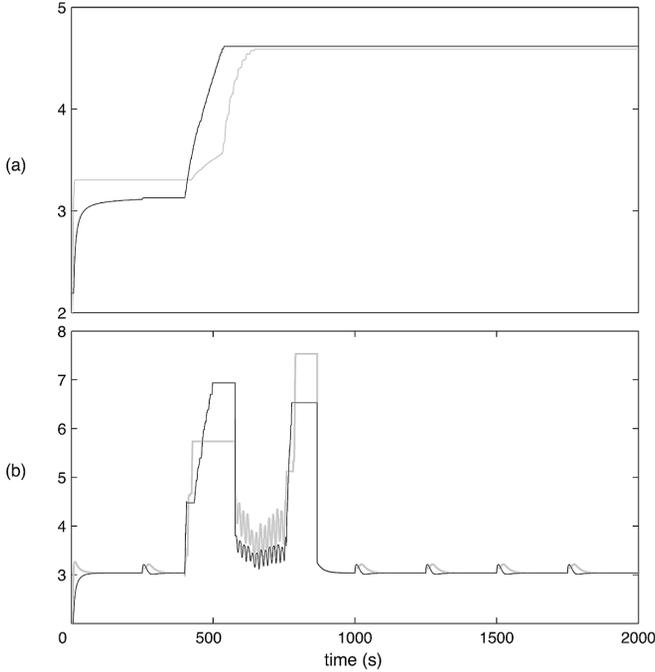


Fig. 5. Test functionals (Π_1 gray line, Π_2 black line). (a) Supervision based on HSL- ∞ . (b) Supervision based on HSL-R.

ensure stability over the whole uncertainty set need not exist, the set of stabilizing controllers changing with the plant. As shown hereafter, nonetheless, such a property holds whenever the plant variations are infrequent or satisfy a slow drift condition.

A. Infrequent Plant Changes

Let $\{\ell_c\}_{c \in \mathbb{Z}_+}$ denote the sequence of time instants at which a plant variation occurs, with $\ell_0 := 0$ by convention. Accordingly, $\mathbb{L}_c := \{\ell_c, \dots, \ell_{c+1} - 1\}$, $c \in \mathbb{Z}_+$, will denote the c th time interval over which the plant is constant. Although we can no longer use Π_{TI}^* in (18) to deduce that the switched system is stable, Lemma 3 ensures that for every $c \in \mathbb{Z}_+$ there exists a candidate index $s \in \underline{N}$ such that

$$\pi_s(t) + h \leq \Pi_{TI}^* + g_3 \|\zeta^{\ell_c - 1}\|_\lambda \lambda^{t - \ell_c + 1}, \quad \forall t \in \mathbb{L}_c \quad (25)$$

where g_3 is as in (18). Thus, for any given accuracy ν and provided that \mathbb{L}_c be large enough, the right hand side of (25) will eventually become smaller than $\Pi_{TI}^* + \nu$. In this respect, let

$$\mathbb{L}_c^\nu := \{t \in \mathbb{L}_c; g_3 \|\zeta^{\ell_c - 1}\|_\lambda \lambda^{t - \ell_c + 1} \leq \nu\}.$$

Then, if at least two resets occur over \mathbb{L}_c^ν , i.e., there is at least one k such that $\mathbb{T}_k \subseteq \mathbb{L}_c^\nu$ one can use (15) to conclude that at time ℓ_{c+1} we have

$$\|\zeta^{\ell_{c+1} - 1}\|_\lambda \leq Z(\Pi_{TI}^* + \nu) \quad (26)$$

where $Z(\cdot)$ is as in (20). Notice that one single reset would not be sufficient since the bound in (15) depends on both Π^k and Π^{k-1} . Although Theorem 1 cannot be invoked to conclude closed-loop stability (since the existence of a finite upper bound for $\{\Pi^k\}_{k \in \mathbb{Z}_+}$ is not apparent from boundedness of $\{\zeta^{\ell_c - 1}\}_{k \in \mathbb{Z}_+}$), this implication actually holds.

Lemma 5: Let the HSL-R switched system Σ_σ be based on the test functionals (7) and reset rule (19). Let A1–A4 hold. Further assume that

$$\forall c \in \mathbb{Z}_+ \quad \exists k \in \mathbb{Z}_+ \quad \text{such that} \quad \mathbb{T}_k \subseteq \mathbb{L}_c^\nu. \quad (27)$$

Then, $\Pi^k \leq \Pi_{TV}^*$ for every $k \in \mathbb{Z}_+$, with

$$\Pi_{TV}^* := \Pi_{TI}^* + g Z(\Pi_{TI}^* + \nu) \quad (28)$$

where $g := \max\{g_3, 2\mu^{-1}\}$. Hence, Σ_σ is stable.

Proof: See the Appendix. \blacksquare

In the light of Lemma 5, a sufficient condition for stability of the switched system is that the minimum interval between two consecutive plant variations (or plant *dwell-time*) be large enough to allow the fulfillment of the condition (27). This happens when the plant dwell-time is such that: 1) the transient term in (25) becomes smaller than ν ; 2) at least two resets occur afterwards. To see this, recall that items i); and 3) above imply (26). Let now ℓ_c^ν denote the first time instant of \mathbb{L}_c^ν , i.e., $\ell_c^\nu := \min\{t : t \in \mathbb{L}_c^\nu\}$. Accordingly, condition (27) amounts to requiring that, for any $c \in \mathbb{Z}_+$, ℓ_{c+1} is always greater or equal to ℓ_c^ν plus the time needed for two resetting to occur. In this respect, using (26) in the definition of \mathbb{L}_c^ν one has

$$\ell_c^\nu - \ell_c \leq \left\lceil \log_\lambda \frac{\nu}{g_3 Z(\Pi_{TI}^* + \nu)} \right\rceil.$$

Moreover, by simple induction argument, if condition (27) is satisfied up to a certain ℓ_c then, by (28), Π_{TV}^* is an upper bound on the smallest test functional over \mathbb{L}_c . In turns, in agreement with Lemma 4, this implies that after at most $2\Delta(\Pi_{TV}^*)$ steps subsequent to ℓ_c^ν the two required reset times occur. Summing the bound on $\ell_c^\nu - \ell_c$ with $2\Delta(\Pi_{TV}^*)$, we have next result.

Theorem 3: Let the HSL-R switched system Σ_σ be based on the test functionals (7) and the resetting rule (19). Let A1–A4 hold. Then, Σ_σ is stable provided that

$$\ell_{c+1} - \ell_c \geq \left\lceil \log_\lambda \frac{\nu}{g_3 Z(\Pi_{TI}^* + \nu)} \right\rceil + 2\Delta(\Pi_{TV}^*) \quad (29)$$

holds for every $c \in \mathbb{Z}_+$. \blacksquare

Notice that the dwell-time L_c depends, through Π_{TI}^* , on the disturbance amplitude. In words, the larger the level of the disturbances, the larger (at least conceptually) the time required to select an appropriate controller, hence the larger the interval required between two successive plant variations. Hence, L_c cannot be known *a priori* unless knowledge on a disturbance upper bound is assumed. Nonetheless, Theorem 3 guarantees that for any disturbance level there exists a sufficiently large plant dwell-time such that stability is not destroyed.

B. Slow Parameter Drift

Let $\theta(t) \in \mathbb{R}^{2n^*}$, $t \in \mathbb{Z}_+$, denote the vector of time-varying parameters composed by the coefficients of $A_t(d)$ and $B_t(d)$. Consistent with this notation, we can rewrite A1 as requiring that $\theta(t) \in \Theta$, $\forall t \in \mathbb{Z}_+$ for some compact set Θ . Assume now that the parameter vector $\theta(t)$ takes values inside Θ with bounded variation rate, i.e.,

$$\theta(t) \in \Theta, \quad |\theta(t+1) - \theta(t)| \leq \delta, \quad \forall t \in \mathbb{Z}_+ \quad (30)$$

where $\delta > 0$ defines the variation rate.

In order to extend the results of the previous section to this setting, it is convenient to define a collection of sets $\{\Theta_1, \dots, \Theta_N\}$, one for each controller index i , with the following properties:

- 1) all the sets $\Theta_i, i \in \underline{N}$ are compact;
- 2) for all $\theta \in \Theta_i$, the frozen-time feedback loop $(P(\theta)/C_i)$ is λ -stable;
- 3) the collection of sets $\{\Theta_1, \dots, \Theta_N\}$ covers the set Θ with overlap $\beta > 0$ in the sense that, for any $\theta \in \Theta$, there exists at least one index $i \in \underline{N}$ such that $\theta \in \Theta_i$ and

$$\text{dist}(\theta, \partial\Theta_i) \geq \beta,$$

where $\partial\Theta_i$ denotes the boundary of Θ_i and $\text{dist}(\cdot, \cdot)$ the Euclidean point-to-set distance.

Thanks to the compactness of the set Θ , by resorting to standard topology arguments, it can be shown that a collection of sets satisfying such properties always exists under the stated assumptions (the existence of a strictly positive β being connected to the existence of a strictly positive Lebesgue number for any open cover of the compact set Θ). More specifically, the following result holds.

Proposition 1: Under assumptions A1 and A2, there always exist an overlap $\beta > 0$ and a collection of sets $\{\Theta_1, \dots, \Theta_N\}$ for which properties 1)–3) hold.

Proof: See the Appendix. ■

Notice that the overlap β can be increased by augmenting the set \mathcal{E} with additional controllers. We proceed now to derive a bound on cost $\pi_i(t)$ that is valid on those time intervals for which the parameter vector $\theta(t)$ belongs to Θ_i . To this end, we exploit the well-known fact that, although stability of all the frozen-time loops $(P(\theta)/C_i)$, $\theta \in \Theta_i$ need not imply stability of the time-varying loop $(P(\theta(t))/C_i)$, $\theta(t) \in \Theta_i$, such a property holds provided that the variation rate δ be small enough. More specifically, the following result can be stated.

Lemma 6: Let the HSL-R switched system Σ_σ be based on the test functionals (7). Let A1–A4 hold and further assume that on a given interval $\{\tau, \tau + 1, \dots, T\}$ the parameter vector $\theta(t)$ always remains inside one of the sets $\Theta_i, i \in \underline{N}$. Then, there exist positive constants $\delta_{\max}, c_0, c_1, c_2$ and c_3 such that, when the variation rate does not exceed the threshold δ_{\max} , i.e., $\delta \leq \delta_{\max}$, cost $\pi_j(t)$ can be upper bounded as

$$\pi_i(t) \leq c_0 + c_1 |\xi_P| \lambda^{t+1} + c_2 \|w^t\|_\lambda + c_3 \|\zeta^{\tau-1}\|_\lambda \lambda^{t-\tau+1} \quad \forall t \in \{\tau, \tau + 1, \dots, T\}.$$

Proof: See the Appendix. ■

Comparing Lemma 6 with its time-invariant counterpart Lemma 3, it must be noted that the constants c_0, c_1, c_2 , and c_3 will now depend on δ and hence, in general, will be greater than the constants g_0, g_1, g_2 , and g_3 in (17) pertaining to the frozen-time analysis.

As shown hereafter, in view of Proposition 1 and Lemma 6, conclusions similar to those derived in Section VI-A hold true in case the plant variations satisfy a slow drift conditions. The idea is to derive an inequality of the type given in (25). To this end, we recursively construct a sequence of time instants $\{\hat{\ell}_q\}_{q \in \mathbb{Z}_+}$ as follows: $\hat{\ell}_0$ is set equal to 0; given $\hat{\ell}_q, \hat{\ell}_{q+1}$ is defined as the

largest time instant such that there exists at least one index $i \in \underline{N}$ for which $\theta(t) \in \Theta_i$ for any $t \in \{\hat{\ell}_q, \hat{\ell}_q + 1, \dots, \hat{\ell}_{q+1} - 1\}$. In words, if $\theta(\hat{\ell}_q)$ belongs to Θ_i then $\hat{\ell}_{q+1}$ denotes the first time instant at which a variation in the plant parameter vector can cause θ to leave Θ_i . In view of property 3) above, we have that $\hat{\ell}_{q+1} - \hat{\ell}_q \geq \lfloor \beta/\delta \rfloor$ for any $q \in \mathbb{Z}_+$. Hence, we have from Lemma 6 that if $\delta \leq \delta_{\max}$ then there exists a candidate index $s \in \underline{N}$ such that

$$\pi_s(t) \leq c_0 + c_1 |\xi_P| \lambda^{t+1} + c_2 \|w^t\|_\lambda + c_3 \|\zeta^{\hat{\ell}_q-1}\|_\lambda \lambda^{t-\hat{\ell}_q+1} \quad \forall t \in \{\hat{\ell}_q, \dots, \hat{\ell}_{q+1} - 1\}.$$

Define

$$\hat{\Pi}_{TI}^* := c_0 + c_1 |\xi_P| + (1 - \lambda^2)^{-1/2} c_2 \|w\|_\infty + h. \quad (31)$$

We therefore obtain $\pi_s(t) + h \leq \hat{\Pi}_{TI}^* + c_3 \|\zeta^{\hat{\ell}_q-1}\|_\lambda \lambda^{t-\hat{\ell}_q+1}$ for all $t \in \{\hat{\ell}_q, \dots, \hat{\ell}_{q+1} - 1\}$. This formula clearly parallels the one in (25) for the case of infrequent plant changes. Accordingly, let $\hat{\Pi}_{TV}^* := \hat{\Pi}_{TI}^* + c Z(\hat{\Pi}_{TI}^* + \nu)$, where $c := \max\{c_3, 2\mu^{-1}\}$ and further assume that

$$\left\lfloor \frac{\beta}{\delta} \right\rfloor \geq \left\lceil \log_\lambda \frac{\nu}{c_3 Z(\hat{\Pi}_{TI}^* + \nu)} \right\rceil + 2\Delta(\hat{\Pi}_{TV}^*) \\ =: \Xi(\hat{\Pi}_{TV}^*). \quad (32)$$

Since (32) implies that $\hat{\ell}_{q+1} - \hat{\ell}_q \geq \Xi(\hat{\Pi}_{TV}^*)$ for all $q \in \mathbb{Z}_+$, we have from Theorem 3 that condition (32) is sufficient for stability to hold. Recall now that in order to derive (32) we used the assumption that $\delta \leq \delta_{\max}$. Combining this latter inequality with the fact that $\beta/\delta \geq \Xi(\hat{\Pi}_{TV}^*)$ implies $\lfloor \beta/\delta \rfloor \geq \Xi(\hat{\Pi}_{TV}^*)$ because $\Xi(\hat{\Pi}_{TV}^*)$ is a positive integer, we have the following stability result in terms of allowed parameter variation rate.

Theorem 4: Let the HSL-R switched system Σ_σ be based on the test functionals (7) and the resetting rule (19). Let A1–A4 hold. Then, Σ_σ is stable provided that

$$\delta \leq \min \left\{ \delta_{\max}, \frac{\beta}{\Xi(\hat{\Pi}_{TV}^*)} \right\}. \quad (33)$$

C. Example 2

We consider a simple model of a robot arm [23]. The transfer function from the control input (motor current) to measurement output (motor angular velocity) is

$$P(s) = \frac{k_m(J_a s^2 + ds + k)}{J_a J_m s^3 + d(J_a + J_m)s^2 + k(J_a + J_m)s}$$

with $J_a \in [0.0001, 0.02]$, $J_m = 0.002$, $d = 0.0001$, $k = 100$, and $k_m = 0.5$. The purpose of the control system is to control the angular velocity responses for all admissible values of the moment of inertia J_a . Two controllers $C_1(d) = (0.0146 - 0.0120d - 0.0025d^2)/(1-d)$ and $C_2(d) = (0.0155 + 0.0113d - 0.0044d^2 - 0.0016d^3)/(1-d)$ have been designed. To illustrate the advantages of HSL-R over HSL- ∞ in the presence of persistent plant variations, we consider the case where J_a switches between 0.0001 (only C_1 is stabilizing) to 0.02 (only

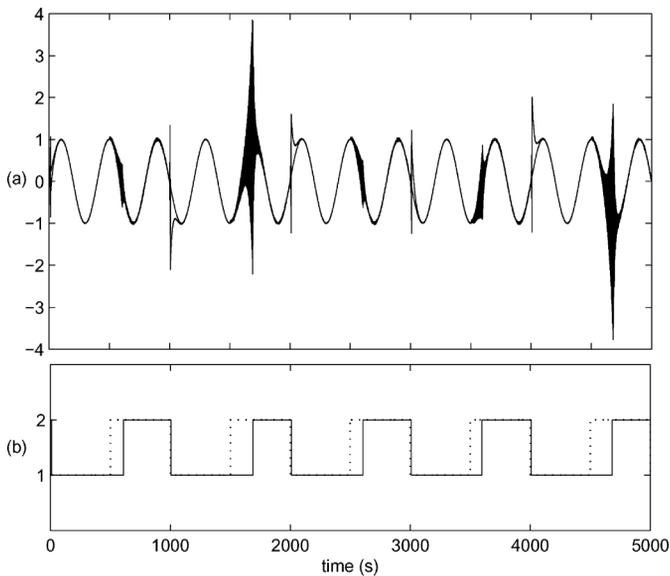


Fig. 6. Supervision based on HSL-R. (a) Plant Output. (b) Plant switching sequence (dotted line) and controller switching sequence (solid line).

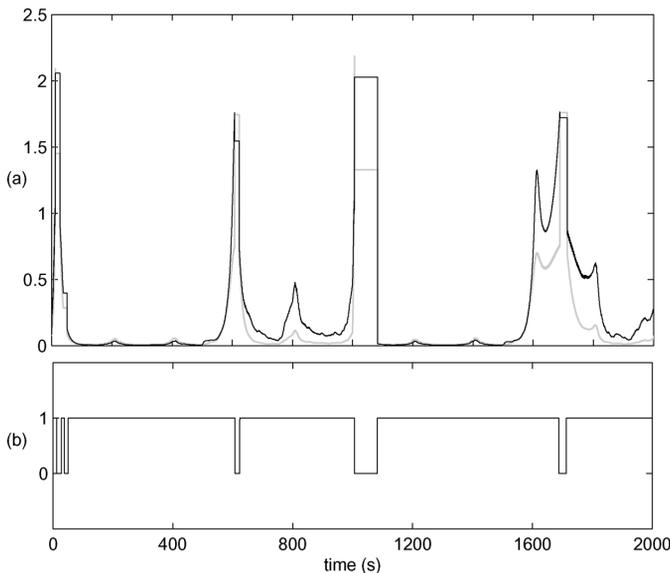


Fig. 7. Detail of HSL-R on the time interval $[0, 2000]$ s. (a) Test functionals (Π_1 gray line, Π_2 black line). (b) Resetting sequence (1 stands for resetting).

C_2 is stabilizing) every 500 s. The parameters h, λ, μ and ϵ are the same as in Example 1. Fig. 6 depicts the plant output response to a sinusoidal reference command and disturbances modeled by taking n_u and n_y to be uniformly distributed in $[-0.01, 0.01]$. In accordance with Fig. 7, the resetting mechanism prevents the test functionals from growing unbounded, thus preserving control reconfiguration. This is not the case for HSL- ∞ . In fact, the ever-growing memory of HSL- ∞ , which is reflected in the monotonicity of the test functionals, implies that upon each plant variation the cost level necessary for falsifying controllers increases. This unavoidably leads to ever-increasing transients, as is apparent from Fig. 8. We finally point out that, although the spikes observed in Fig. 7 are mostly due to the suddenly unstable dynamics, better transients are likely to be achieved via bumpless transfer techniques. This constitutes an important area for future investigation.

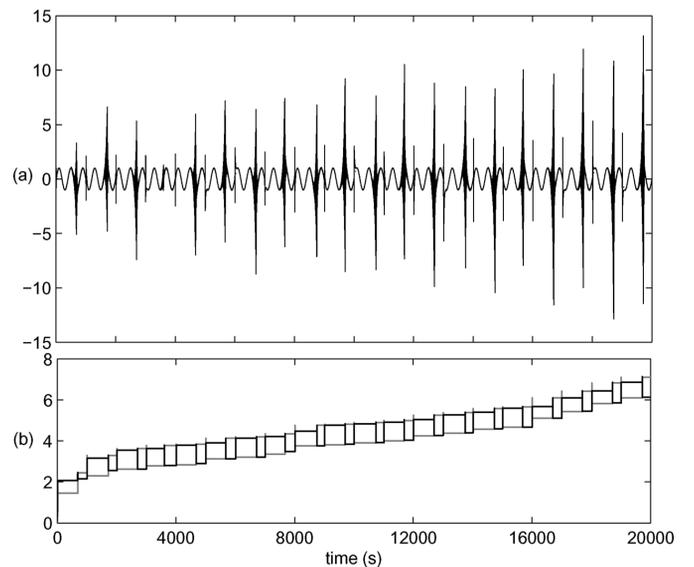


Fig. 8. Supervision based on HSL- ∞ (a) Plant output. (b) Test functionals (Π_1 gray line, Π_2 black line).

VII. CONCLUSION

Consideration has been given to the control of uncertain time-varying plants by means of adaptive switching control techniques. We have introduced a novel class of algorithms based on hysteresis switching, which, when combined with appropriate test functionals, makes it possible to achieve stability for time-varying systems under large plant modeling errors, unmodeled dynamics and persistent disturbances. The characterizing feature of this novel scheme is that the supervisor orchestrates the switching by means of a specially devised mechanism which, from time to time, determines whether past data are still relevant to achieve closed-loop stability. In particular, this mechanism consists of a logic according to which the test functionals are reset to zero whenever the recorded data indicate that the information contained in the test functionals is no longer needed to achieve stability. Although the major emphasis has been on the stabilization of time-varying systems, simulation results indicate that this supervisory scheme compares favorably with HSL- ∞ even when applied to time-invariant systems, since it does not rely on a finite switching stopping-time. These results lend themselves to be extended in various directions. First, the idea of selecting online the memory of the test functionals can be extended to more elaborated rules than a simple resetting logic. As a second point, notice that the motivation for considering supervisory schemes based on unfalsified control was mainly dictated by the goal of achieving robustness against large modeling uncertainties. Nonetheless, the approach here introduced could be used within supervisory schemes alternative to model-free switching control. In this respect, it is known that the adoption of model-based test functionals can significantly improve the transients performance [7], [22]. Hence, a natural question arises on whether resetting logics can be adapted to supervisory schemes based on multiple models.

APPENDIX

Throughout the appendix, we shall make use of the following properties:

- 1) $\|x|_{\alpha}^t\| \leq \|x|_{\alpha}^t\|_{\lambda} \lambda^{-(t-\alpha)}$.
- 2) $\|x|_{\alpha}^t\|_{\lambda} \leq \|x|_{h+1}^t\|_{\lambda} + \|x|_{\alpha}^h\|_{\lambda} \lambda^{t-h}$.
- 3) $\|x|_{\alpha}^{t-h}\|_{\lambda} \leq \|x|_{\alpha}^{t-\ell}\|_{\lambda} \lambda^{-(h-\ell)}$, $\ell \leq h$.

To prove Lemma 2 we use the following result.

Proposition 2: Under A1, there exist finite nonnegative constants g_0, g_1 and g_2 such that

$$\|\zeta^t\|_{\lambda} \leq g_0 |\xi_P| \lambda^{t+1} + g_1 \|w^t\|_{\lambda} + g_2 \|\zeta^{t-1}\|_{\lambda} \quad \forall t \in \mathbb{Z}_+. \quad (34)$$

Proof of Proposition 2: For any LTI feedback loop (P_*/C_i) we have

$$\begin{bmatrix} R_i & -S_i \\ B_* & A_* \end{bmatrix} \zeta = \begin{bmatrix} 0 & 0 & 0 \\ A_* & -B_* & -A_* \end{bmatrix} w. \quad (35)$$

By A1, we also get $|\zeta(t)| \leq c_0 \|\zeta|_{t-q}^{t-1}\| + c_1 \|w|_{t-q}^t\|$, $\forall t \in \mathbb{Z}_+$ for some positive constants c_0 and c_1 , with $q = \max\{n^*, m\}$. By 1), letting $c_2 := c_0 \lambda^{-(q-1)}$ and $c_3 := c_1 \lambda^{-q}$ we get

$$\begin{aligned} \|\zeta^t\|_{\lambda} &\leq |\zeta(t)| + \|\zeta^{t-1}\|_{\lambda} \\ &\leq (c_2 + \lambda) \|\zeta^{t-1}\|_{\lambda} + c_3 \|w|_{t-q}^t\|_{\lambda}, \quad \forall t \in \mathbb{Z}_+ \end{aligned}$$

where the second inequality follows since $\zeta(k) = 0$, $k = -1, -2, \dots$. The proof follows using 2) and recalling that $n_u(k) = u_P(k)$, $n_y(k) = y_P(k)$ $k = -1, \dots, -n^*$. ■

Proof of Lemma 2: Consider an arbitrary $k \in \mathbb{Z}_+$ and let $\mathbb{T}_{k[j]} := \{t_{k[j]}, \dots, t_{k[j+1]} - 1\}$, $t_{k[0]} := t_k$, represent the j th subinterval of \mathbb{T}_k over which the switching signal is constant.

Basic Recursive Equation: First, notice that

$$\begin{aligned} \|\zeta^t\|_{\lambda} &\leq \|\zeta_{\sigma(t)}^t\|_{\lambda} + \|(v_{\sigma(t)} - r)^t\|_{\lambda} \\ &\leq \pi_{\sigma(t)}(t) (\mu + \|r^t\|_{\lambda}) \\ &\quad + (1 + \pi_{\sigma(t)}(t)) \|(v_{\sigma(t)} - r)^t\|_{\lambda}, \quad \forall t \in \mathbb{Z}_+. \end{aligned}$$

Consider now an arbitrary $\mathbb{T}_{k[j]}$. Accordingly, for some $i \in \underline{N}$, one has $\sigma(t) = i$, $\forall t \in \mathbb{T}_{k[j]}$. Let $\delta_i := v_i - r$. By exploiting 2) with respect to δ_i , with $\alpha = 0$ and $h = t_{k[j]} - 1$, we get

$$\begin{aligned} \|\delta_i^t\|_{\lambda} &\leq \|\delta_i|_{t_{k[j]}}^t\|_{\lambda} + \|\delta_i^{t_{k[j]}-1}\|_{\lambda} \lambda^{t-t_{k[j]}+1} \\ &\leq g_i \|\delta_i|_{t_{k[j]}-q}^{t_{k[j]}-1}\|_{\lambda} \lambda^{t-t_{k[j]}+1} + \|\delta_i^{t_{k[j]}-1}\|_{\lambda} \lambda^{t-t_{k[j]}+1} \\ &\leq (\tilde{g}_i \|\delta_i|_{t_{k[j]}-q}^{t_{k[j]}-1}\|_{\lambda} + \|\delta_i^{t_{k[j]}-1}\|_{\lambda}) \lambda^{t-t_{k[j]}+1} \\ &\leq \hat{g}_i \|\delta_i^{t_{k[j]}-1}\|_{\lambda} \lambda^{t-t_{k[j]}+1}, \quad \forall t \in \mathbb{T}_{k[j]}. \quad (36) \end{aligned}$$

In (36), we made use of the following facts: The second inequality follows, for some finite positive constant g_i , because $S_i(d)\delta_i(t) = 0$ for every $t \in \mathbb{T}_{k[j]}$ and $S_i(d)$ is a λ -Hurwitz polynomial; the third inequality is obtained using 1) and letting $\tilde{g}_i := g_i \lambda^{-q}$; the last inequality follows with $\hat{g}_i := 1 + \tilde{g}_i$ since, for all candidate indices, (6) is initialized at time zero from zero initial conditions, *viz.* $\delta_i(k) = 0$, $k = -1, -2, \dots$. By A3 one sees that the map from ζ to δ_i

$$S_i(d) \delta_i(t) = S_i(d) (v_i(t) - r(t)) = [R_i(d) - S_i(d)] \zeta(t)$$

has bounded λ -weighted ℓ_2 -norm. In particular, since (6) is initialized at time zero from zero initial conditions, it readily follows that $\|\delta_i^t\|_{\lambda} \leq \bar{g}_i \|\zeta^t\|_{\lambda}$, $\forall t \in \mathbb{Z}_+$, $\forall i \in \underline{N}$, and some finite nonnegative constant \bar{g}_i . Letting $\hat{g} := \max_{i \in \underline{N}} \{\hat{g}_i\}$ and $\bar{g} := \max_{i \in \underline{N}} \{\bar{g}_i\}$, we finally get

$$\begin{aligned} \|\zeta^t\|_{\lambda} &\leq \pi_{\sigma(t)}(t) (\mu + \|r^t\|_{\lambda}) \\ &\quad + g (1 + \pi_{\sigma(t)}(t)) \|\zeta^{t_{k[j]}-1}\|_{\lambda} \lambda^{t-t_{k[j]}+1} \\ &\quad \forall t \in \mathbb{T}_{k[j]}, \quad j = 0, \dots, \mathfrak{N}_k \quad (37) \end{aligned}$$

where $g := \hat{g} \bar{g}$.

Upper-Bound on Data Between Switching: From Lemma 1, we have $\pi_{\sigma(t+1)}(t) \leq \Pi^k$ for every $t \in \mathbb{T}_k$. Then,

$$\begin{aligned} \|\zeta^t\|_{\lambda} &\leq \Pi^k (\mu + \|r^t\|_{\lambda}) \\ &\quad + g (1 + \Pi^k) \|\zeta^{t_{k[j]}-1}\|_{\lambda} \lambda^{t-t_{k[j]}+1} \\ &\quad \forall t \in \mathbb{T}_{k[j]} \setminus \{t_{k[j+1]} - 1\}, \quad j = 0, \dots, \mathfrak{N}_k. \quad (38) \end{aligned}$$

In words, the performance signal related to the switched-on controller cannot exceed Π^k , apart from the time instants right before switching. Notice that by Lemma 1 this may happen at most \mathfrak{N}_k -times, since, after \mathfrak{N}_k switching, no more switching occurs over \mathbb{T}_k .

Proof of (12): In order to prove (12) we use (38) and exploit the results of Proposition 2 at the time instants $\{t_{k[j+1]} - 1\}$. Without loss of generality, let $g_2, g \geq 1$, g_2 and g as in (34) and, respectively, (38). It is also convenient to let $g_3 := \max\{g_0, g_1\}$ with g_0 and g_1 as in (34), and write in more compact form $L(t, \xi_P, \mu, w) := \mu + |\xi_P| \lambda^{t+1} + \|w^t\|_{\lambda}$. Then, combining (34) and (38), we get

$$\begin{aligned} \|\zeta^t\|_{\lambda} &\leq g_2 \|\zeta^{t-1}\|_{\lambda} + g_3 L(t, \xi_P, \mu, w) \\ &\leq (g_2 \Pi^k + g_3) L(t, \xi_P, \mu, w) \\ &\quad + g_2 g (1 + \Pi^k) \|\zeta^{t_{k[j]}-1}\|_{\lambda} \lambda^{t-t_{k[j]}+1} \\ &\leq G_k (\|\zeta^{t_{k[j]}-1}\|_{\lambda} \lambda^{t-t_{k[j]}+1} + L(t, \xi_P, \mu, w)) \\ &\quad \forall t \in \mathbb{T}_{k[j]}, \quad j = 0, \dots, \mathfrak{N}_k. \quad (39) \end{aligned}$$

In the second inequality we used the fact that $\|r^t\|_{\lambda} \leq \|w^t\|_{\lambda}$, while $G_k := g_3 + g_2 g(1 + \Pi^k)$.

By induction, it is easy to show that

$$\begin{aligned} \|\zeta^t\|_{\lambda} &\leq \left(\sum_{m=1}^{j+1} G_k^m \right) (\|\zeta^{t_{k[j]}-1}\|_{\lambda} \lambda^{t-t_{k[j]}+1} + L(t, \xi_P, \mu, w)) \\ &\quad \forall t \in \bigcup_{m=0}^j \mathbb{T}_{k[m]}, \quad j = 0, \dots, \mathfrak{N}_k. \quad (40) \end{aligned}$$

Hence, (40) holds true for $j+1$, and the proof follows by letting $\mathfrak{g}(\Pi^k) := \sum_{m=1}^{\mathfrak{N}_k+1} G_k^m$. ■

Proof of Lemma 3: Let $\theta(t) \in \mathbb{R}^{2n^*}$, $t \in \mathbb{Z}_+$, denote the vector of time-varying parameters composed by the coefficients of A_t and B_t . Consistently, we can rewrite A1 as requiring that $\theta(t) \in \Theta$, $\forall t \in \mathbb{Z}_+$ for some compact set Θ .

By assumption, $\theta(t) \equiv \theta$ for all $t \in \{\tau, \tau+1, \dots, T\}$ and some $\theta \in \Theta$, *i.e.*, $P = P(\theta) = B(\theta)/A(\theta)$ for all $t \in \{\tau, \tau+1, \dots, T\}$. Consider now that, under assumption A3, the virtual

references in (6) are well-defined. Thus, combining (6) and (1) we can write $\Phi_{\theta/i} \zeta_i = \Psi_\theta w_i$, where

$$\Phi_{\theta/i} := \begin{bmatrix} R_i & -S_i \\ B(\theta) & A(\theta) \end{bmatrix}, \Psi_\theta := \begin{bmatrix} 0 & 0 & 0 \\ A(\theta) & -B(\theta) & -A(\theta) \end{bmatrix}. \quad (41)$$

Consider next that, for any candidate controller C_i such that $(P(\theta)/C_i)$ is λ -stable there exists a polynomial matrix $I_{\theta/i}(d)$ such that

$$\zeta_i(t) = \Phi_{\theta/i}^{-1}(d) I_{\theta/i}(d) | \xi_i|_{\tau-q}^{\tau-1} + \Phi_{\theta/i}^{-1}(d) \Psi_\theta(d) w_i(t) \quad (42)$$

where $\xi_i := [\zeta_i^t \ w_i^t]'$, $q = \max\{n^*, m\}$, and $\Phi_{\theta/i}$ is such that $\det \Phi_{\theta/i}$ is a λ -Hurwitz polynomial. Consider now that, by assumption A2, for any $\theta_j \in \Theta$ there exist a candidate controller C and an open ball $\tilde{\Theta}_j$ around θ_j such that $(P(\theta)/C)$ is λ -stable for all $\theta \in \tilde{\Theta}_j$. Thus, an infinite open cover for Θ exists. In turn, in view of assumptions A1 and A2, this implies the existence of a finite (closed) cover $\Theta_1, \dots, \Theta_N$ for Θ such that, for each j , all the plants $P(\theta)$, $\theta \in \Theta_j$, are stabilized by controller C_j (see the proof of Proposition 1 below).

As a consequence, from (42), it is therefore immediate to conclude that there exist positive reals m_{0j} and m_{1j} such that, for any $\theta \in \Theta_j$

$$\| \zeta_j|_\tau^t \|_\lambda \leq m_{0j} \| \xi_j|_{\tau-q}^{\tau-1} \|_\lambda \lambda^{t-\tau+1} + m_{1j} \| w_j|_\tau^t \|_\lambda \quad \forall t \in \{\tau, \tau+1, \dots, T\} \quad (43)$$

holds true where $m_{0j} := \sup_{\theta \in \Theta_j} \| \Phi_{\theta/j}^{-1} I_{\theta/j} \|_{\infty, \lambda}$, $m_{1j} := \sup_{\theta \in \Theta_j} \| \Phi_{\theta/j}^{-1} \Psi_\theta \|_{\infty, \lambda}$ and $\| H \|_{\infty, \lambda}$ denotes the λ -weighted \mathcal{H}_∞ norm of H . Hence, (43) holds true over Θ with finite positive constants $m_0 := \max_{j \in \underline{N}} m_{0j}$ and $m_1 := \max_{j \in \underline{N}} m_{1j}$. By ii) with $\alpha = 0$ and $h = \tau - 1$ we have $\| \zeta_s^t \|_\lambda \leq \| \zeta_s|_\tau^t \|_\lambda + \| \zeta_s^{\tau-1} \|_\lambda \lambda^{t-\tau+1}$. By i), we also have

$$\begin{aligned} \| \xi_s|_{\tau-q}^{\tau-1} \| &\leq \| \xi_s|_{\tau-q}^{\tau-1} \|_\lambda \lambda^{-(q-1)} \\ &\leq \left(\| \zeta_s|_{\tau-q}^{\tau-1} \|_\lambda + \| w_s|_{\tau-q}^{\tau-1} \|_\lambda \right) \lambda^{-(q-1)}. \end{aligned} \quad (44)$$

Consider next that $\| \zeta_s|_{\tau-q}^{\tau-1} \|_\lambda = \| \zeta_s^{\tau-1} \|_\lambda$ because of the zero initial condition constraint, while

$$\begin{aligned} \| w_s|_{\tau-q}^{\tau-1} \|_\lambda &\leq \| w_s^{\tau-1} \|_\lambda + \| w_s|_{\tau-q}^{\tau-1} \|_\lambda \lambda^\tau \\ &\leq \| w_s^{\tau-1} \|_\lambda + |\xi_P| \lambda^\tau \end{aligned}$$

the first inequality being obtained from 2) with $\alpha = \tau - q$ and $h = -1$. Overall, we get

$$\begin{aligned} \| \zeta_s^t \|_\lambda &\leq m_1 \| w_s^t \|_\lambda + m_2 \| \zeta_s^{\tau-1} \|_\lambda \lambda^{t-\tau+1} \\ &\quad + m_3 |\xi_P| \lambda^{t+1} + m_3 \| w_s^{\tau-1} \|_\lambda \lambda^{t-\tau+1} \end{aligned} \quad (45)$$

where $m_2 := 1 + m_0 \lambda^{-(q-1)}$ and $m_3 := m_0 \lambda^{-(q-1)}$. Noting that $\| \zeta_s^{\tau-1} \|_\lambda \leq \| \zeta_s^{\tau-1} \|_\lambda + \| v_s^{\tau-1} \|_\lambda + \| w^{\tau-1} \|_\lambda$ and $\| w_s^{\tau-1} \|_\lambda \leq \| v_s^{\tau-1} \|_\lambda + \| w^{\tau-1} \|_\lambda$, and using 3) with respect to v_s and w , with $\alpha = 0$, $h = t - \tau + 1$ and $\ell = 0$, we get

$$\begin{aligned} \| \zeta_s^t \|_\lambda &\leq m_4 \| v_s^t \|_\lambda + m_2 \| \zeta_s^{\tau-1} \|_\lambda \lambda^{t-\tau+1} \\ &\quad + m_3 |\xi_P| \lambda^{t+1} + m_4 \| w^t \|_\lambda \end{aligned} \quad (46)$$

where $m_4 := m_1 + m_2 + m_3$. Hence, (17) immediately follows from the definition of the π_i 's. ■

Proof of Lemma 4: Consider the same notation as in the proof of Lemma 2. Since by assumption $\Pi^k \leq \Pi^*$ for every $k \in \mathbb{Z}_+$, the number of switching on every interval \mathbb{T}_k is upper bounded by \mathfrak{N}_* . Now, using (37) and (38) we obtain

$$\begin{aligned} \pi_*(t) &\leq \pi_{\sigma(t+1)}(t) + \epsilon_{k[j]}(t) \frac{1}{\mu + \| r^t \|_\lambda} \\ &\quad \forall t \in \mathbb{T}_{k[j]} \setminus \{t_{k[j+1]} - 1\} \end{aligned} \quad (47)$$

for every $j \leq \mathfrak{N}_k$, where

$$\begin{aligned} \epsilon_{k[j]}(t) &:= g \left(1 + \pi_{\sigma(t+1)}(t) \right) \| \zeta^{t_{k[j]}-1} \|_\lambda \lambda^{t-t_{k[j]}+1} \\ &< G_k \| \zeta^{t_{k[j]}-1} \|_\lambda \lambda^{t-t_{k[j]}+1} \\ &\quad \forall t \in \mathbb{T}_{k[j]} \setminus \{t_{k[j+1]} - 1\}. \end{aligned}$$

We first derive an upper bound for $\epsilon_{k[j]}$. By (40)

$$\begin{aligned} \| \zeta^{t_{k[j]}-1} \|_\lambda &\leq \left(\sum_{m=1}^{\mathfrak{N}_k} G_k^m \right) \left(\| \zeta^{t_k-1} \|_\lambda \lambda^{t_{k[j]}-t_k} \right. \\ &\quad \left. + L(t_{k[j]} - 1, \xi_P, \mu, w) \right) \end{aligned}$$

and, hence,

$$\begin{aligned} \epsilon_{k[j]}(t) &< \mathfrak{g}(\Pi^k) \left(\| \zeta^{t_k-1} \|_\lambda \lambda^{t_{k[j]}-t_k} \right. \\ &\quad \left. + L(t_{k[j]} - 1, \xi_P, \mu, w) \right) \lambda^{t-t_{k[j]}+1} \\ &\leq \mathfrak{g}(\Pi^k) \left\{ (\Pi^{k-1} + \epsilon) (\mu + \| r^{t_k-1} \|_\lambda) \lambda^{t_{k[j]}-t_k} \right. \\ &\quad \left. + L(t_{k[j]} - 1, \xi_P, \mu, w) \right\} \lambda^{t-t_{k[j]}+1} \end{aligned}$$

where the second inequality follows from the definition of admissible resetting times [see (15)]. Consider further that $\| r^{t_k-1} \|_\lambda \leq (1 - \lambda^2)^{-1/2} \| w \|_\infty$, and that $L(t, \xi_P, \mu, w) \leq \mu + |\xi_P| + (1 - \lambda^2)^{-1/2} \| w \|_\infty$ for every t , from which we finally obtain

$$\begin{aligned} \epsilon_{k[j]}(t) &< \lambda^{t-t_{k[j]}+1} \left\{ \mathfrak{g}(\Pi^*) |\xi_P| \right. \\ &\quad \left. + (1 - \lambda^2)^{-1/2} \mathfrak{g}(\Pi^*) (1 + \Pi^* + \epsilon) (\mu + \| w \|_\infty) \right\} \\ &= \lambda^{t-t_{k[j]}+1} Z(\Pi^*) \end{aligned}$$

where $Z(\cdot)$ is as in (20). Consider now that, because of (47), we have $\epsilon_{k[j]}(t) > \epsilon\mu$ for every $t \in \mathbb{T}_{k[j]} \setminus \{t_{k[j+1]} - 1\}$, and, hence,

$$t_{k[j+1]} \leq t_{k[j]} + \left\lceil \log_\lambda \frac{\epsilon\mu}{Z(\Pi^*)} \right\rceil. \quad (48)$$

Indeed, if the latter condition were not satisfied we would then have a reset before $t_{k[j+1]}$, contradicting the fact that $t_{k[j+1]} \leq t_{k[\mathfrak{N}_k+1]} = t_{k+1}$. Applying (48) recursively, we get $t_{k[j]} \leq t_k + j \lceil \log_\lambda \epsilon\mu/Z(\Pi^*) \rceil$. The claim follows by recalling that the number of switching is upper bounded by \mathfrak{N}_* . ■

Proof of Lemma 5: Consider the subsequence $\{q_c\}_{c \in \mathbb{Z}_+}$ of resets $q_c := \max_{k \in \mathbb{Z}_+} \{t_k; t_k \leq \ell_c\}$. Notice that, under (27), $\{q_c\}_{c \in \mathbb{Z}_+}$ exists well-defined since $q_0 = 0$ and $q_i < q_j$ if $i < j$. Let $\mathbb{Q}_c := \{q_c, \dots, q_{c+1} - 1\}$ and note that $\cup_{c \in \mathbb{Z}_+} \mathbb{Q}_c = \mathbb{Z}_+$. Thus, in order to prove that the switched system is stable, it is

sufficient to prove that the upper bound Π_{TV}^* holds over each interval \mathbb{Q}_c since this implies that the same upper bound will hold over each \mathbb{T}_k .

Decompose \mathbb{Q}_c as $\mathbb{Q}_c = (\mathbb{Q}_c \cap \mathbb{L}_{c-1}) \cup (\mathbb{Q}_c \cap \mathbb{L}_c)$ and first assume that the interval $\mathbb{Q}_c \cap \mathbb{L}_{c-1}$ is nonempty. By definition q_c necessarily belongs to \mathbb{L}_{c-1} . Then, from (25) it follows that on this interval one has the upper bound $\Pi_{TI}^* + \nu$. Thus, (12) implies

$$\|\zeta^t\|_\lambda \leq \mathfrak{g}(\Pi_{TI}^* + \nu) (\mu + |\xi_P| \lambda^{t+1} + \|w^t\|_\lambda + \|\zeta^{q_c-1}\|_\lambda \lambda^{t-q_c+1}) \quad \forall t \in \mathbb{Q}_c \cap \mathbb{L}_{c-1}. \quad (49)$$

Indeed, (49) follows immediately by extending the conclusions of Lemma 2 to any truncation of a resetting interval (\mathbb{Q}_c in this case). In addition, by (27) there exists a resetting interval, say \mathbb{T}_{k-1} , right before \mathbb{Q}_c , which is contained into \mathbb{L}_{c-1} and therefore such that $\Pi^{k-1} \leq \Pi_{TI}^* + \nu$. Since by virtue of (19) the sequence of reset times is admissible, the sequence $\{\zeta^{q_c-1}\}_{c \in \mathbb{Z}_+}$ is such that $\|\zeta^{q_c-1}\|_\lambda \leq (\Pi_{TI}^* + \nu + \epsilon)(\mu + \|r^{q_c-1}\|_\lambda)$. Combining this inequality with (49), it follows immediately that $\|\zeta^t\|_\lambda \leq Z(\Pi_{TI}^* + \nu)$ for every $t \in \mathbb{Q}_c \cap \mathbb{L}_{c-1}$. From definition of π_i we have $\pi_i(t) \leq 1 + \mu^{-1} \|\zeta^t\|_\lambda + \mu^{-1} \|w^t\|_\lambda$ for every $t \in \mathbb{Z}_+$. From definition of $Z(\cdot)$ we therefore obtain $\pi_i(t) \leq 2\mu^{-1} Z(\Pi_{TI}^* + \nu)$ for every $t \in \mathbb{Q}_c \cap \mathbb{L}_{c-1}$. Finally, substituting (26) into (25) it follows that for some index s we have that $\pi_s(t) + h \leq \Pi_{TI}^* + g_3 Z(\Pi_{TI}^* + \nu)$ holds on the interval $\mathbb{Q}_c \cap \mathbb{L}_c$ from which the upper bound (28) follows. If instead $\mathbb{Q}_c \cap \mathbb{L}_{c-1} = \emptyset$, then $\mathbb{Q}_c \subseteq \mathbb{L}_c$ and $q_c = \ell_c$. In such a case (28) follows immediately by (25) and (26). ■

Proof of Proposition 1: Consider first that, by assumption A2, for any $\theta_j \in \Theta$ there exist a candidate controller $C_{\rho(\theta_j)}$, $\rho(\theta_j) \in \underline{N}$ and an open ball $\tilde{\Theta}_j$ around θ_j such that $(P(\theta)/C_{\rho(\theta_j)})$ is λ -stable for all $\theta \in \tilde{\Theta}_j$. Thus, an infinite open cover for Θ exists. Recall now that, by the Heine–Borel theorem, under assumption A1 any infinite open cover of Θ has a finite open subcover. Then, this implies the existence of a finite collection of open sets $\{\bar{\Theta}_1, \dots, \bar{\Theta}_{\bar{N}}\}$ that covers Θ and such that, for each j , all the plants $P(\theta)$, $\theta \in \bar{\Theta}_j$, are stabilized by a common controller $C_{\rho(j)}$, $\rho(j) \in \underline{N}$. Further, as well known, the Lebesgue’s number lemma ensures that there exists a number $\eta > 0$ such that every subset of Θ having diameter less than η is contained at least in one of the sets $\bar{\Theta}_j$, $j \in \{1, \dots, \bar{N}\} := \bar{N}$. This implies that the cover $\bar{\Theta}_1, \dots, \bar{\Theta}_{\bar{N}}$ is overlapping in the sense that for any $\theta \in \Theta$ there exists at least one index $j \in \bar{N}$ such that $\theta \in \bar{\Theta}_j$ and $\text{dist}(\theta, \partial\bar{\Theta}_j) \geq \eta$. In order to conclude the proof, it is now sufficient to pick a positive real $\bar{\eta} < \eta$ and define, for each $i \in \underline{N}$, the closed set

$$\Theta_i = \bigcup_{j \in \bar{N}; \rho(j)=i} \{\theta \in \bar{\Theta}_j : \text{dist}(\theta, \partial\bar{\Theta}_j) \leq \bar{\eta}\}.$$

In fact, it is immediate to verify that, with such a definition, the collection of sets $\{\Theta_1, \dots, \Theta_N\}$ satisfies properties 1)–2) by construction. As for 3), let $\beta := \eta - \bar{\eta}$. From the overlapping property of the cover $\bar{\Theta}_1, \dots, \bar{\Theta}_{\bar{N}}$, we have that for any given θ there exists $j \in \bar{N}$ such that the open ball $\mathcal{B}(\theta, \eta) := \{x; \|x -$

$\theta\| < \eta\}$ is contained in $\bar{\Theta}_j$. This, in turn, implies that the open ball $\mathcal{B}(\theta, \beta)$ is contained in the closed set

$$\{\theta \in \bar{\Theta}_j; \text{dist}(\theta, \partial\bar{\Theta}_j) \leq \bar{\eta}\}$$

which is obtained by shrinking the original open set $\bar{\Theta}_j$. Then, by definition of Θ_i , $\mathcal{B}(\theta, \beta)$ is also contained in the set Θ_i with $i = \rho(j)$, which implies that $\text{dist}(\theta, \partial\Theta_i) \geq \beta$. ■

Proof of Lemma 6: Following the same lines as in the proof of Lemma 3, it can be seen that the signal ζ_i satisfies the recursion

$$\Phi_{\theta(t)/i}(d) \zeta_i(t) = \Psi_{\theta(t)}(d) w_i(t). \quad (50)$$

Let $\tilde{\zeta}_i(t) := \zeta_i(t) \lambda^{-t}$ and $\tilde{w}_i(t) := w_i(t) \lambda^{-t}$. Then (50) can be rewritten as

$$\tilde{\Phi}_{\theta(t)/i}(d) \tilde{\zeta}_i(t) = \tilde{\Psi}_{\theta(t)}(d) \tilde{w}_i(t) \quad (51)$$

where $\tilde{\Phi}_{\theta(t)/i}(d) := \Phi_{\theta(t)/i}(\lambda^{-1}d)$ and $\tilde{\Psi}_{\theta(t)/i}(d) := \Psi_{\theta(t)/i}(\lambda^{-1}d)$. Suppose now that, in a given interval $\mathcal{I} := \{\tau, \tau + 1, \dots, T\}$, one has $\theta(t) \in \Theta_i$. Then, for any $t \in \mathcal{I}$, the frozen-time characteristic polynomial $\det \Phi_{\theta(t)/i}(d)$ of (50) is λ -Hurwitz. This, in turn, implies that, for any $t \in \mathcal{I}$, the frozen-time characteristic polynomial $\det \tilde{\Phi}_{\theta(t)/i}(d)$ of (51) is Hurwitz. More specifically, since the set Θ_i is compact, we have that when $\theta(t) \in \Theta_i$ all the roots of $\det \tilde{\Phi}_{\theta(t)/i}(d)$ always lie outside a disk of radius $1 + \varepsilon_i$ with ε_i strictly positive. As a consequence, we can invoke classical results on slowly time-varying systems [24] and conclude that, when the variation rate δ does not exceed a given threshold $\delta_{\max,i}$, then the system (51) is exponentially stable in the interval \mathcal{I} . Thus, it is immediate to conclude that, in this case, there exist positive reals \tilde{m}_{0i} and \tilde{m}_{1i} such that, for any $t \in \mathcal{I}$

$$\|\tilde{\zeta}_i|_\tau^t\| \leq \tilde{m}_{0i} \|\tilde{\xi}_i|_{\tau-q}^{\tau-1}\| + \tilde{m}_{1i} \|\tilde{w}_i|_\tau^t\| \quad (52)$$

where $\tilde{\xi}_i := [\tilde{\zeta}_i^t \ \tilde{w}_i^t]'$. Again, the finiteness of $\delta_{\max,i}$, \tilde{m}_{0i} , and \tilde{m}_{1i} stems from the compactness of Θ_i . Note now that, by construction

$$\begin{aligned} \|\tilde{\zeta}_i|_\tau^t\| &= \lambda^{-t} \|\zeta_i|_\tau^t\|_\lambda, & \|\tilde{\xi}_i|_{\tau-q}^{\tau-1}\| &= \lambda^{-\tau+1} \|\xi_i|_{\tau-q}^{\tau-1}\|_\lambda, \\ \|\tilde{w}_i|_\tau^t\| &= \lambda^{-t} \|w_i|_\tau^t\|_\lambda. \end{aligned} \quad (53)$$

Hence, inequality (52) can be rewritten as $\|\zeta_i|_\tau^t\|_\lambda \leq \tilde{m}_{0i} \|\xi_i|_{\tau-q}^{\tau-1}\|_\lambda \lambda^{t-\tau+1} + \tilde{m}_{1i} \|w_i|_\tau^t\|_\lambda$. The rest of the proof follows along the same lines of the proof of Lemma 3. ■

REFERENCES

- [1] A. S. Morse, “Control using logic-based switching,” in *Trends in Control: An European Perspective*, A. Isidori, Ed. London, U.K.: Springer, 1995, pp. 69–113.
- [2] F. M. Pait and F. Kassab, “On a class of switched, robustly stable, adaptive systems,” *Int. J. Adapt. Control Signal Process.*, vol. 15, pp. 213–238, 2001.
- [3] D. Angeli and E. Mosca, “Lyapunov-based switching supervisory control of nonlinear uncertain systems,” *IEEE Trans. Autom. Control*, vol. 47, no. 3, pp. 500–505, Mar. 2002.

- [4] M. G. Safonov and T. C. Tsao, "The unfalsified control concept and learning," *IEEE Trans. Autom. Control*, vol. 42, no. 6, pp. 843–847, Jun. 1997.
- [5] R. Wang, A. Paul, M. Stefanovic, and M. G. Safonov, "Cost detectability and stability of adaptive control systems," *Int. J. Robust Nonlinear Control*, vol. 17, pp. 549–561, 2007.
- [6] M. Stefanovic and M. G. Safonov, "Safe adaptive switching control: Stability and convergence," *IEEE Trans. Autom. Control*, vol. 53, no. 9, pp. 2012–2021, Sep. 2008.
- [7] S. Baldi, G. Battistelli, E. Mosca, and P. Tesi, "Multi-model unfalsified adaptive switching supervisory control," *Autom.*, vol. 46, pp. 249–259, 2010.
- [8] G. Battistelli, E. Mosca, M. G. Safonov, and P. Tesi, "Stability of unfalsified adaptive switching control in noisy environments," *IEEE Trans. Autom. Control*, vol. 55, no. 10, pp. 2425–2429, Oct. 2010.
- [9] I. Al-Shyouchk and J. Shamma, "Switching supervisory control using calibrated forecasts," *IEEE Trans. Autom. Control*, vol. 54, no. 4, pp. 705–716, Apr. 2009.
- [10] K. Narendra and J. Balakrishnan, "Adaptive control using multiple models," *IEEE Trans. Autom. Control*, vol. 42, no. 2, pp. 171–187, Feb. 1997.
- [11] P. V. Zhivoglyadov, R. H. Middleton, and M. Fu, "Localization based switching adaptive control for time-varying discrete-time systems," *IEEE Trans. Autom. Control*, vol. 45, no. 4, pp. 752–755, Apr. 2000.
- [12] B. D. O. Anderson, T. S. Brinsmead, F. D. Bruye, J. P. Hespanha, D. Liberzon, and A. S. Morse, "Multiple model adaptive control, Part 2: Switching," *Int. J. Robust Nonlinear Control*, vol. 11(5), pp. 479–496, 2001.
- [13] J. P. Hespanha, D. Liberzon, and A. S. Morse, "Hysteresis-based switching algorithms for supervisory control of uncertain systems," *Autom.*, vol. 39, pp. 263–272, 2003.
- [14] J. P. Hespanha and A. S. Morse, "Stability of switched systems with average dwell-time," in *Proc. IEEE Conf. Dec. Control*, 1999, pp. 2655–2660.
- [15] D. Liberzon and L. Vu, "Switching supervisory control for time-varying plants," *IEEE Trans. Autom. Control*, vol. 53, pp. 2012–2021, 2010.
- [16] M. Fu and B. R. Barmish, "Adaptive stabilization of linear systems via switching control," *IEEE Trans. Autom. Control*, vol. AC-31, no. 12, pp. 1097–1103, Dec. 1986.
- [17] P. Rosa, J. Shamma, C. Silvestre, and M. Athans, "Stability overlay for adaptive control laws," *Autom.*, vol. 47, pp. 1007–1014, 2011.
- [18] A. S. Morse, "Supervisory control of families of linear set-point controllers, Part 1: Robustness," *IEEE Trans. Autom. Control*, vol. 42, no. 11, pp. 1500–1515, Nov. 1997.
- [19] D. Angeli and E. Mosca, "Adaptive switching supervisory control of nonlinear systems with no prior knowledge of noise bounds," *Autom.*, vol. 47, pp. 449–457, 2004.
- [20] A. S. Morse, D. Q. Mayne, and G. C. Goodwin, "Applications of hysteresis switching in parameter adaptive control," *IEEE Trans. Autom. Control*, vol. 37, no. 9, pp. 1343–1354, Sep. 1992.
- [21] A. Datta, "Robustness of discrete-time adaptive controllers: An input-output approach," *IEEE Trans. Autom. Control*, vol. 38, no. 12, pp. 1652–1657, Dec. 1993.
- [22] A. Dehghani, B. Anderson, and A. Lanzon, "Unfalsified adaptive control: A new controller implementation and some remarks," in *Proc. Eur. Control Conf.*, Kos, Greece, 2007.
- [23] K. J. Astrom and B. Wittermark, *Adaptive Control*, 2nd ed. : Addison-Wesley, 1995.
- [24] C. A. Desoer, "Slowly varying discrete system $x_{t+1} = A_t x_t$," *Electron. Lett.*, vol. 6, no. 11, pp. 339–340, 1970.



Giorgio Battistelli was born in Genoa, Italy, in 1975. He received the Laurea degree in electronic engineering and the Ph.D. degree in robotics, both from the University of Genoa, Genoa, Italy, in 2000 and 2004, respectively.

From 2004 to 2006, he was a Research Associate in the Dipartimento di Informatica, Sistemistica e Telematica, University of Genoa. He is currently an Assistant Professor of automatic control in the Dipartimento di Ingegneria dell'Informazione, University of Florence. His research interests include

adaptive control, linear and nonlinear estimation, hybrid systems, and data fusion.

Dr. Battistelli is currently an editor of the IFAC journal *Engineering Applications of Artificial Intelligence*, an associate editor of the IEEE TRANSACTIONS ON NEURAL NETWORKS and *Learning Systems*, and a member of the IEEE Control Systems Society Conference Editorial Board.



João P. Hespanha (S'95–M'98–SM'02–F'08) received the Licenciatura in electrical and computer engineering from the Instituto Superior Técnico, Lisbon, Portugal, in 1991 and the Ph.D. degree in electrical engineering and applied science from Yale University, New Haven, CT, USA, in 1998.

From 1999 to 2001, he was Assistant Professor at the University of Southern California, Los Angeles, CA, USA. He moved to the University of California, Santa Barbara, in 2002, where he currently holds a Professor position with the Department of Electrical and Computer Engineering. He is an Associate Director for the Center for Control, Dynamical-Systems, and Computation (CCDC), Vice-Chair of the Department of Electrical and Computer Engineering, and a member of the Executive Committee for the Institute for Collaborative Biotechnologies (ICB). His current research interests include hybrid and switched systems, the modeling and control of communication networks, distributed control over communication networks (also known as networked control systems), the use of vision in feedback control, and stochastic modeling in biology.

Dr. Hespanha is the recipient of the Yale University's Henry Prentiss Becton Graduate Prize for exceptional achievement in research in Engineering and Applied Science, a National Science Foundation CAREER Award, the 2005 best paper award at the 2nd International Conference on Intelligent Sensing and Information Processing, the 2005 Automatica Theory/Methodology best paper prize, the 2006 George S. Axelby Outstanding Paper Award, and the 2009 Ruberti Young Researcher Prize. Dr. Hespanha is an IEEE distinguished lecturer since 2007. From 2004 to 2007, he was an Associate Editor for the IEEE TRANSACTIONS ON AUTOMATIC CONTROL.



Edoardo Mosca (S'63–M'65–SM'95–F'97–LF-06) received the Dr. Eng. degree in electronics engineering from the University of Rome "La Sapienza," Rome, Italy, in 1963.

He then spent four years in the aerospace industry where he worked on research and development of advanced radar systems. Thereafter, from 1968 to 1972, he held academic positions at the University of Michigan, Ann Arbor, MI, USA, and McMaster University, Hamilton, ON, Canada. Since 1972, he has been with the University of Florence, Florence, Italy: until 1974 as an Associate Professor; from 1975 as a full Professor of Control Engineering; and, from 2012, an Emeritus Professor. He has been active in various diversified research fields such as radar signal synthesis and processing, radio communications, system identification, adaptive, predictive, switching supervisory control, and detection of performance degradation in feedback-control systems. He is the author of a book, *Optimal, Predictive, and Adaptive Control* (Prentice Hall, 1995). He has been a journal editor of the *European Journal of Control* and of the *IET Control Theory and Applications*; and, currently, of the *International Journal of Adaptive Control and Signal Processing*. He is the Italian NMO representative in IFAC where he was a Council member from 1996 to 2002, and until 1998 a Council member of EUCA (*European Union Control Association*). He is a Fellow of IFAC.



Pietro Tesi received the Laurea degree and the Ph.D. degree in computer and control engineering, both from the University of Florence, Florence, Italy, in 2005 and 2010, respectively.

In 2006, he worked in the automation industry on research and development of data acquisition systems. During the Ph.D. degree, he was a short-term Ph.D. Scholar at the University of California, Santa Barbara, CA, USA. Since 2010, he has been with both the University of Florence and the University of Genoa, Italy. His current research interests include

adaptive control, predictive control, hybrid systems, and adaptive optics.