ECE 528: Analysis of Nonlinear Systems[1]

1 Lecture 1

1.1 Topological Concepts in $\mathbb{R}^n$

Definition 1.1. *Given a point $x \in \mathbb{R}^n$, its ϵ neighborhood for $\epsilon > 0$ is the set $\{y \in \mathbb{R}^n : |x-y| < \epsilon\}$.*

Definition 1.2. *A set $D \subset \mathbb{R}^n$ is open if D contains some ϵ neighborhood for every $x \in D$.*

Definition 1.3. *A set $D \subset \mathbb{R}^n$ is closed if for every sequence of points $\{x_n\} \subset D$ which converges to a point $x \in \mathbb{R}^n$, $x \in D$ already.*

Definition 1.4. *The boundary ∂D of $D \subset \mathbb{R}^n$ is the set of points $x \in \mathbb{R}^n$ such that every ϵ-neighborhood of x contains points both of D and D^c (i.e. if $B(r,x) \cap D \neq \emptyset$ and $B(r,x) \cap D^c \neq \emptyset$ for every r, then $x \in \partial D$). The interior of D, $\mathring{D} := D \setminus \partial D$.*

If D is open then $D \cap \partial D = \emptyset$. If D is closed then $\partial D \subset D$.

Definition 1.5. *A set D is convex if for every $x, y \in D$ and for all $\alpha \in [0,1]$, the point $\alpha x + (1-\alpha)y \in D$. D is connected if for any two points $x, y \in D$, there is an arc lying entirely in D joining x and y.*

Definition 1.6. *A cover $\mathcal{O}$ of a set D is a set $\{U_\alpha\}$ such that $\bigcup_\alpha U_\alpha \supset D$.*

Definition 1.7. *D is compact if any (and therefore in $\mathbb{R}^n$ all) of the following are satisfied:*

1. *for every cover $\mathcal{O}$ there is a finite subcover $\mathcal{O}_f \subset \mathcal{O}$.*
2. *D is closed and bounded.*
3. *Every sequence has a convergent subsequence with limit in D. In other words, D is sequentially compact.*

Note: in Euclidean space $\mathbb{R}^n$, the conditions in this definition are all equivalent. In general, $2 \not\Rightarrow 1$ (for example, unit sphere in ∞-dimensional space is not compact).

Definition 1.8. *A function $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuous at a point $x \in \mathbb{R}^n$ if $f(x_k) \to f(x)$ whenever $x_k \to x$. More precisely, for every $\epsilon > 0$ there is a $\delta > 0$ such that $|y - x| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$. Important: x is fixed and y varies. If we allow both to vary, we have uniform continuity, i.e. $|x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$, where $x, y \in D$.*

We say that f is continuous on $D \subseteq dom(f)$ if f is continuous at every point in D.

[1] Typeset by James Schmidt. *Disclaimer*: these notes are **not** edited.

Also, f is continuous on D if for every open set U in the range of f, the preimage $f^{-1}(U)$ is open in the domain of f.

Note that for general continuity, $\delta = \delta(\epsilon, x)$ whereas for uniform continuity, $\delta = \delta(\epsilon)$, independent of x. In general, continuity does not imply uniform continuity but on a compact domain, continuity *does* imply uniform continuity.

Proof. Recall that if f is continuous and D is compact, then $f(D)$ is compact too. Therefore, the image of f is compact. Let $\epsilon > 0$ be given and cover $Im(f)$ with $B(\epsilon, y)$ as y ranges over the image of f. Since $f(dom(f))$ is compact, there is a finite subcover. Since f is continuous, the pre-image of each of these open ϵ balls is open. Take δ to be such that a δ ball is contained in each one. □

Theorem 1.1 (Weierstrass). *A continuous function over a compact set attains its extreme points. In other words, there is an $x, y \in dom(f)$ such that $f(x) = \sup(\{f(y) : y \in dom(f)\})$.*

Definition 1.9. *A function f is injective on D if $f(x) = f(y) \Rightarrow x = y$ for every $x, y \in D$; in other words, f^{-1} is well defined on $f(D)$.*

Definition 1.10. *A function $f : [a, b] \subset \mathbb{R} \to \mathbb{R}$ is piecewise continuous if it is continuous everywhere except for a countable number of points in its domain, and has left and right limits $f(x^-)$ and $f(x^+)$ everywhere.*

Definition 1.11. *A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable if its partial derivatives $\frac{\partial f_i}{\partial x_j}$ exists, and is continuously differentiable if $\frac{\partial f_1}{\partial x_j}$ are continuous.*

Example: $f(x) = \begin{cases} x^2 \sin(1/x) & \text{if} x \neq 0 \\ 0 & \text{else} \end{cases}$

$C^n(D_1, D_2)$ is defined as the space of N-times continuously differentiable functions from $D_1 \to D_2$; e.g. C^0 are continuous functions, etc.

For $f : \mathbb{R}^n \to \mathbb{R}$, its gradient vector $\nabla f(x) = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{pmatrix}$ (also sometimes denoted $\frac{\partial f}{\partial x}$). If $f : \mathbb{R}^n \to \mathbb{R}^m$, the Jacobian matrix is again denoted $\frac{\partial f}{\partial x}$ or $J(f)$ with $J(f)_{ij} = \frac{\partial f_i}{\partial x_j}$.

Theorem 1.2 (Mean Value Theorem (for scalar valued functions)). *Suppose a function $f : \mathbb{R}^n \to \mathbb{R}$ is C^1. Then for every $x, y \in \mathbb{R}^n$, there is a point $z \in \theta x + (1 - \theta)y$ for some $\theta \in [0, 1]$, such that $f(y) - f(x) = \nabla f(z) \cdot (y - x)$.*

In general for a vector valued function $f : \mathbb{R}^n \to \mathbb{R}^m$, the Mean Value Theorem must be applied component wise, i.e. $f_i(y) - f_i(x) = \nabla f_i(z_i) \cdot (y - x)$, for $i = 1, \ldots, m$.

Theorem 1.3 (Bellman-Gronwall Lemma). *Let $\lambda : [a, b] \to \mathbb{R}$ be continuous, $\mu : [a, b] \to \mathbb{R}$ continuous and nonnegative, and $y : [a, b] \to \mathbb{R}$ is continuous. Suppose that $y(t) \leq \lambda(t) + \int_a^t \mu(s)y(s)ds$, then $y(t) = \lambda(t) + \int_a^t y(s)\mu(s)e^{\int_s^t \mu(\tau)d\tau}ds$.*

Special case: when $\lambda \equiv c \in \mathbb{R}$, then $y(t) \leq \lambda \exp(\int_a^t \mu(\tau)d\tau)$. If also $\mu \equiv d \in \mathbb{R}$, then $y(t) \leq \lambda e^{\mu(t-a)}$. Note that in the hypothesis, y appears on both sides, so even though the conclusion of the theorem is 'more complicated', the result is that y is bounded by an explicit function independent of y.

Applications: take $\dot{y} = \mu y$, $y(t_0) = y_0$. This is equivalent to the integral equation

$$y(t) = y_0 + \int_{t_0}^t \mu y(s) ds \tag{1.1}$$

and *this* equation gives the implicit relation of the sort in the statement of the theorem. Of course, the solution is $y(t) = y_0 e^{\mu(t-t_0)}$ which is exactly the form of the inequality relation when λ and μ are constant; the B-G lemma is more general in that it gives the inequality version of this.

Proof for $\lambda \equiv c$. Define $v(t) := y(t)\exp(-\int_a^t \mu(\tau)d\tau)$. By hypothesis, $v(t) \leq q(t) := \lambda(\lambda + \int_a^t \mu(s)y(s)ds)\exp(-\int_a^t \mu(\tau)d\tau)$. We differentiate, to get $\dot{q}(t) = \mu(t)y(t)\exp(-\int_a^t \mu(\tau)d\tau) - \mu(t)q(t) = \mu(t)(y(t)\exp(-\int_a^t \mu(\tau)d\tau) - q(t)) = \mu(t)(v(t) - q(t)) \leq 0$ because $v \leq q$ and $\mu \geq 0$.

So $q(t) \leq q(a) = \lambda$, which implies that $v(t) \leq \lambda$ as well. Now just multiply through by $\exp(\int_a^t \mu(\tau)d\tau)$ to get the result. $\square$

Definition 1.12. *A sequence $\{x_k\} \subset X$- $(X, ||\cdot||)$ a normed vector space- is called Cauchy if $||x_n - x_m|| \to 0$ as $n, m \to 0$. More precisely, for every $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that $n, m > N \Rightarrow ||x_n - x_m|| < \epsilon$.*

If $\{x_k\}$ is convergent (i.e. $x_k \to x$), then it is Cauchy. Proof is obvious: $|x_n - x_m| = |x_n - x + x - x_n| \leq |x_n - x| + |x - x_m| < \epsilon/2 + \epsilon/2$ for appropriately chosen N and $n, m > N$.

The converse is not always true, for example in noncomplete spaces.

Definition 1.13. *A normed vector space which is complete is called Banach, i.e. if every Cauchy sequence is convergent (has its limit in the space).*

Examples of Banach spaces:

1. $(\mathbb{R}^n, ||\cdot||_p)$ for $1 \leq p \leq \infty$.
2. $(C([a,b], \mathbb{R}^n), ||\cdot||_\infty$.

2 Lecture 2

2.1 Contraction Mapping

Let $(X, ||\cdot||)$ be a Banach space; a map $P : X \to X$ is called a *contraction* [mapping] if there is a $\rho < 1$ such that $||P(x) - P(y)|| \leq \rho||x - y||$ for all $x, y \in X$. Comments: $\rho \geq 0$ is automatic because $im(||\cdot||) \geq 0$. Also, P is automatically uniformly continuous on X (i.e. for every $\epsilon > 0$ there's a $\delta > 0$ such that $||x - y|| < \delta \Rightarrow ||P(x) - P(y)|| < \epsilon$, for all $x, y \in X$); for contraction, let $\delta := \epsilon/\rho$.

Theorem 2.1 (Contraction Mapping Theorem)**.** *Let $S \subset X$ be closed and let $P : S \to S$ be a contraction. Then*

1. *There exists a unique fixed point $x^* \in S$ of P, i.e. a point x^* such that $P(x^*) = x^*$.*
2. *x^* is the limit of successive approximations of x_0, $x_1 := P(x_0), \ldots, x_n := P^n(x_0), \ldots$ for all $x_0 \in S$.*

Proof. Step 1: want to show that the limit of $P^k(x_0)$ exists for every x_0. Step 2: show that the limit is a fixed point. Step 3: show that the fixed point found in step 2 is unique.

Step 1: It suffices to show that the sequence $\{P^k(x_0)\}$ is Cauchy for any $x_0 \in S$, because the existence of limit follows [by definition] from the completeness of X. Let $\epsilon > 0$ be given, then $||P^k(x) - P^{k+1}(x)|| = ||P^k(x) - P^k(P(x))|| \leq \rho^k||x - P(x)|| = \rho^k C$, where $C := ||x - P(x)||$. Then $\rho^k \to 0$ as $k \to 0$; let N be such that $\rho^k < \epsilon/C$ for any $k > N$. To show Cauchy, in general, need to show $||P^k(x_0) - P^{k+r}(x_0)|| = ||P^k(x_0) - P^k(P^r(x_0))|| \leq \rho^k||x_0 - P^r(x_0)||$, and the same argument as when $r = 1$ works; alternatively, apply triangle inequality to $||P^k(x) - P^{k+1}(x)|| = ||P^k(x) - P^{k+1}(x) + P^{k+1}(x) - \ldots + P^{k+r-1}(x) + P^{k+r}(x)||$ to get $(\rho^{k+r-1} + \ldots + \rho^k)||P(x_0) - x_0|| = \rho^k||P(x_0) - x_0||(\sum_{k=0}^{\infty} \rho^k) = \frac{\rho^k}{1-\rho}||P(x_0) - x_0|| \to 0$ as $k \to \infty$.

Note that $x^* \in S$ since S is closed.

Step 2: $||P(x^*) - x^*|| = 0$ by the previous argument; since $P(x^*) = P(\lim_{k\to\infty} P^k(x_0) = \lim_{k\to\infty} P^{k+1}(x_0) = x^*$, where the second equality holds by continuity of P.

Step 3 is obvious: $||P(x) - P(y)|| \leq \rho||x - y||$; but $P(x) = x$ and $P(y) = y$ implies that $||P(x) - P(y)|| = ||x - y||$. □

2.2 Fundamental Properties of Dynamical Systems

A dynamical system looks like:

$$\dot{x} = f(t, x), \;\; x(t_0) = x_0 \tag{2.1}$$

where $x \in \mathbb{R}^n$, $t \in [t_0, \infty)$, t_0 denotes initial time, $x_0 \in \mathbb{R}^n$ denotes the initial state, and $f : [t_0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$.

Special case: the system is time invariant (or autonomous), namely

$$\dot{x} = f(x) \tag{2.2}$$

$\dot{x}$ stands for $\frac{d}{dt}x(t)$ where $x(t)$ is a 'solution' of the system. We need to set out conditions for when such a solution exists, is well defined, is unique, etc.

2.3 Existence and Uniqueness of Solutions

Want to know: from every chosen initial condition x_0 that there exists a unique $x(t)$ satisfying $x(t_0) = x_0$ and $\dot{x}(t) = f(x)$ satisfies the dynamic equation defining the system. We need to clarify exactly what this means (e.g. what regularity conditions we impose on the system).

Example 1. Let $\dot{x} = ax$ where $a \in \mathbb{R}$, and $x \in \mathbb{R}$. For initial condition $x(t_0) = x_0$, the solution is given by

$$x(t) = x_0 e^{a(t-t_0)} \tag{2.3}$$

Notice that this solution exists (is defined) for all $t > t_0$. It decays for $a < 0$ and grows for $a > 0$.

Example 2. Let

$$\dot{x} = \sqrt{x} \tag{2.4}$$

with $x_0 = 0$. Or you could take $x(t) = t^2/4$. The problem: solutions to this equation are uniquely determined by the initial condition. What's wrong? $f(x) = \sqrt{x}$ has infinite slope (therefore, nonexistent derivative, i.e. $f'(0)$ is not well defined).

Example 3. Let

$$\dot{x} = x^2 \tag{2.5}$$

with $x(0) = x_0 \neq 0$; can solve this by integrating:

$$\frac{dx}{x^2} = dt \Rightarrow \int_{x_0}^{x} \frac{dx}{x^2} = \int_0^t dt \Rightarrow \frac{-1}{x}|_{x_0}^{x} = t \Rightarrow x(t) = \frac{1}{\frac{1}{x_0} - t} \tag{2.6}$$

What does this look like? If $x_0 > 0$, then at time $t = 1/x_0$, $x(t) = \infty$; we call this phenomenon 'finite escape time'. Thus the solution exists only locally around the initial condition. The growth here for the solution is faster than the growth of exponential growth (which reflects possible difference between linear and nonlinear dynamics).

2.4 Some Notions

First, consider autonomous system

$$\dot{x} = f(x) \tag{2.7}$$

Definition 2.1. *A function $f : \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz if for any compact set $D \subset \mathbb{R}^n$, there is a constant $L_D \in |R$ satisfying:*

$$||f(x) - f(y)|| \leq L||x - y|| \, \forall x, y \in D \tag{2.8}$$

Definition 2.2. *A function $f : \mathbb{R}^n \to \mathbb{R}^n$ if there exists such an L satisfying the same property above, except for all $x, y \in \mathbb{R}^n$ (not just compact subsets).*

Examples

1. A contraction mapping is trivially Lipschitz.
2. $f(x) = \sqrt{x}$ is *not* locally Lipschitz at 0. This is obvious since $f'(0) = \infty$. On the other hand, it is at any other point. Note, also, that f *is* uniformly continuous on $\mathbb{R}^{\geq 0}$. Therefore, uniform continuity does *not* imply that the Lipschitz condition is satisfied.
3. If f is $\mathcal{C}^1$ and f' is bounded on its domain, then f is globally Lipschizt.

   Proof. $MVT \Rightarrow |f(x) - f(y)| \leq |f'(z)||x - y|$ for some z. □

4. $f(x) = x^2$ is locally Lipschitz. Follows from previous, since f' is bounded on every compact subset. On the other hand, it's not globally Lipschitz. The same idea applies; since $f'(x) = 2x$ is unbounded on $\mathbb{R}$. This doesn't *prove* this, but provides intuitive rationale.

Proposition 2.1. *$\mathcal{C}^1 \Rightarrow$ Local Lipschitz $\Rightarrow$ Continuous.*

The first implication was already shown and the second is obvious.

3 Lecture 3

Recall $\mathcal{C}^1 \Rightarrow$ Locally Lipschitz$\Rightarrow$continuity.

The reverse implications do not hold. For example, $\sqrt{x}$ is continuous but not Lipschitz.

Theorem 3.1 (Radamacher's Theorem)**.** *A locally Lipschitz function is differentiable almost everywhere.*

By default, we take 'Lipschitz' to mean *locally* Lipschitz.

Definition 3.1. *For $\dot{x} = f(t,x)$ we say that f is locally Lipschitz in x uniformly over t (for $t \in [a,b]$ or $[a,\infty)$), if for every compact subset $D \subset \mathbb{R}^n$, there is an $L > 0$ sucxh that $|f(t,x) - f(t,y)| \leq L|x-y|$ for every $x, y \in D$, and every $t \in [a,b]$.*

Similar definition applies for global Lipschitz.

Theorem 3.2. *$\dot{x} = f(t,x)$, with $x(t_0) = x_0$, and $x \in \mathbb{R}^n$, $t \in [t_0, t_1]$ (when $t_1 = \infty$ the right bracket is open). Suppose that f is piecewise continuous in t for each fixed x, and that f is locally Lipschitz in x uniformly over t. Then, for arbitrary initial x_0, there exists a unique solution $x : [t_0, t_0 + \delta] \to \mathbb{R}^n$ for some $\delta > 0$. Moreover, if f is globally Lipschitz, then the solution is correspondingly global ($\delta = \infty$).*

Remark: in general, δ will depend on the initial condition. Also, we may not always be interested in uniqueness (existence alone suffices), in which case local Lipschitzness can be replaced by continuity of f (a much weaker assumption); take for example $\dot{x} = \sqrt{x}$).

Before the proof, let's clarify what we mean by 'solution'. Our options:

1. x is differentiable everywhere and satisfies the ordinary differential equation $\dot{x} = f(x)$ for all t.
2. x is differentiable almost everywhere and satisfies the ordinary differential equation $\dot{x} = f(x)$ for all t.

Consider, for example, $\dot{x} = f(t)$ where f is a square wave. The solution is a sawtooth, whose derivative exists almost everywhere but not everywhere (it is continuous everywhere, though, and satisfies the ODE everywhere that $\dot{x}$ is defined). Nevertheless, we expect our results to apply in this situation as well.

Instead of the ordinary differential equation $\dot{x} = f(t,x)$, and $x(t_0) = x_0$, we can consider the integral equation $x(t) = x_0 + \displaystyle\int_{t_0}^{t} f(s, x(s))ds$. The upshot is that this equation is that it doesn't have any derivatives. For the proof we'll work with it instead. By 'solution' we mean a function $x(\cdot)$ which satisfies this integral equation. Since f is piecewise continuous in t, and x is its integral, it follows that x is differentiable *a.e.* and satisfies $\dot{x} = f(t,x)$ a.e. Such functions are called absolutely continuous.

Proof. Strategy: Looking for a map $P : X \to X$ which is a contraction and such that its fixed point is a solution to the integral equation, where $X = \{x : [t_0, t_0+\delta] \to \mathbb{R}^n\} = \mathcal{C}^0([t_0, t_0+\delta], \mathbb{R}^n)$. We need to figure out what P is; Given $x(\cdot)$, defined $P(x)$ to be the function $(Px)(t) := x_0 + \int_{t_0}^{t} f(s, x(s))ds$. A function $x(\cdot)$ is a fixed point of P iff x is a solution in the sense which we defined above.

Now the name of the game is simply to apply the contraction mapping theorem; in order to do so, we need a closed subset $S \subset X$ such that $P : S \to S$ is a contraction. Given x_0 define $S = \{x : [t_0, t_0 + \delta] \to \mathbb{R}^n \in \mathcal{C}([t_0, t_0 + \delta], \mathbb{R}^n) : ||x - x_0|| < r\}$ where $r > 0$ is fixed and arbitrary, δ is to be chosen and $||\cdot||$ is the maximum norm on $\mathcal{C}^0$, namely $||x - x_0|| = \max_{t_0 \le t \le t_0+\delta} |x(t) - x_0|$. Note that here $x_0(t) \equiv x_0$ (where x_0 denotes a function on left and a value in $\mathbb{R}^n$ on the right). To enforce that $S \subset X$, we need to change the function domain in x to $[t_0, t_0 + \delta]$. Essentially, the rest of the proof is an exercise in verifying that the conditions of Contraction Mapping are satisfied.

The first thing is that X must be complete. That $(\mathcal{C}^0, ||\cdot||)$ with the maximal norm is a Banach space is a fact from basic analysis. But what we need is for S to be a Banach space. Indeed, this will follow immediately from the fact that S is a closed subset of X, a Banach space. Need to show that if $x_k \to \overline{x}$ is a sequence of function in S which converge to some function $\overline{x}$, that $\overline{x} \in S$, i.e. that $||\overline{x} - x_0|| \le r$. But this is obvious.

Now check that $P : S \to S$, which is equivalent to that $|\int_{t_0}^{t} f(s, x(s))ds| \le r$ for all $t \in [t_0, t_0 + \delta]$. Here is where we have to specify δ. Well, $|\int_{t_0}^{t} f(s, x(s))ds| \le \int_{t_0}^{t} |f(s, x(s)) - f(s, x_0) + f(s, x_0)|ds \le \int_{t_0}^{t} |f(s, x(s)) - f(s, x_0)ds + \int_{t_0}^{t} |f(s, x_0)|ds \le L|x(s) - x_0|\delta + \int_{t_0}^{t} |f(s, x_0)|ds$, where the last inequality follows immediately from our local uniform Lipschitz assumption. The second term $\int_{t_0}^{t} |f(s, x_)|ds \le (t - t_0) \cdot \max_{t_0 \le s \le t_0+\delta} |f(s, x_0)| =: h(t - t_0) \le h\delta$. Putting this all together, our first integral is bounded by $\delta(Lr + h)$ which we want to be less than r. So Take δ to be such that

$$\delta \le \frac{r}{Lr + h} \tag{3.1}$$

□

4 Lecture 4

Proof Cont'd. We have $x(t) = x_0 + \int_{t_0}^{t_1} f(s, x(s))ds =: (Px)(t)$. Define $S := \{x : [t_0, t_0 + \delta] \to \mathbb{R}^n : |x(t) - x_0| \le r \forall t\}$. We checked that $P : S \to S$ if $\delta \le \frac{r}{Lr+h}$ where LL is a Lipschitz constant on r ball $h := \max_{t_0 \le s \le t_1} |f(s, x_0)|$.

What is left to show is that P is a contraction. Consider $||Px - Py|| = \max_{t_0 \le t \le t_0+\delta} ||(Px)(t) - (Py)(t)|| = || \int_{t_0}^{t} f(s, x(s)) - f(s, y(s))ds|| =\le \int_{t_0}^{t} |f(s, x(s)) - f(s, y(s))|ds \le \int_{t_0}^{t} L|x(s) - y(s)|ds \le L\delta||x - y||$. We need $L\delta < 1$, so if $\delta < \frac{1}{L}$ then P is a contraction.

Combining these results, if $\delta < \min\{t_1 - t_0, \frac{r}{Lr+h}, \frac{1}{L}\}$, then Contraction Mapping provides the desired conclusion.

In the global case, we need to use the fact that f is globally Lipschitz; we still have that $\delta < \min\{t_1 - t_0, \frac{r}{Lr+h}, \frac{1}{L}\}$, but since L is a universal Lipschitz constant, the only thing left to verify is that h doesn't grow too fast. But notice that $\frac{r}{Lr+h} \to \frac{1}{L}$ as $r \to \infty$. Pick some $\delta < 1/L$, at each step for a given h we can pick r sufficiently large so that $\frac{r}{Lr+h}$ is close enough to $\frac{1}{L}$ so that $\delta \frac{r}{Lr+h}$. Then we will have a constant δ which can be used in repeated application in the Local case to give a global solution. □

Remark: global Lipschitz is generally a very strong condition, and sometimes unnecessarily stronger than needed to prove global existence of solutions. In practice, we'll use Lyapunov functions.

4.1 Continuous Dependence on Initial Conditions and Parameters

Let $\dot{x} = f(t, x, \lambda)$ with $t \in [t_0, t_1]$, $x \in \mathbb{R}^n$ and $\lambda \in R^p$ a constant parameter, and initial condition $x(t_0) = x_0$.

Theorem 4.1. *Suppose that f is continuous in each argument and locally Lipschitz in x uniformly over $t \in [t_0, t_1]$ and over $\lambda \in B_r^p(\lambda_0)$ some λ_0, $r > 0$ and $B^p \subset \mathbb{R}^p$. Suppose that $\dot{x} = f(t, x, \lambda_0)$, and $x(t_0) = x_0$ has a solution $x(\cdot)$ defined (and therefore automatically unique) on the whole interval $[t_0, t_1]$. Then for every $\epsilon > 0$ there is a $\delta > 0$ such that $|x_0 - \overline{x}_0| < \delta$ and $|\lambda - \lambda_0| < \delta$, then $\dot{\overline{x}} = f(t, \overline{x}, \lambda)$, $\overline{x}(\lambda_0) = \overline{x}_0$ has a unique solution on $[t_0, t_1]$ satisfying $|x(t) - \overline{x}(t)| < \epsilon$ for all $t \in [t_0, t_1]$.*

Consider, for elucidation, a counterexample; namely a dynamical system with bifurcation, where the dynamics of the solution depend on discontinuous function. As a remark, it should be enough to have piecewise continuity in t.

Before proving the theorem, first a lemma.

Lemma 4.1. *Consider two systems* $\dot{x} = f(t, x)$ *with i.c.* $x(t_0) = x_0$, $\dot{y} = f(t, y) + g(t, y)$ *with* $y(t_0) = y_0$); f *satisfies 'usual hypotheses' (those which guarantee existence and uniqueness of solution) and* g *satisfies* $|g(t, y)| \leq \mu$, *some* $\mu > 0$. *Suppose that the corresponding solutions* $x(\cdot)$ *and* $y(\cdot)$ *are defined on* $[t_0, t_1]$ *and belong to some bounded set* W. *Then they satisfy the following bound:*

$$|x(t) - y(t)| \leq |x_0 - y_0| \exp(L(t - t_0)) + \frac{\mu}{L}(\exp(L(t - t_0)) - 1) \tag{4.1}$$

for all $t \in [t_0, t_1]$ *where* L *is the Lipschitz constant on* W.

Proof. Rewrite in integral form:

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds \tag{4.2}$$

and

$$y(t) = y_0 + \int_{t_0}^{t} f(s, y(s)) + g(s, y(s))ds \tag{4.3}$$

Subtracting, we get

$$|x(t) - y(t)| \leq |x_0 - y_0| + \int_{t_0}^{t} |f(s, x(s)) - f(s, y(s))| + |g(s, y(s))|ds \tag{4.4}$$

where the inequality follows from about a hundred applications of triangle inequality and passage under the integral. Then we use the bound on g and local Lipschitzness of f with constant L; we thus obtain

$$|x_0 - y_0| + \int_{t_0}^{t} |f(s, x(s)) - f(s, y(s))| + |g(s, y(s))|ds \leq |x_0 - y_0| + \mu(t - t_0) + \int_{t_0}^{t} L|x(s) - y(s)|ds \tag{4.5}$$

Now use Bellman-Gronwell Lemma:

$$|x(t) - y(t)| \leq |x_0 - y_0| + \mu(t - t_0) + \int_{t_0}^{t} (|x_0 - y_0| + \mu(s - t_0))|Le^{L(t-s)}ds \tag{4.6}$$

The last thing to notice is that $Le^{L(t-s)} = -\frac{d}{ds}e^{L(t-s)}$ and integrate by parts to obtain the result. Heavy computation, but nothing extra conceptually. □

Proof of Theorem. Strategy: reduce the statement in the theorem to the statement in the previous lemma; then apply lemma.

Define epsilon tube $U_\epsilon \subset [t_0, t_1] \times \mathbb{R}^n \subset \mathbb{R}^{n+1}$ defined by $U_\epsilon := \{(t, x) : t \in [t_0, t_1] \, |x - x(t)| \leq \epsilon\}$, a compact subset of $\mathbb{R}^{n+1}$. We want to show that a solution to the system above will be contained in the epsilon tube, or more precisely, that $(t, \overline{x}(t)) \in U_\epsilon$ for all $t \in [t_0, t_1]$. We can write $f(t, \overline{x}, \lambda) = f(t, \overline{x}, \lambda_0) + f(t, \overline{x}, \lambda) - f(t, \overline{x}, \lambda_0)$; the difference between the last two terms we can treat as g. So write $f(t, \overline{x}, \lambda) = f(t, \overline{x}, \lambda_0) + g(t, \overline{x})$. By continuity (since we assume that f is continuous in λ), for every $\mu > 0$ there is a $\beta > 0$ such that $|lambda - \lambda_0| < \beta$ and $(t, x) \in U_\epsilon$ implies that $|f(t, x, \lambda) - f(t, x, \lambda_0)| < \mu$.

Now apply lemma with here $y = \overline{x}$. As long as $(t, \overline{x}(t)) \in U_\epsilon$, the lemma implies that $|x(t) - \overline{x}(t)| \leq |x_0 - \overline{x}_0|e^{L(t-t_0)} + \frac{\mu}{L}(e^{L(t-t_0)} - 1)$. We want to show that the term in the left hand side is less than ϵ. The time horizon is fixed but we can take $|lambda - \lambda_0|$ and $|x_0 - \overline{x}_0|$ to be as small as we like (δ), and μ arbitrarily small as well; and the right combination of small δ and μ □

5 Lecture 5

5.1 Continuous Dependence on Parameters and Initial Conditions

Proof Cont'd. $x(t)$ solution of $\dot{x} = f(t, x, \lambda_0)$ with initial condition x_0 and similarly for $\overline{x}$ except that $\dot{\overline{x}} = f(t, \overline{x}, \lambda_0) + g(t, \overline{x})$. We arrived at a bound $|x(t) - \overline{x}(t)| \leq |x_0 - \overline{x}_0| e^{L(t-t_0)} + \frac{\mu}{L}(e^{L(t-t_0)} - 1)$. We pick $|x_0 - \overline{x}_0|$ and μ small enough. Since $|t_1 - t_0|$ is bounded, we can bound the right hand side by ϵ, no matter what ϵ is. □

Remark: this argument only works for finite time interval, unless we have some stability property as time goes to infinity. We only need $|g| \leq \mu$ in tube.

5.2 Differentiability of Solutions; Sensitivity Functions

Let $\dot{x} = f(t, x, \lambda)$ where f is $\mathcal{C}^1$ w.r.t. x and λ (in addition to what we normally have). Then solution $x(t, \lambda)$ is differentiable w.r.t. to λ. TO see this, first convert to integral equation

$$x(t, \lambda) = x_0 + \int_{t_0}^{t} f(s, x(s, \lambda), \lambda) ds \tag{5.1}$$

with x_0 fixed. Then we can differentiate this with respect to λ.

We write

$$\frac{\partial}{\partial \lambda} x(t, \lambda) = x_\lambda(t, \lambda) = \int_{t_0}^{t} \frac{\partial f}{\partial x}(s, x(s, \lambda), \lambda) x_\lambda(s, \lambda) + \frac{\partial f}{\partial \lambda}(s, x(s, \lambda), \lambda) ds \tag{5.2}$$

Therefore $x_\lambda(t, \lambda)$ is a solution of the differential equation $\frac{\partial}{\partial t} x_\lambda(t, \lambda) = \dot{x}_\lambda(t, \lambda) = x_\lambda(t_0, \lambda) + \frac{\partial f}{\partial x}(t, x(t, \lambda), \lambda) x_\lambda(t, \lambda) + \frac{\partial f}{\partial \lambda}(t, x(t, \lambda), \lambda) = \frac{\partial f}{\partial x}(t, x(t, \lambda), \lambda) x_\lambda(t, \lambda) + \frac{\partial f}{\partial \lambda}(t, x(t, \lambda), \lambda)$ because $x(t_0, \lambda) \equiv x_0$ (alternatively, could see this from the integral). Let $S(t) := x_\lambda(t, \lambda_0)$, $A(t) := \frac{\partial f}{\partial x}(t, x(t, \lambda_0), \lambda_0)$, $B(t) := \frac{\partial f}{\partial \lambda}(t, x(t, \lambda_0), \lambda_0)$. We generally call this $S(t)$ the 'sensitivity function' and it satisfies the sensitivity equation:

$$\dot{S}(t) = A(t) S(t) + B(t) \tag{5.3}$$

The significance is the following: up to first order we can write

$$x(t, \lambda) = x(t, \lambda_0) + S(t)(\lambda - \lambda_0) + o(\lambda) \tag{5.4}$$

5.3 Comparison Principle

Let u and v be two scalar valued signals satisfying both

$$\dot{u} = f(t, u), \quad u(t_0) = u_0 \tag{5.5}$$

$$\dot{v} \leq f(t, v), \quad v(t_0) = v_0 \tag{5.6}$$

Then $v(t) \leq u(t)$ as long as the two signals exist.

To show this rigorously, can consider $\dot{\overline{u}} f(t, \overline{u}) + \lambda$ for smlal $\lambda > 0$. If v crosses above u, then by continuous dependence, v also crosses above $\overline{u}$ for small λ, which is a contradiction.

5.3.1 Some Examples

Example 1 $\dot{x} = -x - x^4 \leq -x$. By comparison principle, solution of $\dot{x} = -x$ with same initial condition, $e^{-t}c_0$ gives an upper bound for $\dot{x} = -x - x^4$.

Example 2 $\dot{x} = -x - x^3$. Instead of considering x directly, let's look at $v(x) := x^2$. Then $\dot{v} = 2x\dot{x} = -2x^2 - 2x^4 \leq -2x^2 = -2v$, and now we have reduced this problem to the last example with $\dot{v} \leq -2v$; namely $v(t) \leq e^{-2t}v(t_0)$. This implies that $x^2(t) \leq e^{-2t}x^2(t_0) \Leftrightarrow |x(t)| \leq e^{-t}|x(t_0)|$.

5.4 Lyapunov Stability

5.4.1 Stability Definitions

First the autonomous case where

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n \tag{5.7}$$

and f is locally Lipschitz. Assume that $x = 0$ is an equilibrium, namely $f(0) = 0$.

Definition 5.1. *A local equilibrium $x_e = 0$ is Lyapunov stable if for every $\epsilon > 0$ there is a $\delta > 0$ such that $|x(0)| < \delta$ implies that $|x(t)| < \epsilon$ for all $t \geq 0$.*

Definition 5.2. *A local equilibrium point $x_e = 0$ is attractive if there is a $\delta > 0$ such that for all initial conditions x_0 satisfying $|x_0| < \delta$, $x(t) \to 0$ as $t \to \infty$. More formally, for every $x_0 \in \mathbb{R}^n$, there is a δ such that for all $\epsilon > 0$ there exists a T (a function of x_0 and ϵ) for which $|x(t)| < \epsilon$ whenever $t \geq T$ and $|x_0| < \delta$.*

Definition 5.3. *A local equilibrium point $x_e = 0$ is [locally] asymptotically stable if it is stable and locally attractive.*

An example of stable but not attractive system is

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{5.8}$$

For the converse non-implication, consider Vinograd's counterexample

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \frac{x^2(y-x)+y^5}{(x^2+y^2)(1+(x^2+y^2)^2)} \\ \frac{y^2(y-2x)}{(x^2+y^2)(1+(x^2+y^2)^2)} \end{pmatrix} \tag{5.9}$$

here f is indeed Lipschitz and the system is attractive, and it's not Lyapunov stable.

6 Lecture 6

Definition 6.1. *Let $\dot{x} = f(x)$ with origin the equilibrium. We say the equilibrium is Lyapunov stable if for every $\epsilon > 0$ there is a $\delta > 0$ such that $|x(0)| < \delta \Rightarrow |x(t)| < \epsilon$ for all t, and attractive if there is a δ_2 such that $|x_0| < \delta_2 \Rightarrow \lim_{t\to\infty} x(t) = 0$; finally the equilibrium is asymptotically stable if it is L. stable and attractive.*

Attractivity: there is $\delta_2 > 0$ such that for every x_0 satisfying $|x_0| < \delta$, for all $\epsilon > 0$ there is a $T(x_0, \epsilon)$ such that $|x(t)| < \epsilon$ for every $t > T$.

Definition 6.2. *Region of attraction is the set of all x_0 from which $x(t) \to$ as $t \to \infty$.*

Remark: asymptotic stability has a ball in its definition but the region need not be a ball.

Definition 6.3. *Global Asymptotic Stability: asymptotically stable and region of attraction is entire space.*

Definition 6.4. *Exponentially stable if there are $\delta, c, \lambda > 0$ such that $|x(0)| < \delta \Rightarrow |x(t)| \le c|x_0|e^{-\lambda t}$ for all $t \ge 0$.*

Remark: in a time invariant system, we can WLOG take any arbitrary equilibrium to be zero by change of coordinates and that initial time is zero.

Definition 6.5. *Globally Exponential Stability: Exponential Stability for all $\delta > 0$.*

Now for time varying case

$$\dot{x} = f(t, x), \quad x(t_0) = x_0 \tag{6.1}$$

We still assume that origin is equilibrium, namely $f(t, 0) = 0$.

Definition 6.6. *Equilibrium point is Lyapunov stable if for every $\epsilon > 0$ and every initial time t_0 there is a $\delta_1(\epsilon, t_0)$ such that $|x_0| < \delta_1 \Rightarrow |x(t)| < \epsilon$ for all $t \ge t_0$.*

Definition 6.7. *Attractive: if for every t_0 there is a δ_2 such that $|x_0| < \delta_2 \Rightarrow x(t) \to 0$, namely for every t_0, there is a $\delta_2(t_0)$ such that for every x_0 such tht $|x_0| < \delta_2$ and for every $\epsilon > 0$ there is a $T(t_0, x_0, \epsilon)$ such that $|x(t)| < \epsilon$ for all $t \ge t_0 + T$.*

As before, asymptotic stability is stability with attractivity. The rest generalize in the obvious way; just include initial time where needed (e.g. $|x(t)| \le c|x_0|e^{-\lambda(t-t_0)}$ for all $t \ge t_0$).

With these definitions, convergence can get slower as we increase t_0; also there is included the possibility of a lack of robustness w.r.t. disturbances. Therefore, we seek stability properties which are uniform with respect to t_0.

Definition 6.8. *0 is uniformly stableif for every $\epsilon > 0$ there is a $\delta_1 > 0$ (independent of t_0) such that for all t_0, for all x_0 satisfying $|x_0| < \delta_1$ we have that $|x(t)| < \epsilon$ for all $t \ge t_0$.*

(These new definitions, as usual with uniformity concepts, simply permute the order of some quantifiers from previous definitions.)

Definition 6.9. *Uniformly attractive: if there is a* $\delta_2 > 0$ *such that for all* $\epsilon > 0$ *there is a* T *such that for all* t_0*,* $|x_0| < \delta_2 \Rightarrow |x(t)| < \epsilon$ *for all* $t \geq t_0 + T$*.*

Note that $T = T(\delta_2, \epsilon)$ does not depend on t_0 or specific choice of x_0 in the δ_2 ball.

Uniformly Stable + Uniformly Attractive = Uniformly Asymptotically Stable

Similarly for GUAS (global uniform asymptotic stability): for all $\delta_2 > 0$ and $\epsilon > 0$, there is a $T(\delta_2, \epsilon)$ such that $|x_0| < \delta \Rightarrow |x(t)| < \epsilon$ for all $t \geq t_0 + T$. GUES: $|x(t)| \leq c|x_0|e^{-\lambda(t-t_0)}$ for all $t \geq t_0$, already uniform w.r.t. t_0, because convergence depends only on $t - t_0$, not t_0 itself. Already uniform w.r.t. x_0 (in δ_2 ball) because it involves only the norm $|x_0|$.

6.1 Comparison Functions

Definition 6.10. *A function* $\alpha : [0, \infty) \to [0, \infty)$ *is called* $\mathcal{K}$ *if it's continuous,* $\alpha(0) = 0$ *and monotonically increasing.*

Note that we can also define α on a finite interval to be of class $\mathcal{K}$ if it has the same property. Examples: $\sqrt{x}$, x.

Definition 6.11. *If* $\alpha : [0, \infty) \to [0, \infty)$ *is of class* $\mathcal{K}$ *and* $\alpha(r) \to \infty$ *as* $r \to \infty$ *then* α *is of class* $\mathcal{K}_\infty$*.*

Definition 6.12. *A continuous* $\beta : [0, \infty) \times [0, \infty) \to [0, \infty)$ *is called* $\mathcal{KL}$ *if* $\beta(\cdot, t)$ *is of class* $\mathcal{K}$ *for each fixed* t *and* $\beta(t, \cdot)$ *decreases to zero for each fixed* r*.*

Example: $\beta(r, t) = cre^{-\lambda t}$.

Given a system

$$\dot{x} = f(t, x), \quad x(t_0) = x_0, \quad f(t, 0) = 0 \tag{6.2}$$

then we have the following lemma

Lemma 6.1 (Khalil Lemma 4.5)**.** *Uniform Stability [at origin] is equivalent to the existence of* $\alpha \in \mathcal{K}$ *such that* $|x(t)| \leq \alpha(|x_0|)$ *for* $|x_0| < c$ *some* $c \in \mathbb{R}$*. Secondly, uniform asymptotic stability is equivalent to the following: there is a* $\beta \in \mathcal{KL}$ *such that* $|x(t)| \leq \beta(|x_0|, t - t_0)$ *for* $|x_0| \leq c$ *some* $c \in \mathbb{R}$*.*

WLOG, can take $\alpha \in \mathcal{K}_\infty$, by increasing α if necessary.

Note that exponential stability is a special case, with $\beta(r, s) = cre^{-\lambda s}$. Another example of a class $\mathcal{KL}$ function is $\beta(r, t) = \frac{r^2}{1+t}$.

Proof. We prove the second: suppose that $|x(t)| \leq \beta(|x_0|, t - t_0)$, $\beta \in \mathcal{KL}$. Since β is decreasing in the second argument, we have $|x(t)| \leq \beta(|x_0|, 0) = \alpha(|x_0|) \in \mathcal{K}$. Now need to show that this implies uniform stability (for every $\epsilon > 0$ there is a $\delta > 0$ such that $|x_0| < \delta \Rightarrow |x(t)| < \epsilon$). Choose δ such that $\alpha(\delta) = \epsilon$. In other words, $\delta(\epsilon) = \alpha^{-1}(\epsilon)$, the local inverse of α.

Finally, since $|x(t)| < \beta(|x_0|, t - t_0)$ and for each fixed $|x_0|$ we have $\beta(|x_0|, t - t_0) \to 0$ as $t \to \infty$ implies that $x(t) \to 0$; $|x_0| < \delta_2$, to have $\beta(\delta_2, t - t_0) < \epsilon$ need $t - t_0 \geq T$ for large enough T dependent on β. But this follows from the definition of $\mathcal{L}$

□

The point of defining these classes of functions is precisely to encode the information contained in the definition of stability and attractivity.

6.2 Lyapunov Functions

Given a system $\dot{x} = f(x)$, $x \in \mathbb{R}^n$, then a continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}$, $V(x)$ is positive definite for all $x \in \mathbb{R}^n \setminus 0$, and whose derivative is negative semi definite $\dot{v}(x) \leq 0$, radially unbounded ($V(x) \to \infty$ whenever $||x|| \to \infty$) is said to be Lyapunov function.

7 Lecture 7

Let $V : \mathbb{R}^n \to \mathbb{R}$ be $\mathcal{C}^1$ function for system

$$\dot{x} = f(x) \tag{7.1}$$

$x \in \mathbb{R}^n$. We want derivative of V along solution of $\dot{x} = f(x)$. $V(x(t))$: consider as a function of time for a given solution $x(t)$; then

$$\frac{d}{dt}V(x(t)) = \frac{\partial V}{\partial x}x(t) \cdot \dot{x}(t) = \frac{\partial V}{\partial x} \cdot f|_{x(t)} \tag{7.2}$$

We work with $\frac{\partial V}{\partial x} \cdot f(x)$ a function from $\mathbb{R}^n \to \mathbb{R}$; this is defined independently of a solution to $\dot{x} = f(x)$.

Example: Let $V(x) = x^T Px$, and $\dot{x} = Ax$; then

$$\dot{V}(x) = \frac{\partial V}{\partial x} \cdot f|_x = \dot{x}^T Px + x^T P\dot{x} = (Ax)^T Px + x^T P(Ax) = x^T(PA + A^T P)x \tag{7.3}$$

Theorem 7.1 (Lyapunov's 2nd Method)**.** *Let*

$$\dot{x} = f(x), \quad f(0) = 0 \tag{7.4}$$

with $V : \mathbb{R}^n \to \mathbb{R}$ *positive definite and* $\mathcal{C}^1$. *Then*

1. *The zero equilibrium is stable if*

$$\dot{V}(x) \leq 0 \quad \forall x \in \mathbb{R}^n \setminus 0 \tag{7.5}$$

2. *Asymptotically stable if*

$$\dot{V}(x) < 0 \quad \forall x \in \mathbb{R}^n \tag{7.6}$$

3. *GAS if*

$$\dot{V}(x) < 0 \quad \forall x \neq 0 \tag{7.7}$$

   and V *is radially unbounded.*

Remarks: $\dot{V}(0) < 0$ is not possible since $\dot{V}(0) = \frac{\partial V}{\partial x} \cdot f|_{x=0} = 0$. Secondly, for the first two statements, the same holds locally for $V : D \subset \mathbb{R}^n \to \mathbb{R}$ and we just ask for $\dot{V} \leq 0$ for all $x \in D$.

Proof. We start with the first statement. We need to show that for every $\epsilon > 0$ there is a $\delta > 0$ such that $|x_0| < \delta \Rightarrow |x(t)| < \epsilon$ for every t. Let $\epsilon > 0$ be given. Take positive b such that $b < \min_{|x|=\epsilon} v(x)$ which is well defined by positive definiteness of V. Then let $\delta_1 > 0$ be such that if $|x| < \delta_1$, then $V(x) \leq b$ which exists by continuity of V.

Now we claim that if $|x_0| < \delta_1$, then $|x(t)| < \epsilon$ for all t. Well, $|x_0| \leq \delta_1$ implies that $V(x_0) \leq t$ and $\dot{V}(x(t)) \leq 0$ implies that $V(x(t)) \leq b$ for all $t \geq 0$. Given this, then we have $|x(t)| < \epsilon$ because otherwise, there would be a time $\bar{t}$ such that $|x(\bar{t})| = \epsilon$ which by choice of b implies that $V(x(\bar{t})) > b$, contradiction.

For aysmptotic stability, pick some $\epsilon > 0$ and find $\delta_1 > 0$ for this ϵ as in the first part. Let $|x_0| \leq \delta_1$; we need to show that $x(t) \to 0$. Consider $V(x(t))$: $\dot{V}(x(t)) < 0$ except at 0 and $V(x(t)) \geq 0$ (monotonic) implies that $V(x(t)) =: c$ exists. Two possibilities: $c = 0$ and $c \neq 0$.

Case 1: $c = 0$. Then $x(t) \leq \epsilon$; then $V(x(t)) \to 0$, $V = 0$ only at zero implies that $x(t) \to 0$.

Case 2: $c > 0$ Then $V(x(t)) \geq c > 0$ for all t. As before, there is an $r > 0$ such that for all $|x| \leq r$ we have $V(x) \leq c$. However, $x(t)$ a solution to the system cannot enter the r-ball, i.e. $0 < r \leq |x(t)| \leq \epsilon$. The set $\{x : r \leq |x| \leq \epsilon\}$ is a compact set. Take $\max \dot{V}$ in this set and call it $-d < 0$. For all t, then, we have $V(x(t)) \leq V(x_0) - dt$ for all $t \geq 0$. But then $V < 0$ at some time T, contradcting positive definiteness of V. So $c > 0$ isn't possible.

For global asymptotic stability, we want to show that $\delta_1 \to \infty$ as $\epsilon \to \infty$. As in proof of first statement, as $\epsilon \to \infty$, since V is radially unbounded, we can take $b \to \infty$ and therefore $\delta_1 \to \infty$ as well.

□

Example: $\dot{x} = \varphi(x)$ with $x\varphi(x) > 0$ for all $x \neq 0$. Then set $V = 1/2x^2$ so that $\dot{V} = -\dot{x}\varphi(x) < 0$ for all $x \neq 0$ which implies GAS.

8 Lecture 8

8.1 Laypunov Example

Last time we proved Lyapunov's theorem, namely that $\dot{x} = f(x)$ with $x \in \mathbb{R}^n$, and $f(0) = 0$, for $V : \mathbb{R}^n \to \mathbb{R}$ a $\mathcal{C}^1$ positive definite function and $\dot{V}(x) = \frac{\partial V}{\partial x} \cdot f(x)$ zero at equilibrium point: then the system is L. stable if $\dot{V}(x) \leq 0$ for all x and asymptotically stable if $\dot{V}(x) < 0$ for all $x \neq 0$. Global asymptotic stability holds if V is radially unbounded. Recall that radial unboundedness means that $V(x) \to \infty$ as $|x| \to \infty$. As a counter example, $(x_1 - x_2)^2$ is *not* radially unbounded.

Example:

$$\ddot{x} + \dot{x} + \varphi(x) = 0 \tag{8.1}$$

in state space form this is equivalent to

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -x_2 - \varphi(x_1) \end{pmatrix} \tag{8.2}$$

satisfying $x\varphi(x) > 0$ for all $x \neq 0$. We want to find whether or not this is stable or asymptotically stable. Potential energy is given by $\Phi(x) = \displaystyle\int_0^x \varphi(z)dz$ and kinetic energy is $\frac{1}{2}x_2^2$. For Lyapunov function let's first try

$$V(x_1, x_2) = \frac{1}{2}(x_1^2 + x_2^2) \tag{8.3}$$

Then

$$\dot{V} = x_1 x_2 - x_2(x_2 + \varphi(x_1)) = x_1 x_2 - x_2^2 - x_2\varphi(x_1) \tag{8.4}$$

This function obviously isn't helpful because we know nothing about $\varphi(x_1)x_2$.

Then take V to be the total energy,

$$V(x) = \varphi(x_1) + \frac{1}{2}x_2^2 \tag{8.5}$$

which is positive definite if $x\varphi(x) > 0$ and radially unbounded if $x\varphi(x) > kx^2$ for some $k > 0$. We calculate

$$\dot{V}(x) = \varphi(x_1)x_2 - x_2(x_2 + \varphi(x_1)) = -x_2^2 \tag{8.6}$$

which is negative semindefinite. Therefore, by Lyapunov's 2nd method the origin is L. stable. However it is not negative definite because at $x_2 = 0$, but $x_1 \neq 0$ we still have $V(x) = 0$. Therefore we cannot conclude that the system is asymptotically stable. Nevertheless, our inability to conclude a. stability does *not* imply that the system is not asymptotically stable. In other words, Lyapunov stability is only a sufficient condition for a. stability. Perhaps we can find another Lyapunov function V which would work.

We need to make sure that V is positive definite

$$\begin{pmatrix} a_1 & a_3/2 \\ a_3/2 & a_2 \end{pmatrix} > 0 \tag{8.7}$$

A choice that works:

$$V = \frac{a}{2}x_1^2 + ax_1x_2 + \frac{1}{2}x_2^2 + \Phi(x_1) \tag{8.8}$$

for $0 < a < 1$; then

$$\det\begin{pmatrix} a/2 & a/2 & a/2 & 1/2 \end{pmatrix} = \frac{a-a^2}{4} > 0 \tag{8.9}$$

as long as $a \in (0,1)$. V is positive definite and also radially unbounded; need to check that its derivative is negative definite:

$$\dot{V} = (ax_1+ax_2+\varphi(x_1))x_2-(ax_1+x_2)(x_2+\varphi(x_1)) = ax_1x_2+ax_2^2+\varphi(x_1)x_2-ax_1x_2-ax_1\varphi(x_1)-x_2^2-x_2\varphi(x_1) \tag{8.10}$$

Simplifying, we have

$$\dot{V}(x) = -(1-a)x_2^2 - ax_1\varphi(x_1) < 0 \tag{8.11}$$

and this *is* negative definite.

Lesson: finding Lyapunov functions is hard. There is no algorithmic method for finding one in general, but a general approach is trial and error. Next is another way to find Stability:

8.2 LaSalle's Principle

Goal: conclude global asymptotic stability from only $\dot{V} \leq 0$ with some additional analysis. Let $\dot{x} = f(x)$ and $x = x(t)$ be a fixed solution of the system. For this trajectory $x(t)$, define its positive limit set Γ^+ as the set of all limit (or accumulation) points of $x(t)$; in math lingo:

$$\Gamma^+ = \bigcup_{\{t_k\}\subset 2^{\mathbb{R}^+}} \{z \in \mathbb{R}^n : \lim_{t_k\to\infty} x(t_k) = z\} \tag{8.12}$$

where we stipulate (notationally) that $\lim_{t_k\to\infty} x(t_k) = \emptyset$ whenever the limit doesn't exist (and further we say $\bigcup_{\emptyset} := \emptyset$). Sometimes Γ^+ is denoted as L^+ and also sometimes called the ω-limit set (also going backward in time we have α-limit set). The concept is useful really only for bounded trajectories which don't converge to an equilibrium.

Another definition

Definition 8.1. *A set $M \subset \mathbb{R}^n$ is called [positive] invariant] if $x_0 \in M$ implies that $x(t) \in M$ for all $t > 0$.*

Properties of positive limit set of Γ^+

Proposition 8.1. *Assume that $x(t)$ is a bounded solution of $\dot{x} = f(x)$. Then its positive limit set Γ^+ has the following properties:*

1. $\Gamma^+ \neq \emptyset$.
2. Γ^+ *is bounded.*

3. *Γ^+ is closed (and therefore compact).*

4. *Γ^+ is an invariant set.*

5. *$x(t) \to \Gamma^+$ as $t \to \infty$. More precisely, for $\lim_{t\to\infty} x(t, \Gamma^+) := \lim_{t\to\infty} \inf_{z\in\Gamma^+} |x(t) - z| = 0$.*

Proof. 1. Bolzano-Weierstrass (a bounded set always has a limit point).

2. Γ^+ is a subset of the closure of the evolution set.

3. Need to show that if $x_k \in \Gamma^+$ and $x_k \to x$ then $x \in \Gamma^+$ as well. Define a sequence of t_{kj} such that $x(t_{kj}) \to x$ by taking the diagonal of sequences $\{t_k^j\}$ where t_k^j is the sequence of points t_k for which $x(t_k^j) \to x_j$ for j fixed. Just make sure that $|x(t_{kj}) - x_k| < 1/k$ and $t_{kj} \to \infty$. It converges to x because $|x_k - x| < \epsilon/2$ for k large enough and $|x(t_{kj}) - x_k| < \epsilon$ for j large enough, using the triangle inequality this implies that $|x(t_{kj}) - x| < \epsilon$. Therefore $x \in \Gamma^+$.

4. Need to show that for any $y_0 \in \Gamma^+$, then the corresponding solution $\{y(t) : t \in \mathbb{R}^+\} \subset \Gamma^+$. $x(t_j) \to y_0 = y(0) \xrightarrow{t} y(t)$, then $x(t_j + t) \to y(t)$ too, by continuity of solution w.r.t. initial condition because if we took $x_0 \notin \Gamma^+$ but close enough to y_0, then $x_0(t)$ should be close to $y_0(t)$ as well.

5. If not, then there is an $\epsilon > 0$ and sequence $\{t_k\} \to \infty$ such that $d(x(t_k), \Gamma^+) \geq \epsilon$; in other words, that $|x(t_k) - z| \geq \epsilon$ for all $z \in \Gamma^+$. But $x(t_k)$ bounded implies it is sequentially bounded (subsequence has a limit), call it $\overline{x}$; then by definition $\overline{x} \in \Gamma^+$, contradiction (since $\overline{x} \notin \Gamma^+$ since $|\overline{x} - z| \geq \epsilon$ for *all* $z \in \Gamma^+$).

$\square$

9 Lecture 9

Last time we talked about positive limit set, as the set of limit points of a trajectory, and we enumerated a litany of properties which this limit set satisfies.

9.1 LaSalle's Invariance Principle

Theorem 9.1. *Let $V : \mathbb{R}^n \to \mathbb{R}$ be $\mathcal{C}^1$ positive definite, and satisfy*

$$\dot{V}(x) \leq 0, \quad \forall x \in \mathbb{R}^n \tag{9.1}$$

Let $S := \{x \in \mathbb{R}^n : \dot{V}(x) = 0\}$ and M be the largest invariant subset of S. Then every bounded solution $x(t)$ converges to M as $t \to \infty$.

Corollary 9.1 (Barbashin-Krasovski)**.** *Under the same hypotheses of the previous theorem, if $M = \{0\}$ and all the solutions are bounded, then the system is GAS.*

Proof of Theorem. Fix $x(t)$ a (arbitrary) bounded solution. From Lyapunov's theorem, the hypothesis that $\dot{V}(x) \leq 0$ implies that $x(t)$ is already Lyapunov stable and since $V(x(t))$ is (not necessarily strictly) decreasing, it has a limit as $t \to \infty$, call it $C \geq 0$. Let Γ^+ be the positive limit set of $x(t)$. By definition, every $z \in \Gamma^+$ is the limit of $\{x(t_k)\}_{k\in\mathbb{N}}$ for some sequence of times $\{t_k\} \subset \mathbb{R}$. $V(x) = V(\lim\limits_{k\to\infty} x(t_k)) = \lim\limits_{k\to\infty} V(x(t_k)) = C$ where the second equation holds by continuity. Here z was arbitrary which means this limit is for every $z \in \Gamma^+$, i.e. $V(\Gamma^+) = \{c\}$.

From property 4 (of Γ^+ from last lecture), Γ^+ is invariant which implies that $\dot{V}(x) = 0$ on Γ^+ because $V \equiv c$ on Γ^+. From property 5, $\lim\limits_{t\to\infty} x(t) \in \Gamma^+$. To summarize, $\Gamma^+ \subset M \subset S := \{x : \dot{V}(x) = 0\}$, and $x(t) \to \Gamma^+$, which is exactly what we wanted to show. □

Remarks: LaSalle states that $x(t) \to M$, not $x(t) \to \Gamma^+$ because Γ^+ is hard to find and depends on choice of $x(t)$; M is typically easier to find and works for all solutions $x(t)$.

Boundedness of $x(t)$ is crucial; unbounded solutions, if they exist, may not converge to M (and in general probably will not). IF V is radially unbounded (an additional hypothesis), then $\dot{V} \leq 0$ implies that each solution is bounded.

Locally, for x_0 close enough to zero, solutions are bounded by Lyapunov stability.

Positive definiteness of V in theorem is needed for Lyapunov stability but not needed for $x(t) \to M$ claim; we only need a lower bound on $V(x(t))$, which exists because $x(t)$ is bounded.

We want to use the corollary to show global asymptotic stability. Need to be able to show that $M = \{0\}$ iff there are no nonzero solutions along $\dot{V} \equiv 0$.

Example:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -x_2 - \varphi(x_1) \end{pmatrix} \tag{9.2}$$

satisfying $x_1\varphi(x_1) > 0$ for all $x_1 \neq 0$. Energies are given by

$$K = \frac{1}{2}x_2^2 \tag{9.3}$$

$$U = \Phi(x_1) = \int_0^{x_1} \varphi(z)dz \tag{9.4}$$

We tried a Lyapunov equation as total (internal) energy, $v = \frac{1}{2}x_2^2 + \varphi(x_1)$ and $\dot{V} = \varphi(x_1)x_2 - x_2(x_2 + \varphi(x_1)) = -x_2^2 \leq 0$. Recall that this L. function did not work before using Lyapunov analysis. Let's see how LaSalle can help. The set S is just the x axis, $S = \{(x_1, x_2) \subset \mathbb{R}^2 : x_2 = 0\}$. Suppose there is a nonzero solution along which $\dot{V} = 0$, i.e. one that remains in the x_1 axis, so $x_2 \equiv 0$. But if $x_2 = 0$ then $\dot{x}_2 \equiv 0$, so that by equation of $\dot{x}$, both $\dot{x}_2$ and $\dot{x}_1$ are identically zero. Then $\varphi(x_1)$ must be identically zero which implies that $x_1 \equiv 0$. This establishes that the equilibrium is the only solution for which $\dot{V} = 0$, and by LaSalle's invariance principle (or the corollary of B-K), provided that V is radially unbounded, the system is globally asymptotically stable. Note, however, that V need not be radially unbounded; so it is necessary to stipulate certain conditions on φ to ensure radial unboundedness of V.

Another example from adaptive control:

$$\dot{x} = \theta^* x + u \tag{9.5}$$

where θ^* is fixed but unknown. Idea: introduce a guess $\hat{\theta}$ for value of θ^* and define the control law $u = -(\hat{\theta}+1)x$. If $\theta^* = \hat{\theta}$, then the closed loop $\dot{x} = -x$ which is nicely stable. If they are not equal, tune $\hat{\theta}$ according to

$$\dot{\hat{\theta}} = x^2 \tag{9.6}$$

which will grow fast if x does not converge to zero (thereby eventually $\hat{\theta}$ will dominate and x *will* go to zero). Lets try $V(x, \hat{\theta}) = \frac{1}{2}(x^2 + (\hat{\theta} - \theta^*)^2)$, then $\dot{V} = x\dot{x} + (\hat{\theta} - \theta^*)\dot{\hat{\theta}} = -x^2 \leq 0$.

```
clear all
theta=100;
f=@(t,x)[(theta-x(2)-1)*x(1);(x(1))^2]
figure
hold all
[tp,xp]=ode45(f,[0,2],[1,1]);
plot(tp,xp(:,1));
figure
plot(tp,xp(:,2));
```

$S = \{(x, \hat{\theta}) : \dot{V} = 0\}$ which is the set of points where $x = 0$. $M = \{0\}$ iff there are no nonzero solutions in $\hat{\theta}$ axis; S consists entirely of equilibria, $M = S$, implies that LaSalle tells us nothing about $\hat{\theta}$ (see figure; here θ=100 but $\theta \rightarrow\approx 180$). This is nevertheless a good result: V radially unbounded and $\dot{V} \leq 0$ we can still say that $\hat{\theta}$ remains bounded.

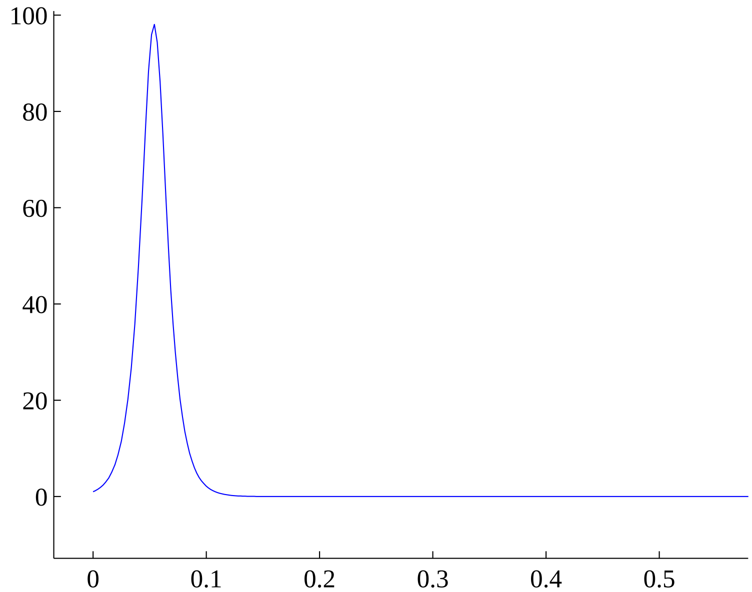

Figure 1: Simulation of behavior of x

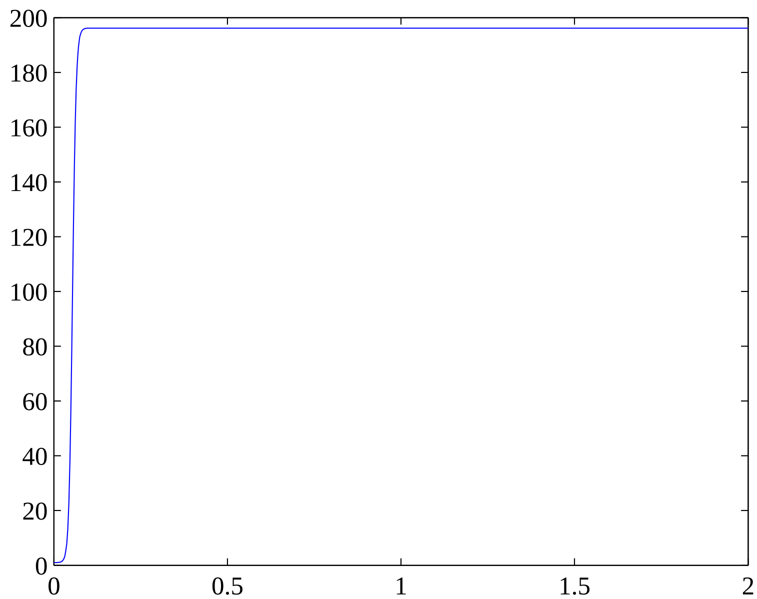

Figure 2: Behavior of Adaptation

10 Lecture 10

10.1 Lyapunov's First Indirect Method

Idea: use quadratic Lyapunov function to show local asymptotic stability.

Let $V(x) = x^T Px$, $P = P^T > 0$ $\mathcal{C}^1$ positive definite and radially unbounded. Also, suppose that $\lambda_{min}(P)|x|^2 \leq V(x) \leq \lambda_{max}(P)|x|^2$;

$$\dot{x} = f(x), \quad f(0) = 0 \tag{10.1}$$

construct linearization; for each component $f_i(x)$, $i = 1, \ldots, n$, apply mean value theorem to write $f_i(x) = f_i(0) + \frac{\partial f_i}{\partial x}(z_i) \cdot x$ where $z_i \in (0, x_i)$. Since 0 is equilibrium point the first term is zero, so we can rewrite this as

$$\frac{\partial f_i}{\partial x} f(0)x + [\frac{\partial f_i}{\partial x}(z_i) - \frac{\partial f_i}{\partial x}(0)] \cdot x \tag{10.2}$$

The temr in brackets approaches zero as $x \to 0$ by continuity. Therefore, we have

$$\frac{\partial f_i}{\partial x}(0) \cdot x + g_i(x) \tag{10.3}$$

where $g_i(x) = o(|x|)$, which means exactly that

$$\lim_{x \to 0} \frac{g_i(x)}{|x|} = 0 \tag{10.4}$$

We can repeat this for each i, and we have

$$f(x) = \frac{\partial f}{\partial x}(0) \cdot x + g(x) \tag{10.5}$$

where $\frac{\partial f}{\partial x}$ denotes the Jacobian matrix

$$\frac{\partial f}{\partial x} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix} \tag{10.6}$$

To avoid cumbersome notation, we call $A :=$ the Jacobian of f, and write $f(x) = Ax + g(x)$ which approximates the behavior of the original system near 0.

Theorem 10.1 (Lyapunov's First Method)**.** *Given system*

$$\dot{x} = f(x), \quad f(0) = 0 \tag{10.7}$$

and linearization $f(x) = Ax + g(x)$ with $g(x) = o(|x|)$. Suppose A is Hurwitz; thne the system is locally asymptotically stable at 0.

Proof. Recall from ECE 515 that A Hurwitz implies that there's a positive definite matrix $P = P^T > 0$ satisfying the Lyapunov equation

$$PA + A^T P = -Q < 0 \tag{10.8}$$

for some positive definite matrix Q. Let $V(x) = x^T Px$. Along solutions of $\dot{x} = f(x) = Ax + g(x)$, we get $\dot{V}(x) = x^T(PA+A^TP)x+2x^TPg(x) = -x^TQx+2x^TPg(x) \leq -\lambda_{min}(Q)|x|^2+2|x|||P|||g(x)|$ but we know that $\frac{|g(x)|}{|x|} \to 0$ as $|x| \to 0$; in other words $|x| < \delta \Rightarrow |g(x)| \leq \epsilon|x|$ for given ϵ and there is such a related $\delta(\epsilon)$ ($\forall\epsilon\exists\delta\ldots$). Then for $|x| < \delta$, we have $\dot{V} \leq -\lambda_{min}(Q)|x|^2 + 2\epsilon||P|||x|^2 = -(\lambda_{min}(Q)-2\epsilon||P||)|x|^2$. Pick ϵ small enough so that $\dot{V} < 0$ for $|x| < \delta$. This implies by Lyapunov's 2nd method local asymptotic stability. □

Comments: $\lambda_{min}(P)|x|^2 \leq x^TPx \leq \lambda_{min}(P)|x|^2$, and $\dot{V} \leq -\rho|x|^2 \leq -2\lambda V$ for some $\lambda > 0$. As $\epsilon \to 0$, $\lambda \to$ decay rate for linearized dynamics. By comparison principle, $V(x(t)) \leq e^{-2\lambda t}V(x_0)$, and this implies thta $\lambda_{min}(P)|x(t)|^2 \leq e^{-2\lambda t}\lambda_{max}(P)|x_0|^2$; divide through by λ_{min} and take square roots to get

$$|x(t)| \leq ce^{-\lambda t}|x_0| \tag{10.9}$$

where $c = \sqrt{\frac{\lambda_{max}(P)}{\lambda_{min}(P)}}$, and this equation means that the system is not only locally Lyapunov stable, but logcally exponentially stable, and the convergence rate near zero is that of $\dot{x} = Ax$.

$$|x(t)| \leq ce^{-\lambda t}|x_0| \tag{10.10}$$

Around the origin the lyapunov function has level sets which can take various shapes, e.g. ellipses, and the more elongated the ellipses are the larger the ratio is between λ_{max} and λ_{min}.

If $A = \frac{\partial f}{\partial x}(0)$ has at least one eigenvalue with positive real part, then the origin is not stable. Idea: unstable eigenvalue of A dominates behavior of $g(x)$. A rigorous proof via (Chetaev) instability theorem can be found in Khalil.

If $Re(\lambda_i) \leq 0$ for all eigenvalues but at least one λ_i has real part exactly 0, then the test is inconclusive and there's nothing we can say from Lyapunov's test regarding stability or instability; sometimes this is called the critical case. We do know, however, in this case that local exponential stability is *not* possible, but nevertheless local asymptotic stability is possible. This will follow later from converse Lyapunov theorem for exponential stability.

10.2 Some Examples

$$\dot{x} = -x - x^3 \tag{10.11}$$

and the linearization is simply

$$\dot{x} = -x \tag{10.12}$$

and this is asymptotically stable, so the original system is locally exponentially stable. In fact, this system is globally asymptotically stable by Lyapunov's second method, since $V(x) = x^2$ gives

$$\dot{V} = -x^2 - x^4 < 0 \quad \forall x \neq 0 \tag{10.13}$$

it happens also to be globally exponentially stable since $\dot{V} \leq -x^2$. On the other hand,

$$\dot{x} = -x + x^3 \tag{10.14}$$

is still locally exponentially stable (by both Lyapunov's 1st and 2nd method (use same V)).

$$\dot{x} = -x^3 \tag{10.15}$$

Here the linearization is zero so Lyapunov's first method doesn't help here; on the other hand, the direct method with $V(x) = x^2/2$ gives $\dot{V} = -x^4 < 0\ \forall x \neq 0$ which implies global asymptotic stability, but convergence rate is not exponential. Solve the ode to get $x(t) \sim \frac{1}{\sqrt{t}}$.

Nonlinear damped spring, revisited

$$\ddot{x} + \dot{x} + \varphi(x) = 0 \tag{10.16}$$

where $x\varphi(x) > 0$ for all $x \neq 0$. Rewriting in state space

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -x_2 - \varphi(x_1) \end{pmatrix} \tag{10.17}$$

Compute linearization

$$A = \frac{\partial f}{\partial x}|_{x=0} = \begin{pmatrix} 0 & 1 \\ -\varphi'(0) & -1 \end{pmatrix} \tag{10.18}$$

and the linearized system is

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\varphi'(0) & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{10.19}$$

and A is Hurwitz iff if $\varphi'(0) > 0$ (for det $\begin{pmatrix} -\lambda & 1 \\ -\alpha & -1-\lambda \end{pmatrix} = \lambda^2 + \lambda + \alpha$ and apply Routh criteria on coefficients). So under this condition, the first method gives local exponential stability. Compare this conclusion with the same example from a previous lecture when we used Lyapunov's second method, from which we got global asymptotic stability with a creatively found V, under the assumpion that $x\varphi(x) > 0$. We also used LaSalle's invariance principle, and that gave us local asymptotic stability under the same conditions and global asymptotic stability with strengthened condition that $x\varphi(x) > kx^2$ for positive k (where here we used $V = \frac{1}{2}x_2^2 + \Phi(x_1)$ radially unbounded). For the first method, we get only local asymptotic stability, under the added condition that $\varphi'(0) > 0$, a local condition of $x\varphi(x) > kx^2$. The tradeoff is that the first method is relatively easy to apply but its conclusion is not as strong as could be with the second method (gives no global conclusions, and gives nothing if $\varphi'(0) = 0$).

10.3 Stability of Nonautonomous Systems

Consider

$$\dot{x} = f(t, x) \tag{10.20}$$

Looking a these systems is relevant for analysis of even autonomous systems. Consider $\dot{x} = f(x)$ and we want to track a reference trajectory call it $x_{ref}(t)$; for this problem we track the error dynamics for $e := x - x_{ref}$ with

$$\dot{e} = \dot{x} - \dot{x}_{ref} = f(x) - \dot{x}_{ref} = f(e + x_{ref}) - \dot{x}_{ref} \tag{10.21}$$

Thus in state space our system is given by

$$\begin{pmatrix} \dot{e} \\ \dot{x} \end{pmatrix} = \begin{pmatrix} f(e + x_{ref}(t)) - \dot{x}_{ref}(t) \\ f(x) \end{pmatrix} \tag{10.22}$$

where these terms describing dynamics of e are time dependent.

The properties that we want from $\dot{x} = f(t, x)$ will be:

1. *Uniform* stability
2. *Uniform* asymptotic stability
3. Global *uniform* asymptotic stability

We'll see that $V(x)$ is not always enough; we will sometimes need to work with $V(t, x)$, from which $\dot{V} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x)$, and we will want this to be negative uniformly over time.

11 Lecture 11

11.1 Stability of Non-Autonomous Systems

$\dot{x} = f(t, x)$ with $x \in \mathbb{R}^n$ and $t \in [t_0, \infty)$, and $f(t, 0) \equiv 0$ an equilibrium at the origin. Given function $V = V(t, x) : [t_0, \infty) \times \mathbb{R}^n \to \mathbb{R}$, and $\dot{V}(t, x) := \frac{\partial V}{\partial t}(t, x) + \frac{\partial V}{\partial x}(t, x) \cdot f(t, x)$, along solution $x(t)$ for $\dot{x} = f(t, x)$ we have

$$\frac{d}{dt} V(t, x(t)) = \dot{V}(t, x(t)) \tag{11.1}$$

Suppose there is a continuous positive definite functions $W_1(x)$, $W_2(x)$ such that

$$W_1(x) \leq V(t, x) \leq W_2(x) \quad \forall t, x \tag{11.2}$$

This implies that $V(t, x)$ is positive definite in the sense that $V(t, 0) = 0$ and $V(t, x) > 0$ for all $x \neq 0$, for all t; property $V(t, x) \leq W_2(x)$ is called decrescent property of V.

Theorem 11.1 (Lyapunov's Direct Method for Time-Varying Systems)**.** *Given system*

$$\dot{x} = f(t, x) \tag{11.3}$$

with equilibrium $f(t, 0) \equiv 0$, and $V(t, x)$ $\mathcal{C}^1$ positive definite and decresecent. Then the following hold:

1. $\dot{V}(t, x) \leq 0 \Rightarrow$ *origin is uniformly stable.*
2. *If $\dot{V}(t, x) \leq -W_3(x) < 0$ for all $x \neq 0$, then origin is uniformly asymptotically stable.*
3. *If condition 2. holds and W_1 is radially unbounded, then we have GUAS.*

Proof. 1 is similar to the proof of 1. for Lyapunov's Theorem in the autonomous case $\dot{x} = f(x)$. We need to show that for every $\epsilon > 0$ there is a $\delta > 0$ such that $|x_0| < \delta \Rightarrow |x(t)| < \epsilon$ for all t_0 and all $t \geq t_0$. Fix $\epsilon > 0$ and pick positive b satisfying

$$b < \min_{|x|=\epsilon} W_1(x) \tag{11.4}$$

and pick $\delta > 0$ such that if $|x| \leq \delta$ we have $V(x) \leq b$. Then if $|x_0| \leq \delta$, from $\dot{V} \leq 0$ we know that $V(t, x(t)) \leq V(t_0, x_0) \leq W_2(x_0) \leq b$. This means that $|x(t)|$ remains less than ϵ because if at some t we have $|x(t)| = \epsilon$, then $W_1(x(t)) > b$ by definition of b, and hence $V(t, x(t)) \geq W_1(x(t)) > b$, which contradicts that $V(t, x(t)) \leq b$ for all t.

For 2., let $\delta = \delta(\epsilon)$ for some $\epsilon > 0$, and let $|x_0| \leq \delta$. There is class $\mathcal{K}$ functions α_1, α_2 on $[0, \epsilon]$ such that $\alpha_1(|x|) \leq W_1(x) \leq V(t, x) \leq W_2(x) \leq \alpha_2(|x|)$. Similarly, $\dot{V} \leq -W_3(x) - \alpha_3(|x|)$, where $\alpha_3 \in \mathcal{K}$ on $[0, \epsilon]$; $\alpha_2 \in K$ has an inverse α_2^{-1}. This gives $\dot{V} \leq -\alpha_3(\alpha_2^{-1}(V)) =: -\alpha(V)$ (a composition of two class $\mathcal{K}$ functions and is therefore also class $\mathcal{K}$).

Claim: the scalar system

$$\dot{y} = -\alpha(y), \quad y \in \mathbb{R} \tag{11.5}$$

, for α continuous positive definite, is asymptotically stable: $\exists \beta \in \mathcal{KL}$ such that $|y(t)| \leq \beta(|y_0|, t-t_0)$ (at least for $|y_0| \leq \delta$ some δ). Proof of claim: use $V(y) = \frac{1}{2}y^2$; then $\dot{V}(y) = -y\alpha(y) < 0,\ \forall y > 0$. Note that β can be explicitly constructed from α (Lemma 4.4 K). Use comparison principle:

$$\dot{y} = -\alpha(y), \quad y(t_0) = V(t_0, x_0) \geq 0 \tag{11.6}$$

$$\dot{V} \leq -\alpha(V) \tag{11.7}$$

it tells us that

$$V(t, x(t)) \leq \beta(V(t_0, x_0), t - t_0) \tag{11.8}$$

Then

$$\alpha_1(|x(t)|) \leq V(t, x(t)) \leq \beta(V(t_0, x_0), t - t_0) \leq \beta(\alpha_2(|x_0|), t - t_0) \tag{11.9}$$

Applying α_1^{-1} to the outer terms, we have

$$|x(t)| \leq \alpha_1^{-1}(\beta(\alpha_2(|x_0|), t - t_0)) =: \overline{\beta}(|x_0|, t - t_0) \in \mathcal{KL} \tag{11.10}$$

This shows UAS.

3. has the same proof as in the autonomous case. □

Remark: If α_1, α_2, α_3 are scalar multiples of some power of $|x|$,

$$k_1|x|^a \leq V(t, x) \leq k_2|x|^a \tag{11.11}$$

and

$$\dot{V}(t, x) \leq -k_3|x|^a \tag{11.12}$$

for some $k_{1,2,3}, a > 0$, then 0 is uniformly exponenentially stable (and globally so if the inequalities holds globally).

We already made a similar observation earlier for quadratic bounds $a = 2$. Then

$$\dot{V} \leq -\frac{k_3}{k_2}V \Rightarrow V(t, x(t)) \leq e^{-\frac{k_3}{k_2}(t-t_0)}V(t_0, x_0) \tag{11.13}$$

where the implication holds by the comparison principle. This implies that

$$k_1|x(t)|^a \leq e^{-\frac{k_3}{k_2}(t-t_0)}k_2|x_0|^a \tag{11.14}$$

and then

$$|x(t)| \leq e^{-\frac{k_3}{ak_2}(t-t_0)}(\frac{k_2}{k_1})^{1/a}|x_0| =: ce^{-\lambda(t-t_0)}|x_0| \tag{11.15}$$

where the last term is the desired UES estimate.

For example: given linear time varying dynamics (LTV),

$$\dot{x} = A(t)x \tag{11.16}$$

$V(t, x) = x^T P(t)x$ where $0 < c_1 I \leq P(t) \leq c_2 I$ for all t and some $c_1, c_2 > 0$. This makes V positive definite and decrescent. The derivative of V then is

$$\dot{P}(t) + P(t)A(t) + A^T(t)P(t) \leq -Q(t) \leq -c_3 I,\ c_3 > 0 \tag{11.17}$$

Then remark applies with $a = 2$ and we have $GUES$. In fact, for linear (even time-varying) $UAS \Rightarrow GUES$.

Briefly, Lyapunov's First (indirect) method for time varying systems

$$\dot{x} = f(t,x) = A(t)x + g(t,x) \tag{11.18}$$

where $A(t) = \frac{\partial f}{\partial x}(t,0)$. If $\frac{\partial f}{\partial x}$ is uniformly continuous in t for each x, then $g(t,x) = o(|x|)$. If $\dot{x} = A(t)x$ is asymptotically stable (or GUES, automatic for linear systems), then $\dot{x} = f(t,x)$ is locally GUES. There is no simple eigenvalue test for the non-autonomous case to check for this condition.

11.2 LaSalle-Like Theorems for Time Varying Systems

There is some information which can be gotten from

$$\dot{V}(t,x) \leq -W_3(x) \leq 0 \tag{11.19}$$

where W_3 is positive semi definite). However, the conclusions are not as strong as Lasalle's results from time invariant systems.

12 Lecture 12

12.1 LaSalle Like Stability Theorem for Time Varying Systems

Theorem 12.1. *Let*

$$\dot{x} = f(t, x) \tag{12.1}$$

satisfy the usual conditions, and let $f(t, 0)$ be uniformly bounded, namely $f(t, 0) \leq C$ for some $C > 0$ all $t \geq 0$. Let $V(t, x)$ be positive definite decrescent, i.e. there exists w_1 and w_2 positive definite satisfying

$$w_1(x) \leq V(t, x) \leq w_2(x) \tag{12.2}$$

and that there is a w_3 continuous positive semi definite:

$$\dot{V}(t, x) \leq -w_3(x) \leq 0, \ \ \forall x \tag{12.3}$$

Then all bounded solutions converge to the set $S := \{x : w_3(x) = 0\}$.

Example: Consider

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} e^{-2t} x_2 \\ -e^{-2t} x_1 - x_2 \end{pmatrix} \tag{12.4}$$

and let $V(x_1, x_2) = \frac{1}{2}(x_1^2 + x_2^2)$, then $\dot{V} = -x_2^2 \leq 0$, so $S = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 = 0\}$; applying the usual procedure, $x_2 \equiv 0 \Rightarrow \dot{x}_2 \equiv 0$ which implies by the second equation that $x_1 \equiv 0$ and this implies that there are no nonzero solutions in $S \Rightarrow M = \{0\}$. Nevertheless it is not true that x converges to 0. To see this, pick x_0 such that $x_1(t_0) > 0$ and $x_2(t_0) = 0$. The first observation, is that a solution $x(t)$ will remain in the $B_{||x_0||}$ ball about 0. In other words, $|x(t)| \leq |x_0|$ for all t. We solve the equations:

$$x_1(t) = x_1(0) + \int_{t_0}^{t} e^{-2s} x_2(s) ds \tag{12.5}$$

and use $|a - b| \geq |a| - |b|$ to write

$$|x_1(t)| \geq |x_1(t_0)| - |x_1(t_0)| \cdot \int_{t_0}^{t} |e^{-2s}| ds \geq \frac{1}{2} |x_1(t_0)| \tag{12.6}$$

Where the last inequality follows from that the integral, evaluated, is $-\frac{1}{2} e^{-2s}|_{t_0}^{t} \leq -\frac{1}{2}(e^{-2t} - 1) \leq \frac{1}{2}$.

Therefore, $x_1(t) \not\to 0$.

Boundedness of solutions follows locally from $\dot{V} \leq 0$, and global boundedness follows from w_1 being radially unbounded ('forces V to be radially unbounded').

Before proving the theorem, we will need the following lemma:

Lemma 12.1 (Barbalat's Lemma). *Suppose that $x(t)$ is bounded, $\dot{x}(t)$ is bounded, w is continuous, and $\int_{t_0}^{\infty} w(x(t)) dt < \infty$. Then $w(x(t)) \to 0$ as $t \to \infty$.*

Proof. Since $w(x)$ is uniformly continuous over D, we have that for every $\epsilon > 0$ there is a $\delta_x > 0$ such that $|x - y| \leq \delta_x \Rightarrow |w(x) - w(y)| \leq \frac{\epsilon}{2}$. Secondly, since $\dot{x}$ is bounded, $x(t)$ is also uniformly continuous, i.e. given any $\delta_x > 0$ there is a δ_t such that $|t_1 - t_2| \leq \delta_t \Rightarrow |x(t_1) - x(t_2)| \leq \delta_x$. Combining, we have that $|t_1 - t_2| \leq \delta_t \Rightarrow |w(x(t_1) - w(x(t_2))| \leq \epsilon/2$.

Now we show that $w(x(t)) \to 0$ as $t \to \infty$. Suppose not; then there is an $\epsilon > 0$ and sequence $\{t_k\} \to \infty$ such that $w(x(t_k)) \geq \epsilon$ for all k. From uniform continuity, $|t - t_k| \leq \delta_t$, then $w(x(t)) \geq \epsilon/2$. For simplicity, take $w \geq 0$ (in the theorem w_3 has this property). Then we integrate and get $\int_{t_0}^{t} w \geq \sum^{\infty} \delta_t \epsilon/2 = \infty$. □

Proof of Theorem. Consider $\dot{V}(t,x) \leq -w_3(x)$ and integrate from t_0 to t; then $V(t,x(t)) - V(t_0,x_0) = -\int_{t_0}^{t} w_3(x(s))ds$, iff $\int_{t_0}^{t} w_3(x(s))ds \leq V(t_0,x_0) - V(t,x(t)) \leq V(t_0,x_0)$ for all t where the last inequality holds because V is positive definite. Therefore $\int_{t_0}^{\infty} w_3(x(s))ds$ exists and is bounded. We need to show that $w_3(x(s)) \to 0$ as $s \to \infty$. If $x(t)$ is bounded, the image is contained in a compact set D; since $w_3(x)$ is continuous, it is uniformly continuous over D, i.e. for every $\epsilon > 0$ there is a $\delta_x > 0$ such that $|x - y| \leq \delta_x \Rightarrow |w(x) - w(y)| \leq \frac{\epsilon}{2}$.

Now we want to invoke the previous lemma: we have already that $x(t)$ is bounded, that w is continuous and that its integral is finite. But we still have to check that $\dot{x}$ is bounded. In our system, $\dot{x}(t) = f(t,x(t))$ but Lipschitness of f implies that $|f(t,x) - f(t,0)| \leq L|x|$ for all $x \in D$ and all t. We assumed in the statement of the theorem that $f(t,0) \leq c$ for all t, so these two things together imply that $f(t,x)$ must also be bounded for all $x \in D$ and all t as desired. Now we can apply the lemma, to get that $w_3(x(t)) \to 0$ as $t \to \infty$, which is exactly what we wanted to show. □

As a special case, this proof also shows that for $\dot{x} = f(x)$ (time invariant), if $\dot{V} \leq 0$ for all x then every bounded solution converges to $S = \{x : \dot{V} \leq 0\}$. Note that this is weaker than LaSalle, but the proof does not involve properties of limit set Γ^+.

For general time varying systems, the stronger claim that $x(t) \to M$ is not true. It is, however, true for some special classes of time varying systems (see [1]).

13 Lecture 13

Las time we proved LaSalle-Yoschizawa,

$$w_1(x) \leq V(t,x) \leq w_2(x) \tag{13.1}$$

and

$$\dot{V}(t,x) \leq -w_3(x) \leq 0 \tag{13.2}$$

both imply that along bounded solutions

$$\lim_{t\to\infty} w_3(x(t)) = 0 \tag{13.3}$$

We want to know when $w_3(x) \to 0$ implies that $x \to 0$. Define output $y = w_3(x)$, so want $y \to 0 \Rightarrow x \to 0$. This is observability. Interpretation of LaSalle-Yoshizawa, $\dot{V} \leq -w_3(x)$ and observable w.r.t. $y := w_3(x)$ implies $(G)AS$ (G if all solutions are bounded).

Example, consider the linear time varying system

$$\dot{x} = A(t)x \tag{13.4}$$

and

$$V(t,x) = x^T P(t) x \tag{13.5}$$

$$\dot{V}(t,x) = x^T[P(t)A(t) + A^T(t)P(t) + \dot{P}(t)]x \tag{13.6}$$

$$Q(t) = C^T(t)C(t) \tag{13.7}$$

for some $C(t)$ such that the following equation holds

$$P(t)A(t) + A^T(t)P(t) + \dot{P}(t) \leq -C^T(t)C(t) \leq 0 \tag{13.8}$$

Define $y := C(t)x$. Then $\dot{V} \leq -y^T y$. Observability Gramian:

$$M(t_0, t_0+\delta) = \int_{t_0}^{t_0+\delta} \Phi^T(t,t_0)C^T(t)C\Phi(t,t_0)dt \tag{13.9}$$

and we say that the system is uniformly observable if there is a positive $c \in R$ such that $M(t_0, t_0 + \delta) \geq cI$ for all t_0.

Uniform observability and equation 13.8 imply GES. See [1, Khalil] for proof.

13.1 Converse Theorems

We had several versions of Lyapunov's stability theorems. Conclusions, e.g.

1. $\dot{V} < 0 \,\forall\, x \neq 0 \Rightarrow$ AS
2. $\dot{V} < 0 \,\forall\, x \neq 0$ and V radially unbounded $\Rightarrow$ GAS.
3. $K_1|x|^2 \leq V(x) \leq k_2|x|^2$ and $\dot{V} \leq -k_3|x|^2 \Rightarrow$ GES.
4. $\dot{x} = f(t,x)$, $V(t,x)$ satisfying $-\dot{V} \leq -w_3(x) < 0 \forall\, x \neq 0$ implies UAS.

Recall from 515, if $A \in M_n(\mathbb{R})$ is Hurwitz, then there is, for every positive definite symmetric Q, a unique $P = P^T > 0$ such that

$$PA + A^T P = -Q \tag{13.10}$$

For $V = x^T P x$, $\dot{V} = x^T(PA + A^T P)x = -x^T Q x < 0 \forall x \neq 0$, and P is in fact given by

$$P = \int_0^\infty e^{A^T\tau} Q e^{A\tau} d\tau \tag{13.11}$$

There is also an extension of this result to LTV systems: $\dot{x} = A(t)x$ and assume that it's GUES, i.e.

$$||\Phi(\tau,t)|| \leq ce^{-\lambda(\tau-t)}, \quad c, \lambda > 0 \tag{13.12}$$

which means

$$|x(t)| = |\Phi(\tau,t)x_0| = ||\Phi(\tau,t)|||x_0| \leq c|x_0|e^{-\lambda(\tau-t)} \tag{13.13}$$

Then there is a quadratic Lyapunov function $x^T P(t) x$ where

$$P(t) = \int_0^\infty \Phi^T(\tau,t)Q(\tau)\Phi(\tau,t)d\tau \tag{13.14}$$

where $Q(\cdot)$ is chosen to satisfy for some $a, b > 0$,

$$0 < aI \leq Q(t) \leq bI \tag{13.15}$$

The Lyapunov function is

$$V(t,x) = x^T P(t) x = \int_t^\infty x^T \Phi^T(\tau,t)Q(\tau)\Phi(\tau,t)x d\tau \tag{13.16}$$

Notice that the $\Phi(\tau,t)x$ term is the solution at time τ of the system starting at time t. We can rewrite this as

$$\int_t^\infty \varphi^T(\tau;t,x)Q(\tau)\varphi(\tau;t,x)d\tau \tag{13.17}$$

where $\varphi(\tau;t,x)$ denotes solution at time τ which starts in state x at time t.

Claim: $\dot{V}(t,x) < 0$, and in fact $c_1|x|^2 \leq V \leq c_2|x|^2$ and $\dot{V} \leq -c_3|x|^2$ which proves GES. This will follow from a more general converse theorem for GES nonlinear systems.

Theorem 13.1 (Converse Lyapunov For Exponential Stability)**.** *Take a nonlinear LTV*

$$\dot{x} = f(t, x),\ f(t, 0) \equiv 0, f \in \mathcal{C}^1 \tag{13.18}$$

and there are $L, r > 0$ such that

$$||\frac{\partial f}{\partial x}|| \leq L, \quad \forall\, t, \forall\, |x| \leq r \tag{13.19}$$

Assume UES (U automatic), namely

$$|x(t)| \leq c|x_0|e^{-\lambda(t-t_0)},\ \forall\, |x_0| \leq r/c \tag{13.20}$$

Then locally in r/c-ball about 0, there is a function $V(t, x)$ satisfying the following three properties

1. $c_1|x|^2 \leq V(t, x) \leq c_2|x|^2$ *with* $c_1, c_2 > 0$.
2. $\dot{V} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \leq -c_3|x|^2$ *with* $c_3 > 0$.
3. $|\frac{\partial V}{\partial x}| \leq c_4|x|$, $c_4 > 0$ *(V is 'quadratic-like').*

This V can be defined by

$$V(t, x) = \int_t^{t+\delta} \varphi^T(\tau; t, x)\varphi(\tau; t, x)d\tau \tag{13.21}$$

for $\delta > 0$ sufficiently large. Moreover, if hypotheses hold for $r = \infty$ (globally) and the system is GES, then this V also works for all x.

Note that this formula for V is not constructive because it requires knowledge of solutions. If the system is time-invariant

$$\dot{x} = f(x) \tag{13.22}$$

then the solutions $\varphi(\tau; t, x) \mapsto \overline{\varphi}(\tau - t, x)$ and V becomes $(s = \tau - t)$

$$V = \int_0^{\delta} \overline{\varphi}^T(s, x)\overline{\varphi}(s, x)ds \tag{13.23}$$

which is independent of t: $V = V(x)$.

Proof. 1. $V(t, x) = \displaystyle\int_t^{t+\delta} |\varphi(\tau; t, x)|^2 d\tau \leq \int_t^{t+\delta} c^2 e^{-2\lambda(\tau-t)} d\tau |x|^2$ using the exponential stability estimate. The right hand side is computable; it is $\frac{c^2}{2\lambda}(1 - e^{-2\lambda\delta})|x|^2 =: c_2|x|^2$. This gives the upper bound in the first statement. Using $||\frac{\partial V}{\partial x}|| \leq L$ by the Mean Value Theorem we have $|f(t, x)| \leq L|x|$ since $f(t, 0) = 0$. This implies that the solutions cannot decay faster than e^{-Lt}, using the comparison principle. Solutions $|\varphi(\tau; t, x)|^2 \geq |x|^2 e^{-2L(\tau-t)}$ which implies that $V(t, x) \geq \displaystyle\int_t^{t+\delta} e^{-2L(\tau-t)} d\tau |x|^2$ can compute this to get a constant $\frac{1}{2L}(1 - e^{-2L\delta})|x|^2 =: c_1$, and this proves the lower bound in the first statement.

2. Need to differentiate the expression for V:

$$\dot{V}(t,x) = \varphi^T(t+\delta;t,x)\varphi(t+\delta;t,x) - \varphi^T(t;t,x)\varphi(t;t,x) + \int_t^{t+\delta} 2[\varphi^T(\tau;t,x)\varphi_t(\tau;t,x) + \varphi_x(\tau;t,x)f(t,x)]d\tau \tag{13.24}$$

so

$$\dot{V}(t,x) = \varphi^T(t+\delta;t,x)\varphi(t+\delta;t,x) - |x|^2 + \int_t^{t+\delta} 2[\varphi^T(\tau;t,x)\varphi_t(\tau;t,x) + \varphi_x(\tau;t,x)f(t,x)]d\tau \tag{13.25}$$

Claim: $\varphi_t + \varphi_x f(t,x) = 0$; for $\varphi(\tau;t,x) = \varphi(\tau;t+\Delta t, x+\Delta t f(t,x))$ differentiating with respect to Δt gives the result $0 = \varphi_t + \varphi_x f$. Therefore we have

$$\dot{V}(t,x) = \varphi^T(t+\delta;t,x)\varphi(t+\delta;t,x) - |x|^2 \leq c^2|x|^2e^{-2\lambda\delta} - |x|^2 = -(1-c^2e^{-2\lambda\delta})|x|^2 \tag{13.26}$$

Then defined $c_3 := (1-c^2e^{-2\lambda\delta})$ for δ large enough (namely $\delta > \frac{\log(c^2)}{2\lambda}$)., proving 2.

Proof of 3. is similar; see Khalil.

□

14 Lecture 14

14.1 Applications of Converse Lyapunov Theorems

Last time we proved converse Lyapunov theorem for (L)ES:

(L)ES$\Rightarrow$ that there is a function V satisfying $c_1|x|^2 \le V(t,x) \le c_2|x|^2$, $\dot{V} \le -c_3|x|^2$ and $|\frac{\partial V}{\partial x}| \le c_4|x|$.

Application to Lyapunov's 1st (indirect) method. $\dot{x} = f(x)$, linearization, $\dot{x} = Ax$ then $f(x) = Ax + g(x)$ where $g(x) = o(|x|)$. We have $\dot{x} = Ax$ exponentially stable implies that $\dot{x} = f(x)$ is LES (realied on converse Lyapunov functoin for $\dot{x} = Ax$). In the other direction, we use converse Lyapunov for LES for nonlinear systems. We can in fact prove this implication in Lyapunov's 1st method for more general time-varying case, i.e. we'll show that if $\dot{x} = f(x)$ is locally exponentially stable, then its linearization $\dot{x} = A(t)x$ must be exponentially stable.

Have: $\dot{x} = A(t)x$ the linearization, and $\dot{x} = f(t,x) = A(t)x + g(t,x)$ with $|g(t,x)| = o(|x|)$ (uniformly in t).

Since $\dot{x} = f(t,x)$ is locally exponentially stable, [by converse Lyapunov for exponential stability] there is a V such that $\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f \le -c_3|x|^2$, and $|\frac{\partial V}{\partial x}| \le c_4|x|$. Now let's compute $\dot{V}$ along the linearization $\dot{x} = Ax = f(t,x) - g(t,x)$, and $g \sim o(|x|)$ means that for every $\epsilon > 0$ there is a $\delta > 0$ such that if $|x| < \delta$ then $\frac{|g(t,x)|}{|x|} < \epsilon$. Then $\dot{V}|_{\dot{x}=Ax} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f - \frac{\partial V}{\partial x} g \le -c_3|x|^2 - \frac{\partial V}{\partial x} g \le -c_3|x|^2 + |\frac{\partial V}{\partial x}||g| \le -c_3|x|^2 + c_4\epsilon|x|^2$ when $|x| < \delta(\epsilon)$. By taking ϵ small enough (i.e. $\epsilon < \frac{c_3 - k}{c_4}$), for $0 < k < c_3$, we get that

$$\dot{V}|_{\dot{x}=A(t)x} \le -k|x|^2 \tag{14.1}$$

This together with $c_1|x|^2 \le V \le c_2|x|^2$ implies that the linearization is exponentially stable, which is what we wanted to prove.

14.2 Vanishing Perturbations (§9.1[1])

Given system

$$\dot{x} = f(t,x) + g(t,x) \tag{14.2}$$

with f the nominal system and g perturbation. Suppose that the nominal system $\dot{x} = f(t,x)$ is exponentially stable (either locally or globally), and the perturbation is vanishing the following sense: $|g(t,x)| \le \epsilon|x|$ for all t for some $\epsilon > 0$. By converse Lyapunov for exponential stability, there is a V satisfying the same conclusions as we had before:

1. $c_1|x|^2 \le V(t,x) \le c_2|x|^2$
2. $\dot{V}_f = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \le -c_3|x|^2$
3. $|\frac{\partial V}{\partial x} \le c_4|x|$

with all $c_1, \ldots, c_4 > 0$. Now we consider $\dot{V}_{f+g} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) + \frac{\partial V}{\partial x} g(t,x)$ and this we can bound, as before, by $-c_3|x|^2 + c_4\epsilon|x|^2$. The only difference between here and before is that ϵ is fixed, not *arbitrarily* small. Nevertheless, if $\epsilon < c_3/c_4$, then there is a $k > 0$ such that $\epsilon < \frac{c_3-k}{c_4}$ and this implies that $\dot{V}_{f+g} \leq -k|x|^2$, and therefore $\dot{x} = f + g$ is locally exponentially stable. Conclusion: exponential stability is persevered under 'small enough' vanishing perturbations. (Note that if we only had asymptotic stability, we would not be able to make this argument.) Also, this doesn't give us a very constructive means of analyzing robustness; it merely guarantees a "sense" of robustness qualitatively.

14.3 Non-vanishing Perturbations (§9.2[?])

Same basic system

$$\dot{x} = f(t,x) + g(t,x) \tag{14.3}$$

with nominal dynamics $\dot{x} = f(t,x)$ exponentially stable but this time the perturbation satisfies only

$$|g(t,x)| \leq \epsilon \tag{14.4}$$

a small but does not vanish as $x \to 0$. Proceed as before: converse lyapunov theorem gives V with the usual properties, $\dot{V}_f \leq -c_3|x|^2$ and $\frac{\partial V}{\partial x}| \leq c_4|x|$ and look at $\dot{V}_{f+g} \leq -c_3|x|^2 + c_4\epsilon|x|$. The only difference with what we had before is that the perturbation is linear, not quadratic. The reason this is a problem is that for small x the dominating term will be the perturbation. We will show, then, that though we can't get a handle on what happens near zero, at a large scale the system still behaves nicely. Thus

$$\dot{V}_{f+g} \leq -c_3|x|^2 + c_4\epsilon|x| = -c_3|x|(|x| - \frac{c_4}{c_3}\epsilon) < 0 \tag{14.5}$$

if $|x| > \frac{c_4}{c_3}\epsilon =: r$. We want to claim that a level set which touches the ball B_r is an attractive invariant set, but not the ball itself. Consider then $\max\limits_{|x|=r} V(x) =: c_M$ and the level set $\Omega_{c_M} := \{x : V(x) = c_M\}$, the smallest level set of V which contains the r ball B_r, i.e. $|x| \leq r \Rightarrow V \leq c_2 r^2$. Outside of this set V is decreasing, and once we enter this set we will remain there. If we want a *ball* which is attractive, then take $r = \max_{x\in\Omega_{c_M}} ||x||$. In other words, $V \leq c_2 r^2 \Rightarrow |x| \leq \sqrt{\frac{c_2}{c_1}} r$. Conclusion: eventually, $|x(t)| \leq \sqrt{\frac{c_2}{c_1}} \frac{c_4}{c_3}\epsilon$. This is called *ultimate boundedness*: exponential or even asymptotic stability is *not* preserved under non-vanishing perturbations.

More on such analysis later (input to state stability). Note: recall vanishing perturbations looked like $|g| \leq \epsilon|x|$ and non-vanishing looked like $|g| \leq \epsilon$. Near origin vanishing is a stronger property, but far away it's weaker. Therefore, we have the same conclusion for g such that $g \leq \epsilon$ for small x and $g \leq \epsilon|x|$ for larger x.

14.4 Converse Lyapunov Theorems for Asymptotic Stability

We will discuss but not prove the following theorems (the first is 4.16[1]).

Theorem 14.1 (Massera)**.** *$\dot{x} = f(t,x)$, $x = 0$ equiliberium, f is $\mathcal{C}^1$, for $|x| \leq r$: $\frac{\partial f}{\partial x}$ is bounded uniformly over t and assume that the system is UAS: i.e. there is a $\beta \in \mathcal{KL}$ such that $|x(t)| \leq \beta(|x_0|, t - t_0)$, for every $|x_0| \leq r_0$ such that $\beta(r_0, 0) \leq r$. Then there is a $\mathcal{C}^1$ function V satisfying:*

1. $\alpha_1(|x|) \leq V(t,x) \leq \alpha_2(|x|)$
2. $\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \leq -\alpha_3(|x|)$
3. $|\frac{\partial V}{\partial x}| \leq \alpha_4(|x|)$

with $\alpha_{1,\ldots,4} \in \mathcal{K}$ on $[0, r_0]$.

If $f = f(x)$ (time invariant) then V can be chosen independently of t.

Comments: Massera's construction of V looks like $V(t,x) = \displaystyle\int_t^\infty G(|\varphi(\tau;t,x)|)d\tau$ where G is class $\mathcal{K}$ with some other properties.

Bound $|\frac{\partial V}{\partial x}| \leq \alpha_4(|x|)$ useful for perturbation analysis: for $\dot{x} = f + g$, we get (if $|g| < \epsilon$), $\dot{V}_{f+g} \leq -\alpha_3(|x|) + \alpha_4(|x|)\epsilon < 0$ if $\alpha_3(|x|) > \alpha_4(|x|)\epsilon$; if $|x| \leq r$ and $|x| \geq \alpha_3^{-1}(\alpha_4(r)\epsilon)$ then $\dot{V} < 0$.

The next one is Theorem 4.17[1], and is only for time invariant systems.

Theorem 14.2 (Kurzweil)**.** *$\dot{x} = f(x)$ asymptotically stable around 0 equilibrium and f is locally lipschitz. Then there is a continuously differentiable function $V(x)$ satisfying*

1. *V is positive definite*
2. *$\frac{\partial V}{\partial x} f(x) \leq -\alpha_3(|x|) < 0$ for all $x \neq 0$ (α_3 positive definite)*
3. *if $\dot{x} = f(x)$ is GAS, then V is radially unbounded*

Comments; Kurzweil's construction looks like $V(x) = \sup_{t \geq 0}\{g(\varphi(t;x))k(t)\}$ (recall that $\varphi(t;x)$ is the solution at time t starting at state x). With this construction, there is no integration; it is global and gives radially unbounded V for GAS while Massera's construction works only locally around zero. On the other hand, Kurzweil's construction does not give a bound on the gradient, $|\frac{\partial V}{\partial x}|$.

Reference: L. Vu, D. Liberzon, in *Sytems and Control Letters*, 54 (2005), pp 405-416, available at `http://liberzon.csl.illinois.edu/research/comm_sys_jn.pdf`

15 Lecture 15

15.1 Stability of Interconnected Systems: Small Gain and Passivity

We consider internal stability of $\dot{x} = f(x)$ as it relates to inputs, outputs, feedback, $\dot{x} = f(x, u)$, $y = h(x)$.

Consider system $\Sigma_1 \xrightarrow{u} (+d_1)\Sigma_1 \xrightarrow{y} (+d_2)\Sigma_2$.

Absolute stability problem: Special case Σ_1 is LTI system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{15.1}$$

and Σ_2 is static nonlinearity $u = \varphi(y)$. Closed loop system: $\dot{x} = Ax + B\varphi(Cx)$ has much more structure than $\dot{x} = f(x)$.

15.2 Small Gain Theorems

View the following system $u \to \Sigma \to y$ from some input space U to output space Y passing through some 'dynamics' Σ. Fix initial condition x_0. Need to pick U and Y normed vector spaces. Generally we'll take $U = \mathcal{L}_p$ and $Y = \mathcal{L}_q$ for some $1 \le p, q \le \infty$, $p = q$ or $p \ne q$. For example, if $p = q = 2$, then $U = Y = \mathcal{L}_2$ and $||u||^2_{\mathcal{L}_2} = \int_0^\infty |u(t)|^2 dt$. Similarly, $p = q = \infty$, then in $\mathcal{L}_\infty$, $||u||_{\mathcal{L}_\infty} = \sup_{t \in [0,\infty)} |u(t)|$. In general, we have

$$||u||_{\mathcal{L}_p} = \sqrt[p]{\int_0^\infty ||u||^p dt} \tag{15.2}$$

Definition 15.1. *We will say that system Σ has finite $\mathcal{L}_p$ to $\mathcal{L}_q$ gain $\gamma > 0$ if $||y||_{\mathcal{L}_q} \le \gamma ||u||_{\mathcal{L}_p} + c$ where $c \ge 0$ depends on choice of x_0 and equals 0 when $x_0 = 0$.*

By convention, we take the infimum of all γ which work in the definition; we call it the induced gain, $\gamma = \sup_{u \ne 0} \frac{||y||_{\mathcal{L}_q}}{||u||_{\mathcal{L}_p}}$. More generally, could have $||y|| \le \alpha(||u||)$ with $\alpha \in \mathcal{K}_\infty$.

When $p = q$, instead of $\mathcal{L}_p$ to $\mathcal{L}_q$ gain, we'll say merely '$\mathcal{L}_p$ gain'. Most often used: $\mathcal{L}_2$ gain (or just gain, by default it's $\mathcal{L}_2$). Relation to stability: if $u \equiv 0$, then $||y||$ is bounded for all x_0 and in particular, if we set zero initial condition, then $y \equiv 0$. Under observability property, can conclude that $x \to 0$.

Want: Lyapunov (sufficient) conditions for induced gain.

Lyapunov sufficient condition for finite $\mathcal{L}_2$ gain: suppose there is a function V such that $\dot{V} \le -|y|^2 + \gamma^2 |u|^2$. We claim that then the system's $\mathcal{L}_2$ induced gain exists and is at most γ.

Proof. Similar to proof of LaSalle-Yoshizawa. We integrate from 0 to t:

$$V(t) - V(0) \leq -\int_0^t |y|^2 dt + \gamma^2 \int_0^t |u|^2 dt \tag{15.3}$$

and rearranging:

$$\int_0^t |y|^2 dt \leq \gamma^2 \int_0^t |u|^2 dt + V(0) - V(t) \leq \gamma^2 \int_0^t |u|^2 dt + V(0) \tag{15.4}$$

where the last inequality holds from positive definiteness of V. Now we take square roots, using that $\sqrt{a^2 + b^2} \leq a + b$ (for $a, b \geq 0$) to get

$$\sqrt{\int_0^t |y|^2 dt} \leq \gamma \sqrt{\int_0^t |u|^2 dt} + \sqrt{V(0)}, \ \ \forall t \tag{15.5}$$

As this holds for all time, we can take the limit as $t \to \infty$ to get

$$\lim_{t \to \infty} \sqrt{\int_0^t |y|^2 dt} = ||y||_{\mathcal{L}_2} \leq \gamma ||u||_{\mathcal{L}_2} + c \tag{15.6}$$

where $c = \sqrt{V(x_0)}$.

□

Alternative Lyapunov condition for finite $\mathcal{L}_2$ gain: $\mathcal{L}_2$ gain is less than or equal to γ if for every $\hat{\gamma} > \gamma$ there is an $\epsilon > 0$ and $V(x)$ positive definite and $\mathcal{C}^1$ such that

$$\dot{V} \leq -|y|^2 + \hat{\gamma}^2 |u|^2 - \epsilon |x|^2 \tag{15.7}$$

.

Some properties/facts of induced gain:

1. A linear (possibly time varying) GES system has finite $\mathcal{L}_2$ gain and in fact finite $\mathcal{L}_p$ to $\mathcal{L}_q$ gains for any p, q. To show this, use variation of constants formula, i.e. for $\dot{x} = A(t)x + B(t)u, \ \ y = C(t)x$ we have
$$y(t) = C(t)[\Phi(t, t_0)x_0 + \int_{t_0}^t \Phi(t, s)B(s)u(s)ds] \tag{15.8}$$
and GES iff $||\Phi(t, s)|| \leq ce^{-\lambda(t-s)}||$ for $c, \lambda > 0$.
2. For LTI systems, $\mathcal{L}_2$ gain equals $\sup_\omega ||G(i\omega)||_2 = \sup_\omega \sigma_{\max}(G(i\omega))$ where G denotes the transfer matrix, and $\sigma_{\max}$ denotes the maximal singular value.
3. For LTI systems with A Hurwitz, $\mathcal{L}_2$ gain $\leq \gamma$ iff Eq 15.7 holds.

Consider $u_1 \xrightarrow{-y_2} \Sigma_1 \to y \xrightarrow{+u_2} \Sigma_2 \to 2 \xrightarrow{+u_1} \Sigma_1$; Σ_1, Σ_2 act on $\mathcal{L}_2$ for simplicity (could have considered in more generality $\Sigma_1 : \mathcal{L}_p \to \mathcal{L}_q$, and similarly for Σ_2). Assume the following

1. Σ_1 has gain γ_1: $||y_1|| \leq \gamma_1 ||e_1|| + c_1$
2. Σ_2 has gain γ_2: $||y_2|| \leq \gamma_2 ||e_2|| + c_2$
3. small gain condition: $\gamma_1\gamma_2 \leq 1$

Then the overall system has finite gain from $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ to $\begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$ or what amounts to exactly the same from $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ to $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$.

In particular, if there are no external inputs u_1 and u_2, then y_1 and y_2 have finite norms.

Proof. Show finite gain from $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ to $\begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$; we have $e_1 = u_1 - y_1$, and $e_2 = u_2 + y_1$, and we want to show finite gain from $u's$ to $e's$; then $||e_1|| \leq ||u_1|| + ||y_2|| \leq ||u_1|| + \gamma_2 ||e_2|| + c_2$ where we use the gain from Σ_2. Since $e_2 = u_2 + y_1$ we can use triangle inequality again to get

$$||e_1|| \leq ||u_1|| + \gamma_2 ||u_2|| + \gamma_2 ||y_1|| + c_2 \tag{15.9}$$

Now using the second gain condition for Σ_1, we have

$$||e_1|| \leq ||u_1|| + \gamma_2 ||u_2|| + \gamma_2\gamma_1 ||e_1|| + \gamma_2 c_1 + c_2 \tag{15.10}$$

Recall that $\gamma_1\gamma_2 < 1$ we can solve for norm of e_1 by rearranging to get

$$||e_1|| \leq \frac{1}{1 - \gamma_1\gamma_2}(||u_1|| + \gamma_2 ||u_2|| + \gamma 2 c_1 + c_2) \tag{15.11}$$

which is exactly what we wanted to show, for e_1. A similar calculation proves the claim for e_2 and combine to get

$$|| \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} || \leq \gamma || \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} || + c \tag{15.12}$$

□

Special case of linear system + static nonlinearity, absolute stability scenario: $u \to \Sigma_1 \to y \to \varphi \to u$, with $\Sigma_1 : \begin{matrix} \dot{x} = Ax + Bu \\ y = Cx \end{matrix}$ and φ is confined to region determined by two lines $y = \pm ax$. Assume that LTI system Σ_1 has finite $\mathcal{L}_2$ gain γ and that static nonlinear φ satisfies sector condition $|\varphi(y)| \leq \frac{1}{\overline{\gamma}}|y|$ for some $\overline{\gamma} > \gamma$. The the closed loop system

$$\dot{x} = Ax + B\varphi(Cx) \tag{15.13}$$

is asymptotically stable.

Proof. Recall: because Σ_1 has gain γ, such that Equation 15.7 holds. For closed loop system, plug in $u = \varphi(y)$ with $|\varphi(y)| \leq \frac{1}{\overline{\gamma}}|y|$ and we get

$$\dot{V} \leq -|y|^2 + \overline{\gamma}^2 \frac{1}{\overline{\gamma}^2}|y|^2 - \epsilon |x|^2 < 0, \ \forall x \neq 0 \tag{15.14}$$

which implies asymptotic stability (and GAS if V is radially unbounded). This connects Equation 15.7 with external signals to standard Lyapunov condition. □

16 Lecture 16

16.1 Passivity

$u \to \Sigma \to y$ with

$$\begin{aligned} \dot{x} &= f(x,u) \\ y &= h(x) \end{aligned} \tag{16.1}$$

Definition 16.1. *A system is passive if there is a function $V = V(x)$ such that*

$$\dot{V} \leq u^T y \tag{16.2}$$

We call V a storage function and $u^T y$ the suppy rate.

For example, electric circuit, u is the voltage, y the current, and uy the power inflow. Passive system dissipates (does not generate) energy/power.

Integrating, $\dot{V} \leq u^T y$, we get

$$V(T) - V(0) \leq \int_0^t u^T(s) y(s) ds \tag{16.3}$$

Note that we could define passivity in terms of $\int u^T y \geq 0$ (for $x_0 \geq 0$) and look for storage function.

Variant of passivity: if we have equality $\dot{V} = u^T y$, we say that the system is lossless. We could have something like $\dot{V} \leq -u\varphi(u) + u^T y$, with $u\varphi(u) > 0$ for all $u \neq 0$, then this is called input-strictly passive. Also, could have $\dot{V} \leq -y\varphi(y) + u^T y$, called output strictly passive. Finally, with $\dot{V} \leq -W(x) + u^T y$ with W positive definite, we call this strictly passive (or state strictly passive).

Examples:

$$\begin{aligned} \dot{x} &= -ax + u, \quad && a > 0 \\ y &= x && x \in \mathbb{R}^1 \end{aligned} \tag{16.4}$$

Try for a candidate storage function $V(x) = \frac{1}{2}x^2$ so that $\dot{V} = -ax^2 + ux = -ax^2 + uy$ where the second term is the supply rate and the term ax^2 is our positive definite W term, so this is [state] strictly passive.

16.2 Passivity of System Interconnections

$\Sigma_1 \xrightarrow{y} \Sigma_2 \xrightarrow{-} u \to \Sigma_1$

Suppose that

1. Σ_1 is strictly passive, so $\dot{V} \leq -W(x) + u^T y$ (with x the state at Σ_1 and W is positive definite
2. Σ_2 is a state feedback function. For example, Σ_2 can be a linear map ($u = -ky$, with $k \geq 0$). Then for the closed loop system, $\dot{V} \leq -W(x) - y^T ky \leq -W(X) < 0$ for all $x \neq 0$, which implies that the closed loop system is asymptotically stable.

   Alternatively, a static nonlinearity, $u = -\varphi(y)$, with "sector condition" $y^T\varphi(y) > 0$ for all $y \neq 0$ (or simply $y\varphi(y) \geq 0$, $\forall y$). For scalar u, y, this is the usual picture where the graph of φ passes through third and first quadrants. Recall, for small gain we had a graph confined to the region enclosed by two symmetric lines of positive and negative slopes ($\varphi(y)| \leq \frac{1}{\gamma}|y|$). For closed loop, $\dot{V} \leq -W(x) - y^T\varphi(y) \leq -W(x)$ and therefore the closed loop system is again asymptotically stable.

Note that if instead of strict passivity of Σ_1 we had output strict passivity, then we'd need LaSalle observability to show asymptotic stability of closed loop, because it tells us that if we get $\dot{V} \leq -W(y)$, we need to know something about how y relates to x (i.e. what is given by those observability conditions).

Can view static map $u = \varphi(y)$ as passive because $\displaystyle\int u^T y \geq 0$. More generally, we can have any passive system as Σ_2 and stability will follow from the next result.

$\Sigma_1(x_1) \to y \xrightarrow{+u_2} e_2 \to \Sigma_2(x_2) \to y_2 \xrightarrow{+u_1-(y_2)} e_1 \to \Sigma_1$ Same setup as we had last lecture for Small Gain Theorem.

Theorem 16.1. *Feedback connection of two (strictly) passive systems is (strictly) passive.*

$\dot{V}_1(x_1) \leq -W(x_1) + e_1^T y_1$ and similarly, $\dot{V}_2(x_2) \leq -W_2(x_2) + e_2^T y_2$. Claim: closed loop is strictly passive from $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ to $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ (note that we could also have chosen e for the output).

Proof. Need to show that there is a storage function $V(x_1, x_2)$ such that the derivative $\dot{V} \leq -W(x_1, x_2) + u_1^T y_1 + u_2^T y_2$ with W positive definite. Try $V = V_1 + V_2$ (heuristically, we can think of this as total stored energy). Then $\dot{V} = \dot{V}_1 + \dot{V}_2 \leq -W_1(X_1) - W_2(x_2) + e_1^T y_1 + e_2^T y_2$, and we will define $W(x_1, x_2) := W_1(x_1) + W_2(x_2)$ which is positive definite. However, we need to manipulate the expression to get ey in terms of uy; to do this we go back to the diagram, and write $e_1 = u_1 - y_2$, and $e_2 = u_2 + y_1$, and plugging into the previous expression, we get for $e_1^T y_1 + e_2^T y_2$:

$$(u_1 - y_2)^T y_1 + (u_2 + y_1)^T y_2 = u_1^T y_1 + u_2^T y_2 \tag{16.5}$$

which is exactly what we wanted. □

Corollary 16.1. *If $u_1 = u_2 = 0$ (no external inputs), then instead of strict passivity, we have that the closed loop system is asymptotically stable.*

For once we get rid of input we have that the storage function V becomes Lyapunov function.

16.3 Absolute Stability Problem

Σ_1 LTI and Σ_2 state nonlinearity, so we have $\Sigma_1 \to y \to \Sigma_2 \xrightarrow{-} u \to \Sigma_1$; Specifically,

$$\begin{aligned} &\Sigma_1 : \dot{x} = Ax + bu,\ y = c^T x \\ &\Sigma_2 : u = -\varphi(y) \end{aligned} \tag{16.6}$$

a single input single output system with sector nonliniearity, which satisfies $k_1 y^2 \leq y\varphi(y) \leq k_2 y^2$ where $0 \leq k_1 < k_2 \leq \infty$. The absolute stability problem is to find conditions on (A, b, c, k_1, k_2) guaranteeing global or local asysmptotic stability of all closed loop systems obtained in this way.

Aizerman's conjecture (1949): system is absolutely stable if the linear system obtained from $u = -ky$ is asymptotically stable for all $k \in [k_1, k_2]$. The statement of the conjecture is not true. Nevertheless, this [not true] conjecture was one of Aizerman's most influential contribution to the field.

This connects with passivity; for we already know that if LTI system Σ_1 is (strictly) passive and sector condition holds with $k_1 = 0$ and $k_2 = \infty$, then the closed loop system is asymptotically stable. (In fact, in that earlier result we did not require Σ_1 to be linear). Still, there is a disconnect, because this result only gives us conditions given k_1 and k_2 fixed at 0 and ∞ (which indeed is the best possible pair for k_1 and k_2) but it tells us nothing about conditions on (A, b, c) for LTI system to be passive. We want: given A, b, c when can we determine that there is a V without actually looking for it. Good news, there are computable/checkable conditions, but unfortunately they require working with the transfer function (i.e. we must step away from state space methods) $g(s) = c^T(Is - A)^{-1}b$.

Definition 16.2. *The transfer function $g(s)$ is called positive real if it satisfies the following two conditions:*

1. *$g(s) \in \mathbb{R}$ for all $s \in \mathbb{R}$.*
2. *$Re(g(s)) \geq 0$ when $Re(s) \geq 0$.*

Note that this second condition is not easy to check. However, when all poles of g (eigenvalues of A) are stable (i.e. in open left hand plane), it is enough to check the second condition along the imaginary axis, i.e. for all $s = i\omega$ with $\omega \in \mathbb{R}$. Notice that this is reminiscent of Nyquist: the second condition means that the Nyquist locus lies in the closed right half plane.

Definition 16.3. *$g(S)$ is strictly positive real if $g(s - \epsilon)$ is PR for some $\epsilon > 0$.*

From 2 above it's clear that SPR is stronger than PR. Connection with passivity.

17 Lecture 17

Recall from last time that we have for a transfer function

$$g(s) = c^T(Is - A^{-1}b \tag{17.1}$$

is positive real (PR) if both

1. $g(s) \in \mathbb{R}$ for all all s
2. $Re(g(s)) \geq 0$ (n the right hand plane ($Re(s) \geq 0$)

when all poles of g have negative real parts, it's enough to check the second condition for $s = i\omega$.

We say, also, that $g(s)$ is strictly positive real (SPR) if $g(s - \epsilon)$ is PR for some $\epsilon > 0$. For $g(s)$ to be PR it's necessary that its relative degree be zero or q, and g must have stable poles and stable zeros (minimum phase).

$g(S) = \frac{q(s)}{p(s)}$, with $rel\deg = \deg(p) - \deg(q)$. For example, $g(s) = \frac{1}{s+a}$, $a > 0$ (which comes from $\dot{x} = -ax + u$, $y = x$), the first condition is satisfied. and for $i = i\omega$, $g(i\omega) = \frac{1}{i\omega + a}$ have real part ≥ 0 for all $\omega \in \mathbb{R}$. In fact, it is also SPR, using $0 < \epsilon < a$.

17.1 Connection Between PR and Passivity

Theorem 17.1. *Consider*

$$\begin{aligned} \dot{x} &= Ax + bu \\ y &= c^T x \end{aligned} \tag{17.2}$$

both controllable and observable (and therefore a minimum realization of $g(s)$). If A is Hurwitz and $g(s) = c^T(Is - A)^{-1}b$ is SPR, then the system is strictly passive.

This is a consequence of KYP Lemma:

Lemma 17.1 (Kalman-Yakubovich-Popov)**.** *For LTI system as in the preceding theorem, if $g(S)$ is SPR, then there is a positive definite symmetric matrix $P = P^T > 0$ which satisfies*

1. $PA + A^T P = -Q < 0$ *for some* $Q > 0$
2. $Pb = c$

The lemma is not trivial to prove, but we'll use it to show the theorem.

Proof of Theorem. Write $V(x) := \frac{1}{2}x^T Px$ with P as in the lemma; differentiating, we have

$$\dot{V} = \frac{1}{2}x^T(PA + A^T P)x + x^T Pbu = -\frac{1}{2}x^T Qx + x^T cu = -\frac{1}{2}x^T Qx + u^T y \tag{17.3}$$

which implies strict passivity by definition. □

Last time, we had $\Sigma_1 \to y \to \Sigma_2 = \varphi(\cdot) \xrightarrow{-} u \to \Sigma_1$ with Σ_1 strictly passive and $\varphi \in [-, \infty)$ sector implied that the system is asymptotically stable, because we had

$$\dot{V} \leq -W(x) + u^T y = -W(x) - y^T \varphi(y) < 0, \; \forall x \neq 0 \tag{17.4}$$

In the absolute stability problem, we were looking at Σ_1 linear. Now we have a method by means of which we can determine passivity of Σ_1.

Proposition 17.1 (Passivity Criterion)**.** *IF $g(S)$ is SPR, A is Hurwitz, and φ is a $[0, \infty)$ sector nonlinearity, then $\dot{x} = Ax - b\varphi(c^T x)$ is GAS- this is absolute stability.*

This means, given $\Sigma_1 : (A, b, c)$ and $\Sigma_2 : (k_1, k_2)$-sector, passivity criterion gives absolute stability when $g(s) = c^T(Is - A)^{-1}b$ is SPR and $k_1 = 0$, $k_2 = \infty$.

17.2 Loop Transformations

[§7.1[1]]

If $\varphi \in [k, \infty$-sector, then we can write $\varphi(y) = ky + \overline{\varphi}(y)$ where $\overline{\varphi}(y) \in [0, \infty)$-sector. In other words, starting with overall system $\dot{x} = Ax - b\varphi(c^T x)$, if we can rewrite function φ as above, then we can incorporate the ky term in linear part of the system, and the nonlinearity that remains will be $[0, \infty)$-sector. So e.g. $\dot{x} = x + \varphi(x) = x + kx + \overline{\varphi}(x) = (1 + k)x + \overline{\varphi}(x)$.

Another example: if $\varphi \in [0, k]$-sector, then $\frac{\varphi}{1 - \frac{1}{k}\varphi} \in [0, \infty)$-sector. Using this idea, we can reduce $[k_1, k_2]$-sector nonlinearities in feedback with an LTI system to $[0, \infty]$-sector nonlinearities in feedback with another LTI system. After such a transformation, passivity criterion can be applied, and it gives:

Proposition 17.2 (Circle Criterion)**.** *Given*

$$\begin{aligned} \Sigma_1 :& \quad \dot{x} = Ax + bu, \; y = c^T x \\ \Sigma_2 :& \quad u = -\varphi(y) \end{aligned} \tag{17.5}$$

assume that $\frac{1+k_2 g(s)}{1+k_1 g(s)}$ is SPR ($g(s) = c^T(Is - A)^{-1}b$) and that $\varphi \in [k_1, k_2]$-sector (i.e. that $k_1 y^2 \leq y\varphi(y) \leq k_2 y^2$). Then $\dot{x} = Ax - b\varphi(c^T x)$ is absolutely stabe.

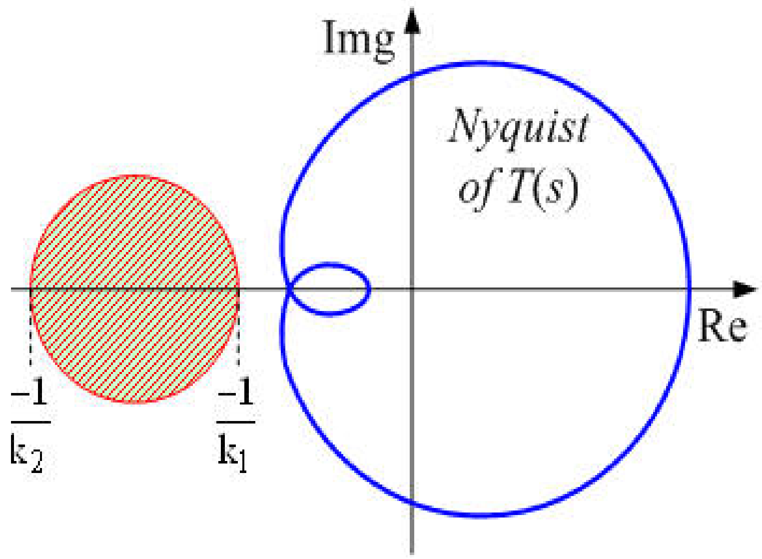

Figure 3: From `http://www.engr.du.edu/vahid/EnPo2.html`

Notes: this still relies on KYP lemma and gives a quadratic Lyapunov function. Also, A need not be Hurwitz, but $A - bk_1c^T$ must be.

Geometric interpretation: Draw a circle with outer points $-1/k_1$ and $1/k_2$. Then $\frac{1+k_2 g}{1+k_1 g}$ SPR means that the Nyqyuist plot of g lies outside this disk and encircles it m times CCW, where m is the number of unstble eigenvalues of A.

Special cases: $k_1 = 0$, $k_2 = \infty$, need g itself to be SPR; Nyquist plot of g must lie in RHP; this matches previous condition in passivity criterion.

Or $k_1 = k_2 = k$; φ becomes linear, $u = -ky$, and disk$\to$ point $-1/k$, and Nyquist plot encircles $-1/k$ m times as in classical control and Nyquist criterion for stability of linear feedback system.

Finally, consider $k_1 < 0$ and $k_2 > 0$ (so nonlinearity confined to [not necessarily symmetric] lines of positive and negative slope). If $k_2 = -k_1$, then this is $|\varphi(y)| < k|y|$ (we saw this in small gain theorem). Then the circle is centered at origin. Now the Nyquist plot must be *inside* the disk; LTI part must have gain$\leq 1/k$.

Consider $\Sigma_1 \to y \to \Sigma_2 \to u \to \Sigma_2$ where Σ_1 is usual LTI system as above and similarly Σ_2 is $[k_1, k_2]$-sector. Circle criterion gives quadratic Lyapunov function.

Proposition 17.3 (Popov Criterion)**.** *Suppose*

1. $g(s) = c^T(Is - A)^{-1}b$ *has one pole at 0 and all other poles in open LHP.*
2. $(1 + \alpha s)g(s)$ *is PR for some* $\alpha \geq 0$

3. $\varphi \in (0, \infty)$ sector: $0 < y\varphi(y)$ for all $y \neq 0$

Then $\dot{x} = Ax - b\varphi(c^T x)$ is absolutely stable.

In the statement above, α is called the 'Popov multiplier'. We're not testing stability of g itself, but a linear function multiple of $g(s)$. Note also that there's a strict inequality in the $(0, \infty)$ sector. Lyapunov function constructed in proof is no longer quadratic:

$$V(x) = x^T P x + \int_0^{c^T x} \varphi(z) dz \tag{17.6}$$

and it depends on choice of nonlinearity φ!

Compare with circle criterion: $V(x) = x^T P x$ did not depend on φ.

There are lots of examples where circle criterion failes: e.g. $\frac{1+k_2 g}{1+k_1 g}$, $k_1 = 0 \to 1 + k_2 g$ SPR.

17.3 ISS STability and Related Notions

[§4.9[1]]

$$\dot{x} = f(x, d) \tag{17.7}$$

x the state and d external input (disturbance and/or control). First we need to make sure that $\dot{x} = f(x, d(t))$ is well-posed (i.e. existence and uniqueness of solutions). We discussed this for $\dot{x} = f(t, x)$. If we let $\overline{f}(t, x) := f(x, d(t))$, we need conditions on $\overline{f}$, like, that $\overline{f}$ is piecewise continuous in t and locally lipschitz in x uniformly over t. What assumptions on $f(\cdot, \cdot)$ on $d(\cdot)$ guarantee the desired conditions on $\overline{f}$?

Once we have these conditions, we will try to understand the notion of gain from d to x, but not in the linear sense, but a notion more suitable for nonlinear systems.

18 Lecture 18

18.1 Input to State Stability

$\dot{x} = f(x, d(t))$ where x is the state and $d(t)$ is disturbance.

Define $\tilde{f}(t, x) = f(x, d(t))$ which satisfies the usual conditions, piecewise continuous in t and $\tilde{f}$ is Lipschitz in x uniformly in t. What properties of $\tilde{f}$ guarantee this? To get the first, we need both that d is piecewise continuous in t and f is continuous in d.

Then $f(x, d(\cdot))$ is piecewise continuous because a continous function acting on piecewise continuous function is still piecewise continuous. On the other hand, piecewise continuity of d is not enough since composition of two piecewise continuous functions is not necessarily piecewise continuous.

To get the second, we need that f is locally Lipschitz in x uniformly over d, so that d locally bounded:

$$|f(x, d(t)) - f(x_2, d(t))| \leq L|x - x_2| \tag{18.1}$$

or we can assume that $f(x, d)$ is locally Lipschitz as a function of (x, d).

Remark: 0-GES means GES under zero input (this is clear for lineary systems with A Hurwitz.)

18.2 Nonlinear Systems

Assume a system is 0-GES: $\dot{x} = f(x, 0)$ is GAS. Does this imply bounded input bounded state?, or convergent input convergent state?

Implications and nonimplications:

1. 0-GAS does not imply BIBS. For example, $\dot{x} = -x + dx$ for $d = 2$ is not bounded.
2. 0-GAS does imply CICS: $\dot{x} = -x + dx$, as $d \to 0$, then $x \to 0$ for this system only.
3. 0-GAS does not imply CICS: $\dot{x} = -x + dx^2$ has finite escape time.

18.3 ISS Definition

Let $\dot{x} = f(x, d(t))$.

1. 0-GAS for $d \equiv 0$, $|x(t)| \leq \beta(|x_0|, t)$ for $\beta \in \mathcal{KL}$.

2. "System Gain": $||x|| \leq \gamma(||d||)$, for $\gamma \in \mathcal{K}_\infty$; Pick ∞ norm: $||d|| = \sup_{0 \leq s \leq t} d(s)$.

   Causality means that $x(t)$ depends on $d(s)$ for $s \leq t$. Rewrite gain condition as

$$|x(t)| \leq \gamma(||d||_{[0,t]}) \tag{18.2}$$

   Caveat: only true for $x_0 = 0$ (as we would otherwise need to add a term for nonzero x_0).

Definition 18.1. *ISS: A system is ISS if its solutions satisfy*

$$|x(t)| \leq \beta(|x_0|, t) + \gamma(||d||_{[0,t]}), \quad \forall t \tag{18.3}$$

where $\beta \in \mathcal{KL}$ *and* $\gamma \in \mathcal{K}_\infty$

ISS implies BIBS, for d bounded implies that $\gamma(\sup(|d|))$ is constant. Also, ISS implies CICS; see Homework.

19 Lecture 19

19.1 Lyapunov Characterization ISS

$$\dot{x} = f(x, d) \tag{19.1}$$

ISS definition: $|x(t)| \leq \beta(|x_0|, t) + \gamma(||d||_{[0,t]})$; note that this is a global property. We could have given a local version as well.

Recall Passivity: $\dot{V} \leq -W(x) + u^T y$ and $\mathcal{L}_2$ gain: $\dot{V} \leq -|y|^2 + \gamma^2 |u|^2$.

An ISS-Lyapunov function is a function $V : \mathbb{R}^n \to \mathbb{R}$ which is positive definite and radially unbounded iff $\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|)$ for $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$.

Now we ask that $\dot{V} := \frac{\partial V}{\partial x} f(x, d) \leq \alpha(|x|) + \chi(|d|)$ with both $\alpha, \chi \in \mathcal{K}_\infty$.

Theorem 19.1. *System is ISS iff there is an ISS Lypaunov function.*

Example: $\dot{x} = -x + xd - x^3$ with both $x, d \in \mathbb{R}$. Try $V(x) = \frac{1}{2}x^2$. Then $\dot{V} = -x^2 + x^2 d - x^4$; we need to manipulate this expression to look like the definition of $\dot{V}$ for ISS Lypunov function; in particular, we need to decouple x and d terms. In this case, we can easily do that by noticing that the expression can be square completed: $\dot{V} = -x^2 - d^2/4 + x^2 d - x^4 + d^2/4 = -(d/2 - x^2)^2 + d^2/4 \leq -x^2 + d^2/4$, and so we have $\alpha = -x^2$ and $\chi = d^2/4$, and hence the system is ISS.

There is an equivalent way of writing the property for $\dot{V}$, and it will be more convenient for proving the theorem:

Lemma 19.1. *V is an ISS Lyapunov function iff there is an $\alpha_3, \rho \in \mathcal{K}_\infty$ such that*

$$|x| \geq \rho(|d|) \Rightarrow \dot{V} \leq -\alpha_3(|x|) \tag{19.2}$$

Often ρ is called the gain margin.

Proof. ($\Rightarrow$): Write $\dot{V} \leq -\alpha(|x|) + \chi(|d|) = -1/2\alpha(|x|) - 1/2\alpha(|x|) + \chi(|d|) \leq -1/2\alpha(|x|)$ *if* $\alpha(|x|) \geq \chi(|d|)$ which happens iff $|x| \geq \alpha^{-1}(2\chi(|d|)) =: \rho(|d|)$. SO we define $\alpha_3(|x|) = -1/2\alpha(|x|)$.

($\Leftarrow$): There are two cases. Case 1:, $|x| \geq \rho(|d|)$; then $\dot{V} \leq -\alpha_3(|x|)$, and we're done automatically.

The second is more involved, $|x| \leq \rho(|d|)$. Take this function $\dot{V} \leq -\alpha_3(|x|)$ and build $\chi \in \mathcal{K}_\infty$ as follow:

$$\chi(r) := \max_{|d| \leq r, |x| \leq \rho(|d|)} \{\dot{V}(x) + \alpha_3(|x|)\} \tag{19.3}$$

which is nondecreasing; it can be upper bounded by a class $\mathcal{K}_\infty$ function.

Then $\dot{V}(x) + \alpha_3(|x|) \leq \chi(|d|)$, and $\dot{V} \leq -\alpha_3(|x|) + \chi(|d|)$, as desired. □

Proof of Theorem. ($\Rightarrow$): This direction relies on converse Lyapunov Theorem for robust stability, and we won't prove this direction.

($\Leftarrow$): We will show that system is ISS if there is a V satisfying the gain margin property. Consider a ball of radius $\rho(||d||)$ in the x state space. The norm of d could either be on $[0, \infty)$ or on $[0, t]$. So either we take d to be bounded and take its sup value on the whole time interval or we consider it on each time interval and the ball will be changing as time evolves. In the end, it doesn't matter (on account of causality).

Consider an evolving trajectory of x; there are three (repeating) stages:

1. $x(t)$ is outside $\rho(||d||)$ ball centered at zero.
2. $x(t)$ is inside the ball.
3. $x(t)$ is outside again.

We will construct β and γ so that the ISS estimate holds at stage 1, 2, and 3 (and then by extension for all future times as well). (Recall that we will be using: $|x| \geq \rho(|d|) \Rightarrow \dot{V} \leq -\alpha_3(|x|)$, and we have $\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|)$.)

Stage 1: $|x(t)| \geq \rho(||d||) \geq \rho(|d(t)|)$, so the gain margin characterization applies here and we have that $\dot{V} \leq -\alpha_3(|x|) \leq -\alpha_3(\alpha_2^{-1}(V))$; still, $\alpha_3 \circ \alpha_2^{-1} \in \mathcal{K}_\infty$. We know that for systems evolving on $\mathbb{R}$, $\dot{y} = -\alpha_3 \circ \alpha_2^{-1}(y)$, $y(0) = V(x(0))$, there is a $\overline{\beta} \in \mathcal{KL}$, $|y(t)| \leq \overline{\beta}(y(0), t)$ (GAS) and by the comparison principle, we get the same for V:

$$V(x(t)) \leq \overline{\beta}(V(x_0), t) \tag{19.4}$$

but we want a bound not on V but on x. Nevertheless, using the α_1, α_2 bounds, we can easily switch between them:

$$|x(t)| \leq \alpha_1^{-1}(\overline{\beta}(V(x_0), t)) \leq \alpha_1^{-1}(\overline{\beta}(\alpha_2(|x_0|), t) =: \beta(|x_0|, t) \tag{19.5}$$

Stage 2: when $x(t)$ is inside the ball, $|x(t)| \leq \rho(||d||)$ and the ISS estimate holds with $\gamma = \rho$ (no β function), so there's nothing to do in stage 2.

Stage 3: if $x(t)$ exits the ball again, the situation we want to consider is the trajectory confined to the smallest (sub)level set of V containing the ρ ball, and find the radius of the smallest ball containing this level set. If $|x| \leq \rho(||d||)$ then $V(x) \leq \alpha_2 \circ \rho(||d||)$. If $V(x) \leq \alpha_2 \circ \rho(||d||)$, then $|x| \leq \alpha_1^{-1} \circ \alpha_2 \circ \rho(||d||) =: \gamma(||d||)$. During stage 3, we have that

$$|x(t)| \leq \gamma(||d||) \tag{19.6}$$

There's another way (using the same approach as in stage 1) to arrive at the same result: we consider x_0 as a point on the boundary of the ρ ball, and get $|x(t)| \leq \beta(\rho(||d||), 0) = \gamma(||d||)$; for all t, $|x(t)| \leq \beta(|x_0|, t) + \gamma(||d||)$ where the β comes from stage 1 and γ from stage 3.

□

Remark, it would be more difficult to prove the theorem if we were working directly with the definition of ISS, namely $\dot{V} \leq -\alpha(|x|) + \chi(|d|)$.

More info on ISS can be found here `http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.78.5962&rep=rep1&type=pdf`

20 Lecture 20

20.1 Some applications of ISS to interconnected systems

ISS: $|x(t)| \leq \beta(|x_0|, t) + \gamma(||d||_{[0,t]})$. Cascade systems:

$$\dot{x} = f(x) \xrightarrow{x} \dot{z} = g(z, x) \xrightarrow{z} \tag{20.1}$$

Question we ask is whether if x and z system are GAS and 0-GAS, respectively, then is the overall cascades system GAS? The answer is no, with counter example

$$\begin{aligned} \dot{z} &= -z + z^2 x \\ \dot{x} &= -x \end{aligned} \tag{20.2}$$

but we need to start with large enough initial condition x_0. $x \to 0$ indeed, but it's too late: z blows up in finite time. Solution: strengthen assumption: keep x system GAS, but make z-system ISS, and then this does imply that the overall system is GAS.

Sometimes this claim is called the 'Cascade Theorem'.

Proof. Since x system is GAS, we can write a bound

$$|x(t)| \leq \beta(|x_0|, t) \tag{20.3}$$

and for the z-system ISS implies that

$$|z(t)| \leq \beta_2(|z_0|, t) + \gamma(||x||_{[0,t]}) \tag{20.4}$$

These are the hypotheses, and we need to show that they imply that the overall system is GAS, which means that we have to show

$$| \begin{pmatrix} x(t) \\ z(t) \end{pmatrix} | \leq \beta(| \begin{pmatrix} x_0 \\ z_0 \end{pmatrix} |, t) \tag{20.5}$$

All that we need is a decay for z, and it is allowed to depend on both the initial condition of z and x but it must decay with time.

We start by manipulating the norm of x in the z ISS equation:

$$||x||_{[0,t]} = \max_{0 \leq s \leq t} |x(s)| \tag{20.6}$$

and we know that $|x(s)| \leq \beta_1(|x_0|, s)$ for each $s \geq 0$. Then

$$||x||_{[0,t]} \leq \max_{0 \leq s \leq t} \beta_1(|x_0|, s) = \beta_1(|x_0|, 0) \tag{20.7}$$

Then plugging in, we have

$$|z(t)| \leq \beta_2(|z_0|, t) + \gamma(\beta_1(|x_0|, 0)) \tag{20.8}$$

As we let time go to infinity, the first term decays but the second does not. So this approach is not going to be fruitful.

Instead, consider $|z(t)| \leq \beta_2(|z_0|, t-t_0) + \gamma(||x||_{[t_0,t]})$. Trick: make both t_0 and t go to infinity but in such a way that they are coupled, so let $t_0 = t/2$; then as $t \to \infty$, $t/2 \to \infty$ as well, *and* (this is the important part), $|t - t_0| \to \infty$ also.

Now, what we have is that

$$|z(t)| \leq \beta_2(|z(t/2)|, t - t/2) + \gamma(||x||_{[t/2,t]}) \tag{20.9}$$

Now we need to bound z at intermediate time, $t/2$, in terms of z at initial time 0:

$$|z(t/2)| \leq \beta_2(|z_0|, t/2) + ||x||_{[0,t/2]} \tag{20.10}$$

We break the latter term into two parts:

$$\begin{aligned} ||x||_{[t/2,t]} &\leq \beta_1(|x_0|, t/2) \\ ||x||_{[0,t/2]} &\leq \beta_1(|x_0|, 0) \end{aligned} \tag{20.11}$$

The first term decays as $t \to \infty$ but the second does not. However, the second is inside the $z(t/2)$ function which is itself the β_2 function which is already decaying, so this non-decay is not a problem.

We use the following fact: if $\alpha \in \mathcal{K}$, then $\alpha(r_1 + r_r) \leq \alpha(2r_1) + \alpha(2r_2)$, and we could continue inductively to get

$$\alpha(\sum^n r_i) \leq \sum \alpha(nr_j) \tag{20.12}$$

To see this e.g. $r_1 + r_2 \leq \max\{r_1, r_2\}$ so $\alpha(r_1 + r_2) \leq \max\{\alpha(2r_1), \alpha(2r_2)\}$.

Rest of the proof will be left as an exercise. □

Ultimately, we're not interested in only cascading systems; we want feedback. Thus we will discuss how to incorporate feedback using the current framework for analysis.

20.2 Nonlinear ISS [Generalization of] Small-Gain Theorem

Consider

$$z \xrightarrow{u} \dot{x} = f(x, z, u) \to x \xrightarrow{v} \dot{z} = g(z, x, v) \to z \tag{20.13}$$

Before in small gain we assumed that both individual subsystems had small gain and we figured out how to combine them to get small gain of the whole system.

Suppose that the x system is ISS w.r.t. inputs both z and u:

$$|x(t)| \leq \beta_1(|x_0|, t) + \gamma(|| \begin{pmatrix} z \\ u \end{pmatrix} ||_{[0,t]}) \tag{20.14}$$

and that we have the same thing for z system w.r.t. inputs x and v:

$$|z(t)| \leq \beta_2(|z_0|, t) + \gamma_2(|| \begin{pmatrix} x \\ v \end{pmatrix} ||_{[0,t]}) \tag{20.15}$$

and that we have a small gain condition

$$\gamma_1 \circ \gamma_2(r) \leq r, \ \forall r \geq 0 \tag{20.16}$$

Note that the small gain condition is sometimes written as

$$(\gamma_1 + \rho_1) \circ (\gamma_2 + \rho_2) \leq id, \ \text{some} \rho_1, \rho_2 \in \mathcal{K}_\infty \tag{20.17}$$

This is needed if we use '+' in ISS definition, but not if we use max.

The conclusion is then: The overall system is ISS w.r.t. (u,v) and in particular it is GAS if there are no u and v.

The proof is similar to the cascade proof but there's a little bit more work; can be found in paper 'Small Gain Theorem for ISS Systems and Applications' by [Jiang-Teel-Praly, 1994] (here `http://link.springer.com/article/10.1007%2FBF01211469`

An alternative proof is with ISS-Lyapunov functions $V_1(x), V_2(x) \to V(x,z) := \max\{V_1(x), \kappa(V_2(z))\}$ for appropriately defined $\kappa \in \mathcal{C}^1$ satisfying $\gamma_1 \circ \gamma_2 \leq id \Rightarrow \gamma_2 \leq \gamma_1^{-1}$ (paper in which this technique is used can be found here `http://www.sciencedirect.com.proxy2.library.illinois.edu/science/article/pii/0005109896000519`).

20.3 ISS related notions not covered in Class

1. Integral input to state stability (iISS). Definition: $\alpha(|x(t)|) \leq \beta(|x_0|, t) + \int_0^t \gamma(|u(s)|)ds$ where $\alpha, \gamma \in \mathcal{K}_\infty$, $\beta \in \mathcal{KL}$. This is actually weaker than ISS; an example is

$$\dot{x} = -x + dx \tag{20.18}$$

   which is not ISS because bounded input does not imply bounded state, but is iISS. (We will see this in more detail in a to-be-distributed past exam)

2. ISS is equivalent to the existence of a function V satisfying $\dot{V} \leq -\alpha(|x|) + \chi(|d|)$ where α is positive definite (not necessarily class $\mathcal{K}$), and $\chi \in \mathcal{K}_\infty$.

3. Dual notions: output to state stability (OSS), which is written analogous to ISS:

$$|x(t)| \leq \beta(|x_0|, t) + \gamma(||y||_{[0,t]}) \tag{20.19}$$

   an observability like notion (detectability)

21 Lecture 21

21.1 Nonlinear Feedback Control

$$\dot{x} = f(x, u),\ x \in \mathbb{R}^n,\ u \in \mathbb{R}^m \tag{21.1}$$

Goal is to achieve asymptotic stabilization to zero by feedback.

We will focus on systems affine in control, i.e. those of the form

$$\dot{x} = f(x) + g(x)u \tag{21.2}$$

where $G(x)$ is an $n \times m$ matrix. Assume that $f(0) = 0$. State feedback means that $u = k(x)$, we will generally stipulate that $k(0) = 0$.

Example 1:

$$\dot{x} = x^2 + xu \tag{21.3}$$

Then a good choice of stabilizing feedback would be e.g. $U = -x - 1$. A better control, which would satisfies $u(0) = 0$, would be $u = -x - x^2$ to give $\dot{x} = -x^3$.

Example 2:

$$\dot{x} = x + x^2 u \tag{21.4}$$

We can stabilize by taking $u = -2/x$ but at zero $u(0) = \infty$ and we can't easily fix it just by defining

$$u(x) = \begin{cases} -2/x & \text{if} \quad x \neq 0 \\ 0 & \text{else} \end{cases} \tag{21.5}$$

(Discontinuity at zero is not removable).

Example 3:

$$\dot{x} = x + x^2(x - 1)u \tag{21.6}$$

A stabilizing feedback $u = -\frac{2}{x(x-1)}$ blows up not only at 0 but also at 1, so the control actually ceases to exist at a finite time. This is worse than the situation in Example 2 since we can't really approach where we want to be.

We want to work with scalar valued $V(x)$ instead of x itself since usually this evolves in $\mathbb{R}^n$ (and therefore is harder to analyze).

We can revisit the previous examples with $V(x) = \frac{x^2}{2}$.

1. $$\dot{V} = x^3 + x^2 u \tag{21.7}$$

   and using $u = -x - x^2$ gives $\dot{V} = -x^4 < 0$ for all $V \neq 0$.

2.
$$\dot{V} = x^2 + x^3 u \tag{21.8}$$
and with $u = -2/x$ we get $\dot{V} = -x^2$. However, since x^3 is small compares to x^2 for $x \sim 0$, u is discountinuous at 0.

3. We have $\dot{V} = x^2 + x^3(x-1)u$ and $x^2 = 1$ at $x = 1$, so there's no stabilizing feedback.

Returning the general system
$$\dot{x} = f(x, u) \tag{21.9}$$
Let $V : \mathbb{R}^n \to \mathbb{R}$ be $\mathcal{C}^1$, positive definite, and optionally radially unbounded (if global results are desired). Then its derivative
$$\dot{V}(x, u) = \frac{\partial V}{dx} f(x, u) \tag{21.10}$$

Definition 21.1 (Artstein, '83)**.** *We call V a control Lyapunov function (CLF) if the following holds:*
$$\inf_{u \in \mathcal{U}} \dot{V}(x, u) < 0, \ \forall x \neq 0 \tag{21.11}$$
with $\mathcal{U} \subset \mathbb{R}^m$ a set of admissible controls. In other words, for each $x \neq 0$ there's a $u \in \mathcal{U}$ such that $\dot{V}(x, u) < 0$.

For the affine system
$$\dot{x} = f(x) + g(x)u \tag{21.12}$$
V a CLF function is equivalent to:

1. $\dot{V}(x, u) = \frac{\partial V}{\partial x} f(x) + \frac{\partial V}{\partial x} g(x)u$
2. CLF condition: For all $x \neq 0$, then $\frac{\partial V}{\partial x} f(x) \geq 0 \Rightarrow \frac{\partial V}{\partial x} g(x) \neq 0$.

Equivalently, for all $x \neq 0$, either the first term is negative or we need to apply some control.

If $\mathcal{U} \subset \mathbb{R}^m$ is bounded, then the above statement no longer holds. Therefore, this equivalent formulation is true only in the special case that $\mathcal{U} = \mathbb{R}^m$.

We say that a CLF V has a small control property (scp) if for every $\epsilon > 0$ there's a $\delta > 0$ such that for all $x \in B^n(0, \delta)$ there is a $u \in \mathcal{U} \cap B^m(0, \epsilon)$ such that $\dot{V}(x, u) < 0$ (this concept describes what we need for $k(0) = 0$ and k to be continuous at 0).

Recall: Example 2 where $\dot{x} = x + x^2 u$ and $V = x^2/2$, $\dot{V} = x^2 + x^3 u$. V is CLF but does not have scp (easy to see).

Theorem 21.1 (Artstein,Songtag)**.** *Consider nonlinear system affine in u*
$$\dot{x} = f(x) + g(x)u \tag{21.13}$$
and suppose there is a CLF function V. Then there is an asymptotically stabilizing feedback $u = k(x) \in \mathcal{C}^1$ away from $x = 0$. If, moreoever, V has the scp, then k can be chosen continuous also at zero with $k(0) = 0$.

Proof idea: V a CLF means by definition that

$$\inf_{u\in\mathcal{U}} \dot{V}(x,u) < 0, \forall x \neq 0 \tag{21.14}$$

so that for all $x \neq 0$, there's a u such that $\dot{V}(x,u) < 0$. Then paste these controls together to get $u = k(x)$ a stabilizing feedback.

What we want to ensure is continuous selection of u w.r.t. x satisfying $\dot{V}(x,u) < 0$, i.e. the resulting functioned obtained by pasting these values together is $\mathcal{C}^1$ except possibly at zero.

Example (in paper by Sontag and Sussmann, 1980):

$$\dot{x} = x[(u-1)^2 - (x-1)][(u+1)^2 + (x-2)] \tag{21.15}$$

Note that this system is *not* affine in control, so if it doesn't agree with the conclusion of the theorem that is no contradiction because the requisite conditions are not the same.

Claim: $V(x) = x^2/2$ is CLF. To see this take

$$\dot{V} = [(u-1)^2 - (x-1)][(u+1)^2 + (x-2)] \tag{21.16}$$

which is negative exactly when one of the terms in square brackets is negative, i.e. when $(u-1)^2 < (x-1)$ and $(u+1)^2 + (x-2) > 0$ or the other way around. Having the first negative gives a region represented by parabola lying on its side with vertex at $(x,u) = (1,1)$ with arms going toward $x = +\infty$ (the region is inside the arms). The other region (corresponding to $(u+1)^2 + (x-2) < 0$) is another parabola lying in its side with vertex at $(x,u) = (2,-1)$ and arms going toward $x = -\infty$. We need for each point on the x axis to have a region of at least one of the parabolas directly above it or below (or both). In other words, the projection of the union of both regions equals the entire x axis. Indeed, in this case that holds, so V is a CLF. However, the parabolas are disjoint and separated by a positive distance (i.e. $\inf_{x\in R_1, y\in R_2} ||x-y|| > 0$), so there is no continuous stabilizing feedback. Any stabilizing feedback will be discontinuosu at some nonzero x.

First a topological fact.

Artstein's Proof.

$$\dot{V} = \frac{\partial V}{\partial x} f(x) + \frac{\partial V}{\partial x} g(x) u \tag{21.17}$$

has for each $x \neq 0$ a u which makes it negative. Take an arbitrary state $x_1 \neq 0$, and for it consider the subset of $\mathcal{U}$ which 'work', i.e. $\mathcal{U}_1 := \{u \in \mathcal{U} : \dot{V}(x_1, u) < 0\} \neq \emptyset$. Take your favorite element from it, call it u_1. Note that $\dot{V}$ is continuous in x (by assumption of $V \in \mathcal{C}^1$). So this u_1 for nearby states will also work: i.e. $\dot{V}(x_1, u_1) < 0 \Rightarrow \dot{V}(x, u_1) < 0$ for all $x \in B(x_1, \delta_1)$ a small enough ball around x_1. (B is open.)

We play the same game, for x_2 and get some other ball $B(x_2, \delta_2)$. Repeating this for each x in the statespace $\mathbb{R}^n \setminus 0$, we generate a cover $B(x_1, \delta_1), B(x_2, \delta_2), \ldots$. $\mathbb{R}^n$ is locally compact so each point has finitely many balls containing it (if there are more we can just throw them away).

Now we use the partition of unity. Namely, there are functions $p_j(x)$ from $\mathbb{R}^n \setminus 0$ to $[0,1]$ which are smooth and satisfy $p_j(x) = 0$ for all $x \notin B(x_j, \delta_j)$ and $\sum p_j(x) = 1$ for all $x \in \mathbb{R}^n \setminus 0$

(this is called a partition of unity *subordinate* to a given cover). See e.g. http://en.wikipedia.org/wiki/Partition_of_unity for more information.

Using a partition of unity, we can define

$$k(x) := \sum_i p_i(x) u_i \tag{21.18}$$

for $x \neq 0$ and $k(0) = 0$. This is $\mathcal{C}^1$ away from zero, and stabilizing since at each point we're taking a convex combination of stabilizing feedback values.

Now we need to show that $\dot{V} < 0$:

$$\begin{aligned} &\dot{V}(x,u) = \tfrac{\partial V}{\partial x} f(x) + \tfrac{\partial V}{\partial x} \sum_j p_j(X) g(x) u_i \\ &= \sum_j p_j(x) \frac{\partial V}{\partial x}(f(x) + g(x)u_i) < 0 \end{aligned} \quad , \ \forall x \neq 0, \forall B(x_j) \tag{21.19}$$

Note that we use convextity of $f + gu$ with respect to u.

□

22 Lecture 22

$$\dot{x} = f(x)_g(x)u \tag{22.1}$$

A CLF function V satisfies $\inf_{u \in \mathcal{U}} \dot{V}(x,u) < 0$ for all $x \neq 0$, with

$$\dot{V}(x,u) = \frac{\partial V}{\partial x} f(x) + \frac{\partial V}{\partial x} g(x)u \tag{22.2}$$

so a CLF means that if $\frac{\partial V}{\partial x} g(x) = 0$ then the drift term is stable.

Theorem 22.1. *If there is a control Lyapunov function V, then there is stabilizing feedback $u = k(x)$, and if V has scp property, then k can be continuous at 0.*

Last time we proved it using partition of unity; the proof was non constructive and abstract (though quite nice, nonetheless). The next one, due to E. Sontag, gives an explicit formula for $u = k(x)$.

Sontag's Proof. To start we need to establish some notation: let $L_f V := \frac{\partial v}{\partial x} \cdot f(x)$; this is called the 'Lie Derivative' of f, and similarly, $L_g V : -\frac{\partial V}{\partial x} g(x)$.

Now define

$$k(x) := \begin{cases} -\frac{L_f V + \sqrt{(L_f(V))^2 + |L_g V|^4}}{|L_g V|^2} (L_g V)^T & \text{if} \quad L_g V \neq 0 \\ 0 & \text{else} \end{cases} \tag{22.3}$$

We have to show two things: first that this feedback actually stabilizes the system, which can be seen by plugging in for $\dot{V}$ and seeing that its negative. Secondly, must show that it's continuous.

To see stability, we consider by hypothesis only when $L_g V \neq 0$ (since we already assumed that the drift is stable):

$$\dot{V}_{cl} = L_f V + L_g V \cdot k(x) = L_f V - \frac{(L_g V)^T L_g V (L_f V + \sqrt{(L_f V)^2 + |L_g V|^4}}{||L_g V||^2} = -\sqrt{(L_f V)^2 + |L_g V|^4} < 0, \tag{22.4}$$

for all nonzero x, where the last inequality holds because $L_f V$ and $L_g V$ are not simultaneously zero except at the origin. Note that other than the choice of k, there's nothing smart going on in the verification of stability.

Next we show regularity; write $k(x) = -\varphi(a,b)(L_g V)^T$ where $a := L_f V$ and $b = |L_g V|^2$. Then $\varphi(a,b) = \begin{cases} \frac{a + \sqrt{a^2+b^2}}{b} & \text{if} \quad b \neq 0 \\ 0 & \text{else} \end{cases}$ We claim that φ is smooth and in fact real analytic on the set $\{(a,b) :\, b > 0 \,\text{or}\, a < 0\}$ (exclude fourth quadrant). Indeed, φ satisfies the following implicit relation

$$b\varphi^2 - 2a\varphi - b = 0 \tag{22.5}$$

which can be checked easily by computation. Taking $F(a,b,\varphi) := b\varphi^2 - 2a\varphi - b$, considering its derivative

$$\frac{\partial F}{\partial \varphi} = 2b\varphi - 2a = \begin{cases} 2\sqrt{a^2+b^2} & \text{if} \quad b \neq 0 \\ -2a & \text{else} \end{cases} \tag{22.6}$$

By Implicit Function Theorem, $\varphi(a,b)$ is $\mathcal{C}^1$ (and actually smooth analytic as well) function.

To summarize, we have mappings $x \mapsto a$ and $\mapsto |L_g|^2$ continuous and mapping of these to $\varphi(a,b)$ which is $\mathcal{C}^\infty$, which by composition implies that $u = k(x)$ depends continuously on x.

Analysis of scp can be found in Sontag's book. □

Remark: there's an interpretation of this formula in terms of linear quadratic optimal control: $u = K(x)$ achieves $L_f V + L_g V k < 0$ or $a + Bk < 0$ where $B := L_g V$ ($b = |B|^2$). This all depends on x, but pretend that x is fixed. In other words, consider an auxiliary scalar linear system

$$\dot{z} = az + Bv \tag{22.7}$$

where z is the state and v the control. Introduce cost $J() := \int_0^\infty v^2 + bz^2$. Optimal control is obtained from Riccati equation:

$$bp^2 - 2ap - b = 0 \tag{22.8}$$

and the optimal control is $k = -B^T p$. This LQR cost forces control to be small when b is small, which forces control to be continuous.

Example 1 from last time

$$\dot{x} = x^2 + xu \tag{22.9}$$

where last time we had that $u = -x - x^2 \Rightarrow \dot{x} = -x^3$ or $u = -x - 1 \Rightarrow \dot{x} = -x$ but u is not zero at zero. Using $V = x^2/2$, $L_f V = x^3$ and $L_g V = x^2$. Sontag's formula gives

$$u = \begin{cases} -\frac{x^3+\sqrt{x^6+x^8}}{x^4}x^2 & \text{if} \quad x \neq 0 \\ 0 & \text{else} \end{cases} \tag{22.10}$$

This formula simplifies to

$$u = -x - |x|\sqrt{1+x^2} \tag{22.11}$$

The closed loop system is then

$$\dot{x} = -x^2 sgn(x)\sqrt{1+x^2} \tag{22.12}$$

This control gives a balance between the two options we previously had, namely between control speed and control effort.

22.1 More on Nonlinear Control Design: Backstepping

Given $\dot{x} = f(x) + g(x)u$ or even a more general system $\dot{x} = f(x, u)$, how to find a CLF? Affine system + CLF gives (by Sontag's formula) stabilizing feedback. We already know that given $\dot{x} = f(x)$ trying to find a Lyapunov function can be difficult. On the other hand, CLF's are more flexible: they *can* be a Lyapunov function, so long as you choose a control in the right way. (Recall that $\forall x \neq 0 \exists u$ such that $L_f V + L_g V u < 0$, which is equivalent to requiring that $L_f V < 0$ *when* $L_g V = 0$). For we only need V to be Lyapunov when the control Lie term vanishes (so in other words, we do not require that the CLF is Lyapunov everywhere.)

Backstepping is a tool for generating CLF for control systems with specific structure (Meilakhs, 1975; Morse, 1976; Krstic-Kannelakopoulos-Kokotovi book, 1990): take

$$\dot{x} = f(x) + g(x)u \tag{22.13}$$

Suppose there exists a stabilizing feedback $u = k(x)$ and corresponding Lyapunov function $V(x)$. Consider an augmented system,

$$\begin{pmatrix} \dot{x} \\ \dot{\xi} \end{pmatrix} = \begin{pmatrix} f(x) + g(x)\xi \\ u \end{pmatrix} \tag{22.14}$$

Can consider this a courser model of a finer system, e.g. kinematic and dynamic system. For example

$$u \to \int \xrightarrow{\xi} \to \dot{x} = f(x) + g(x)u \xrightarrow{x} \tag{22.15}$$

as opposed to $u \to \dot{x} = f(x) + \ldots$

For the augmented system we want to find a CLF $V_a(x, \xi)$ and a stabilizing feedback $u = k_a(x, \xi)$. The idea is to start with $x-$system (simple) and add integrators (or more general dynamics) to build up to the more complicated system. For example

$$\dot{x} = x^2 + xu \tag{22.16}$$

Whcih we could augment as

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_1^2 + x_1 x_2 \\ u \end{pmatrix} \tag{22.17}$$

so what we would do is consider first $\dot{x}_1 = x_1^2 + x_1 u$ pretending that x_2 is u and then go back to the original system. Moreover, as we'll see, we don't need to have pure integrators, it is possible to have e.g.

$$\begin{aligned} \dot{x}_1 &= x_1^2 + x_1 x_2 \\ \dot{x}_2 &= x_3 + \ldots \\ \dot{x}_3 &= u \end{aligned} \tag{22.18}$$

23 Lecture 23

23.1 Backstepping

Suppose for system

$$\dot{x} = f(x) + g(x)u \tag{23.1}$$

that we have stabilizing feedback $u = k(x)$, $k(0) = 0$, and corresponding Lyapunov function $V(x)$

$$\frac{\partial V}{\partial x} f(x) + \frac{\partial V}{\partial x} g(x)k(x) \leq -W(x) < 0 \; \forall x \neq 0 \tag{23.2}$$

We take the augmented system

$$\begin{aligned} \dot{x} &= f(x) + g(x)\xi \\ \dot{\xi} &= u \end{aligned} \tag{23.3}$$

Then we claim that

$$V_a(x, \xi) = V(x) + \frac{(\xi - k(x))^2}{2} \tag{23.4}$$

is a control Lyapunov function for the augmented system; a stabilizing feedback can be defined by $u = k_a(x, \xi) = k'(x)f(x) + k'(x)g(x)\xi - \frac{\partial V}{\partial x} g(x) - \xi + k(x)$; then we compute the derivative

$$\begin{aligned} &\dot{V}_a = \tfrac{\partial V}{\partial x} f(x) + \tfrac{\partial V}{\partial x} g(x)\xi + (\xi - k(x))(u - k'(x)f(x) - k'(x)g(x)\xi) = \\ &\tfrac{\partial V}{\partial x} f(x) + \tfrac{\partial V}{\partial x} g(x)k(x) + (\xi - k(x))(\tfrac{\partial V}{\partial x} g(x) + u - k'(x)f(x) - k'(x)g(x)\xi) \leq \\ &-W(x) + (\xi - k(x))(\tfrac{\partial V}{\partial x} g(x) + u - k'(x)f(x) - k'(x)g(x)\xi) \\ &\leq -W(x) - (\xi - k(x))^2 < 0, \; \forall (x, \xi) \neq (0, 0) \end{aligned} \tag{23.5}$$

To see the last inequality, suppose that $-W(x) - (\xi - k(x))^2 = 0$, then $W(x) = 0 \Rightarrow x = 0 \Rightarrow k(x) = 0 \Rightarrow \xi = 0$.

The control function may not be unique but this technique gives us a systematic way of generating a CLF.

We can extend this to

$$\begin{aligned} \dot{x} &= f(x) + g(x)\xi \\ \dot{\xi} &= F(x, \xi) + G(x, \xi)u, \; G \neq 0 \end{aligned} \tag{23.6}$$

Exercise: carry out procedure to make what we had above work for this (same process). Systems that look like this are said to be in 'strict feedback form'.

23.2 ISS Disturbance Attenuation

From the original system

$$\dot{x} = f(x) \tag{23.7}$$

we took investigated both control problem

$$\dot{x} = f(x) + g(x)u \tag{23.8}$$

and disturbance problem

$$\dot{x} = f(x) + g(x)d \tag{23.9}$$

where we had CLF's and ISS-LF, respectively. Now we want to combine these and consider control design:

$$\dot{x} = f(x) + g_1(x)d + g_2(x)u \tag{23.10}$$

We assume here that dynamics are affine w.r.t. both control and disturbance. The problem we want to consider is how to choose u such that the closed loop with exogenous disturbance d will still have ISS property, we'll call it ISS-CLF.

Recall that for $\dot{x} = f(x,d)$, V is an ISS Lyapunov function if

$$\frac{\partial V}{\partial x} f(x,d) \leq -\alpha(|x|) + \chi(|d|),\ \alpha, \chi \in \mathcal{K}_\infty \tag{23.11}$$

This suggests that we define for a system $\dot{x} = f(x) + g_1(x)d + g_2(x)$ that V is an ISS CLF if

$$\inf_{u\in\mathcal{U}} \frac{\partial V}{\partial x} f(x) + \frac{\partial V}{\partial x} g_1(x)d + \frac{\partial V}{\partial x} g_2(x)u \leq -\alpha(|x|) + \chi(|d|) \tag{23.12}$$

From this we hope that we can find the desired feedback using Sontag's universal formula:

$$\inf_{u\in\mathcal{U}} a(x) + B(x)u < 0, \forall\, x \neq 0 \Rightarrow u = k(x) = \varphi(a, B) \tag{23.13}$$

Now we have

$$\inf_{u} a(x,d) + B(x)u \leq -\alpha(|x|) + \chi(|d|) \tag{23.14}$$

The functional dependence of a on d requires that $u = k(x,d)$ which by hypthesis this is not allowed (generally we don't have control access to disturbances). So before applying Sontag's formula, note that the above ISS-CLF condition can be equivalently rewritten according to gain margin condition:

$$|x| \geq \rho(|d|) \Rightarrow \inf_{u} \frac{\partial V}{\partial x} f(x) + \frac{\partial V}{\partial x} g_1(x)d + \frac{\partial V}{\partial x} g_2(x)u \leq -\alpha_3(|x|) < 0,\ \forall x \neq 0 \tag{23.15}$$

where $\rho, \alpha_3 \in \mathcal{K}_\infty$. But we still need to find a way to deal with the disturbance term inside. Observer, however, that this condition is equivalent to:

$$\inf_{u} \frac{\partial V}{\partial x} f(x) + \left|\frac{\partial V}{\partial x} g(x)\right| + \left|\frac{\partial V}{\partial x}\right| \rho^{-1}(|x|) + \frac{\partial V}{\partial x} g(x)u \leq -\alpha_3(|x|) \tag{23.16}$$

Hence we are choosing our disturbance to be bounded within a ball of a specified size, and the worst case is then

$$\frac{\partial V}{\partial x} g_1(x) \cdot d = \left|\frac{\partial V}{\partial x} g_1(x)\right| \rho^{-1}(|x|) \tag{23.17}$$

Now, finally, we can apply Sontag's formula to find an input to state stabilizing feedback $u = k(x)$.

23.3 Advanced Topics: Perturbation Theory and Averaging

We are going to generalize earlier results of vanishing and nonvanishing perturbations, which as a corollary will give a result on continuity of solutions on infinite intervals. We'll then apply these results to averaging (this material is §§9.1-9.4 and 10.3-10.4,10.6[1]).

Consider again a system of the form

$$\dot{x} = f(t,x) + g(t,x) \tag{23.18}$$

where f is the 'nominal' dynamics and g the perturbation. Goal: assuming that $\dot{x} = f(t,x)$ has 'nice' behavior (in a sense we will later make precise), we want to be able to say something about the perturbed system.

Hypotheses: we assume that the equilibrium $x(0) = 0$ is exponentially stable for the nominal system. Additionally, we need to say something about the disturbance, namely that it is bounded as

$$|g(t,x)| \leq \gamma(t)|x| + \delta(t),\ \forall t, x \tag{23.19}$$

where γ and δ are continuous, nonnegative and δ is bounded. More assumptions on γ will come later.

This combines and generalizes earlier system descriptions which we've seen:

1. vanishing perturbations: $|g(t,x)| \leq \epsilon|x|$ with ϵ small. (In the current situation, we'd have $\gamma(t) \leq \epsilon$ and $\delta \equiv 0$
2. Nonvanishing perturbation: $|g(t,x)| \leq \epsilon$, and this corresponds to $\gamma \equiv 0$ and $|\delta(t)| \leq \epsilon$.

Since $\dot{x} = f(t,x)$ is exponentially stable, by converse Lyapunov theorem for ES, there is a Lyapunov function V such that

$$\begin{array}{l} c_1|x|^2 \leq V(t,x) \leq c_2|x|^2 \\ \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \leq -c_3|x|^2 \\ \left|\frac{\partial V}{\partial x}\right| \leq c_4||x||,\ c_1,\ldots,c_4 > 0 \end{array} \tag{23.20}$$

where these inequalities hold for all $x \in \mathcal{B}(0,r)$, a ball around the origin of radius r. If the system is GES, then take $r = \infty$. The main idea is to differentiate V along solutions of the perturbed system, and use assumptions to get some sort of decay; it won't be exponential, nor necessarily attractive to zero.

24 Lecture 24

Consider perturbed system

$$\dot{x} = f(t,x) + g(t,x) \tag{24.1}$$

and assume that

1. x=0 is exponentially stable for $\dot{x} = f(t,x)$
2. $|g(t,x)| \le \gamma(t)|x| + \delta(T)$, for γ, $\delta \ge 0$, continuous and δ boudned

By converse Lyapunov for exponential stability, we have

$$\begin{array}{l} c_1|x|^2 \le V(t,x) \le c_2|x|^2 \\ \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \le -c_3|x|^2 \\ \left|\frac{\partial V}{\partial x}\right| \le c_4|x| \end{array} \tag{24.2}$$

where these hold locally, i.e. $|x| \le r$ some $r > 0$, and $c_1, \ldots, c_4 \ge 0$.

Then the derivative of V for $\dot{X} = f + g$ is

$$\begin{array}{ll} \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) + \frac{\partial V}{\partial x} g(t,x) & \le -c_3|x|^2 + \left|\frac{\partial V}{\partial x}\right| |g(t,x)| \\ & \le -c_3|x|^2 + c_4\gamma(t)|x|^2 + c_4\delta(t)|x| \\ & \le -[\frac{c_3}{c_2} - \frac{c_4}{c_1}\gamma(t)]V + c_4\delta(t)\sqrt{\frac{V}{c_1}} \end{array} \tag{24.3}$$

Define $W(t) := \sqrt{V(t,x(t))}$ along solutions, so then the derivative of W is given by

$$\begin{array}{ll} \dot{W} & = \frac{\dot{V}}{2\sqrt{V}} \\ & \le -\frac{1}{2}(\frac{c_3}{c_2} - \frac{c_4}{c_1}\gamma(t))W + \frac{c_4}{2\sqrt{c_1}}\delta(t) \end{array} \tag{24.4}$$

By the comparison principle, using

$$\Phi(t,t_0) = \exp\left(-\frac{c_3}{2c_2}(t-t_0) + \frac{c_4}{2c_1}\int_{t_0}^t \gamma(s)ds\right) \tag{24.5}$$

as the state transition matrix from t_0 to t, we have

$$W(t) \quad \le \Phi(t,t_0)w(t_0) + \frac{c_4}{2\sqrt{c_1}}\int_{t_0}^t \Phi(t,\tau)\delta(\tau)d\tau \tag{24.6}$$

Since $\sqrt{c_1}|x| \le W \le \sqrt{c_2}|x|$, we have as a result

$$|x(t)| \le \sqrt{\frac{c_2}{c_1}}\Phi(t,t_0)|x_0| + \frac{c_4}{2c_1}\int_{t_0}^t \Phi(t,\tau)\delta(\tau)d\tau \tag{24.7}$$

Need a bound on γ; assume that

$$\int_{t_0}^t \gamma(s)ds \le \epsilon(t-t_0) + \eta,\ \eta, \epsilon \ge 0 \tag{24.8}$$

Then

$$\Phi(t,t_0) \le \exp([-\frac{c_3}{2c_2} - \frac{c_4}{2c_1}\epsilon](t-t_0)) \exp(\frac{c_4}{2c_1}\eta) \tag{24.9}$$

We want ϵ to be small enough so that the expression in brackets is positive, which would give exponential decay in time: we define this right hand side to be

$$e^{-\alpha(t-t_0)}\rho,\ \alpha > 0,\ \rho \ge 1 \tag{24.10}$$

where ϵ is small enough, namely $\epsilon < c_1c_3/(c_2c_4)$.

Plug this into bound for x, and get

$$|x(t)| \le \sqrt{\frac{c_2}{c_1}}\rho|x_0|e^{-\alpha(t-t_0)} + \frac{c_4\rho}{2c_1}\int_{t_0}^{t} e^{-\alpha(t-\tau)}\delta(\tau)d\tau \tag{24.11}$$

But must make sure that $|x| \le r$. Therefore, we need to make sure that the last expression has a particular bound; we need:

$$\sqrt{\frac{c_2}{c_1}}\rho|x_0|e^{-d(t=t_0)} + \frac{c_4\rho}{2c_1\alpha}(1-e^{-\alpha(t-t_0)})\sup_{t\ge t_0}\delta(t) \le r \tag{24.12}$$

And it can be shown that the left hand side is bounded by

$$\max\{\sqrt{\frac{c_2}{c_1}}\rho|x_0|, \frac{c_4\rho}{2c_1\alpha}\sup(\delta(t))\} \tag{24.13}$$

because $\alpha e^{-\alpha t} + b(1-e^{-\alpha t})$ looks like a convex curve from a to b.

So we need to assume that x_0 and $\delta(\cdot)$ are small enough, and that

$$|x_0| \le \frac{r}{\rho}\sqrt{\frac{c_1}{c_2}} \tag{24.14}$$

$$\delta(t) \le \frac{2rc_1\alpha}{c_4\rho},\ \forall t \tag{24.15}$$

Then the bound refeq:bnd is true for $t \ge t_0$.

We have thus proved the "Perturbation Lemma":

Proposition 24.1. *Let $x = 0$ be an exponential equilibrium of nominal system $\dot{x} = f(t,x)$, let $V = V(t,x)$ be a Lyapunov function provided by the converse Lyapunov theorem for exponential stability*

$$\begin{array}{cc} c_1|x|^2 \le V(t,x) \le c_2|x|^2 & c_1, c_2 \ge 0 \\ \dot{V} \le -c_3|x|^2 & c_3 \ge 0 \\ |\frac{\partial V}{\partial x} \le c_4|x| & c_4 \ge 0 \end{array} \tag{24.16}$$

locally for $|x| \le r$, $r > 0$. Suppose further that

$$g(t,x)| \le \gamma(t)|x| + \delta(t) \tag{24.17}$$

with γ, $\delta \geq 0$ and γ satisfying

$$\int_{t_0}^{t} \gamma(s)ds \leq \epsilon(t-t_0)+\eta,\ 0 \leq \eta < \frac{c_1 c_3}{c_2 c_4},\ \eta \geq 0 \tag{24.18}$$

Suppose that

$$|x_0| \leq \frac{r}{\rho}\sqrt{\frac{c_1}{c_2}},\ \delta(t) \leq \frac{2rc_1\alpha}{c_4\rho},\ \forall t \tag{24.19}$$

Then: the solutions of $\dot{x} = f(t,x)+g(t,x)$ satisfy:

$$|x(t)| \leq \sqrt{\frac{c_2}{c_1}}\rho|x_0|e^{-\alpha(t-t_0)} + \frac{c_4\rho}{2c_1}\int_{t_0}^{t}\delta(\tau)d\tau, \forall t \geq t_0,\ \alpha > 0, \rho \geq 1 \tag{24.20}$$

where α and ρ are given in the above proof (as is the proof of this statement)

$$\alpha := \frac{c_3}{2c_2} - \frac{c_4}{2c_1}\epsilon > 0,\ \rho := e^{\frac{c_4}{2c_1}\eta} \tag{24.21}$$

For the GES case $\dot{x} = f(t,x)$, $r = \infty$ for every $x_0, \delta(\cdot)$.

Proof. Everything that precedes the statement of the proposition. □

Vanishing perturbations: no $\delta(t)$ implies that the integral term in our bound disappears, and we get exponential stability of the perturbed system. Before, we had $|\gamma(t)| \leq \epsilon$, but this is a little bit more general (η allows γ to be large for a finite time period).

Some classes of $\gamma(\cdot)$ satisfying $\int_{t_0}^{t}\gamma(s)ds \leq \epsilon(t-t_0)+\eta$:

1. $|\gamma(t)| \leq \epsilon, \forall t$ (as vanishing perturbation earlier)
2. Any $\mathcal{L}_1$ function $\int_{t_0}^{t}\gamma(s)ds \leq \delta$
3. Any function converging to zero (becomes $\leq \epsilon$ after some time, η handles $\int \gamma$ before that happens, and $\eta(t-t_0)$ upper bounds $\int \gamma$ after)

Example:

$$\dot{x} = [A(t)+B(t)]x \tag{24.22}$$

where $\dot{x} = A(t)x$ is GES. Then GES is preserved if $B(t) \to 0$ or $\int_0^\infty ||B(t)||dt < \infty$ because in this case $\gamma(t) \leftrightarrow B(t)$.

For the perturbation Lemma to give useful bound in presence of $\delta(t)$, we need to know something about $\int_{t_0}^{t} e^{-\alpha(t-\tau)}\delta(\tau)d\tau$. This is response of stable scalar linear system

$$\dot{z} = -\alpha z + \delta(t) \tag{24.23}$$

forced by $\delta(t)$ with initial condition $z_0 = 0$.

We know for example that $\delta(t)$ bounded implies that this is bounded. Also if $\delta(t) \to 0$ then solution also $\to 0$. Staring at the bound in the proposition long enough convinces one of these statements.

In this prolem, we are given the structure of the system $\dot{x} = f + g$, but in a general problem that we may be trying to figure out, we need to decide/find for ourselves exactly how to artificially set things up to apply this result (A similar principle applied in the homework for e.g. $A = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}$, $B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $u = -\varphi(x_1)$, where A is not Hurwitz, but it can be made so by taking $\hat{A} = \begin{pmatrix} 0 & 1 \\ -\alpha & -1 \end{pmatrix}$ for positive α, and $u = -\varphi(x_1) + \alpha x$, basically just adding zero.)

25 Lecture 25

Lemma 25.1 (Perturbation Lemma)**.** *$\dot{x} = f(T, x)$ exponentially stable around $x = 0$, $|g(t, x)| \leq \gamma(t)|x| + \delta(t)$ with $\int_{t_0}^{t} \gamma(s)ds \leq \epsilon(t - t_0) + \eta$, wiht $\epsilon, |x_0|, \delta(t)$ small enough. Then solutions of $\dot{x} = f(t, x) + g(t, x)$ satisfy*

$$|x(t)| \leq c|x_0|e^{-\alpha(t-t_0)} + d\int_{t_0}^{t} e^{-\alpha(t-t_0)}\delta(\tau)d\tau \tag{25.1}$$

25.1 Continuity of Solutions on Infinite Time Horizon

$$\begin{aligned} \dot{x} &= f(t, x),\ x(t_0) = x_0 \\ \dot{y} &= f(t, y) + g(t, y),\ y(t_0) = y_0 \end{aligned} \tag{25.2}$$

Theorem 25.1 (Continuity on Infinite Time Horizon)**.** *Let $f \in \mathcal{C}^1$ and $\frac{\partial f}{\partial x}$ locally LIpschitz in x uniformly in t, $|g(t, y)| \leq \mu$, and $x = 0$ exponentially stable equilibrium of $\dot{x} = f(t, x)$. Then for each compact set inside the exponentially stability region for $\dot{x} = f(t, x)$, if x_0 is in this compact set and if $|x_0 - y_0|$ and μ are sufficiently small, then solutions $x(t)$ and $y(t)$ of $\dot{x} = f(t, x)$ and $\dot{y} = f(t, y) + g(t, y)$ satisfy the following bound*

$$|x(t) - y(t)| \leq ce^{-\alpha(t-t_0)}|x_0 - y_0| + \beta\mu, \forall t \geq 0 \tag{25.3}$$

where $c, \alpha, \beta > 0$.

Proof. In order to prove this we will use the perturbation lemma. Consider the error $e := y - x$ and we want to derive a differential equation of e in the form "nominal +perturbation".

We have

$$\begin{aligned} \dot{e} &= \\ \dot{y} - \dot{x} &= f(t, y) + g(t, y) - f(t, x) \\ &= f(t, e) + \Delta(t, e) + g(t, y) \\ &= f(t, x(t) + e) - f(t, x(t)) - f(t, e) + f(t, 0) \\ &= \frac{\partial f}{\partial x}(t, x(t) + \lambda_1 e)e - \frac{\partial f}{\partial x}(t, \lambda_2, e)e \end{aligned} \tag{25.4}$$

where we want to express the second line as a function of e (plus perturbation), $\Delta(t, e) := f(t, x(t) + e) - f(t, x(t)) - f(t, e)$, $f(t, 0) = 0$ (since 0 is equilibrium of $\dot{x} = f$), and the fifth line follows from the mean value theorem with $\lambda_1, \lambda_2 \in [0, 1]$. Since $\frac{\partial f}{\partial x}$ is Lipschitz, uniformly in time, we have

$$|\Delta(t, e)| \leq L(|e| + |x|)|e| \tag{25.5}$$

for $L \geq 0$.

Then Perturbation term is

$$|\Delta(t, e) + g(t, y)| \leq L(|e| + |x|)|e| + \mu \tag{25.6}$$

which is exactly in the form we need $\gamma(t)|e| + \delta(t)$ as in the Perturbation Lemma, and where $\gamma(t) = L(|e| + |x|)$, $\delta(t) = \mu$. Recall that we assumed that the nominal system is exponentially stable. Recall that $x(t) \to 0$ exponentially fast and $e(0)$ is small . If e_0 is small enough, for arbitrary small, $\nu > 0$, we have $\int_{t_0}^{t} \gamma(s)ds \leq \epsilon(t-t_0) + \eta$, where

$$\eta = \int_{t_0}^{T} |x(s)|ds \tag{25.7}$$

where $|x(t)| < \nu$ for all $g \geq T$ (true so long as $|e(t)| \leq r$ where r comes from Perturbation lemma). Now apply the perturbation lemma,

$$|e(t)| \leq c|e_0|e^{-\alpha(t-t_0)} + d\int_{t_0}^{t} e^{-\alpha(t-\tau)}\mu d\tau \tag{25.8}$$

where the integral term is less than r and the bound is valid for all $t \geq 0$. □

25.2 Periodic Perturbation of Autonomous Systems[Section §10.3 Khalil]

Take

$$\dot{x} = f(x) + \epsilon g(t, x, \epsilon) \tag{25.9}$$

$\epsilon > 0$, $f, g \in \mathcal{C}^1$ w.r.t. x (see Khalil for more details). Assume that $x = 0$ is exponentially stable equilibrium for the nominal system $\dot{x} = f(x)$, that g is bounded and T-periodic in t for $T > 0$, i.e. $g(t+T, x, \epsilon) = g(T, x, \epsilon)$ for every t, x, ϵ.

Previous analysis: if x_0 and ϵ are small, then $x(t) \to$ some neighborhood of zero whose size is proportional to ϵ. What happens inside this neighborhood? When can $x(t)$ be periodic?

Observations: define the following map: $P_\epsilon(x) := \phi(T, 0, x, \epsilon)$ the solution at time T at x at time 0, with value ϵ on right hand side of the system; $P_\epsilon : \mathbb{R}^n \to \mathbb{R}^n$.

Lemma 25.2. *Perturbed system has a T-periodic solution iff the equation $x = P_\epsilon(x)$ has a solution, say $x = p_\epsilon$ (fixed point).*

Proof. Obvious. □

Lemma 25.3. *There are positive numbers k and ϵ^* such that $X = P_\epsilon(x)$ has a unique solution in $\{x : |x| \leq k|epsilon|\}$ for every $|\epsilon| < \epsilon^*$.*

Proof Idea. Use implicit function theorem and use fact that $A := \frac{\partial f}{\partial x}|_{x=0}$ is Hurwitz (from exponential stability assumption and Lyapunov's 1st method). (Remember A, we're gonna need it later.) For $x = 0\epsilon = 0$, we have a trivial periodic solution $x \equiv 0$. □

Lemma 25.4. *If $\overline{x}(t, \epsilon)$ is a T-periodic solution of the perturbed sytsem satisfying the bound $|\overline{x}(t, \epsilon)| < k|\epsilon|$, (where k is from lemma 2), then the solution is exponentially stable.*

Proof Sketch. Set error variable $z := x - \overline{x}(t,\epsilon)$ iff $x = z + \overline{x}(t,\epsilon)$, then

$$\begin{aligned} \dot{z} &= \dot{x} - \dot{\overline{x}} = f(x) + \epsilon g(t,x,\epsilon) - f(\overline{x}) - \epsilon g(t,\overline{x},\epsilon) \\ &= f(z+\overline{x}) - f(\overline{x}) + \epsilon[g(t,z+\overline{x},\epsilon) - g(t,\overline{x},\epsilon)] \\ &= \hat{f}(t,z) \end{aligned} \tag{25.10}$$

Now linearize about $z = 0$ to get

$$\begin{aligned} \tfrac{\partial \hat{f}}{\partial z}|_{z=0} &= \tfrac{\partial f}{\partial x}|_{z=0} + \epsilon \tfrac{\partial g}{\partial x}|_{\epsilon=0} \\ &= A + [\tfrac{\partial f}{\partial x}(\overline{x}) - A] + \epsilon \tfrac{\partial g}{\partial x}(t,\overline{x},\epsilon) \\ &= \epsilon \tfrac{\partial g}{\partial x}(t,\overline{x},\epsilon) \\ &\xrightarrow{\epsilon \to 0} 0 \end{aligned} \tag{25.11}$$

where the first term vanishes by continuity of $\frac{\partial f}{\partial x}$

Then

$$\dot{z} = Az + [\frac{\partial f}{\partial x}(\overline{x}) - A] + \epsilon \frac{\partial g}{\partial x}(t,\overline{x},\epsilon) \tag{25.12}$$

and the second term as we previously saw is a vanishing perturbation.

By result on vanishing perturbations, linearized z system is exponentially stable and by Lyapunov's first method, original z system is exponentially stable. □

Theorem 25.2. *There exists k, ϵ^* such that for every $|\epsilon| < \epsilon^*$, perturbed system has a unique T-periodic solution $\overline{x}(t,\epsilon)$ such that $|\overline{x}(t,\epsilon)| < k|\epsilon|$, and this solution is exponentially stable.*

Note that if $g(t,0,\epsilon) = 0$, then the perturbed system has equilibrium at zero. Then $x \equiv 0$ is this unique periodic solution, and it's still exponentially stable.

Proof. The previous three lemmas. □

26 Lecture 26

26.1 Averaging Theory

$$\dot{x} = \epsilon f(t, x) \tag{26.1}$$

f is T-periodic in t, $f(t+T, x) = f(t, x)$ for all t, x and ϵ small, slow system response.

Suppose we have a small bandwith system (acts like low pass filter, i.e. high frequency input doesn't 'make it through'). We'll define an average for the system and approximate the behavior of original system by that of average system.

Define average

$$f_{av}(x) := \frac{1}{T}\int_0^T f(\tau, x)d\tau \tag{26.2}$$

Average system $\dot{x} = f_{av}(x)\epsilon$ an autonomous system. More generally, given a system

$$\dot{x} = \epsilon f(t, x, \epsilon) \tag{26.3}$$

we define the average

$$f_{av}(x) = \frac{1}{T}\int_0^T f(\tau, x, 0)d\tau \tag{26.4}$$

The goal is to study how the *autonomous* average system approximates the behavior of the original time varying system.

Let $h(t, x) := f(t, x, 0) - f_{av}(x)$, which is T periodic in t and has mean zero. Also let $u(t, x) := \int_0^t h(\tau, x)d\tau$ which is also periodic in T. (Think $\sin(t)$ and $1 - \cos(t)$). Hence $u(t, x)$ is bounded for all t and for all x in a compact set, and

$$\frac{\partial u}{\partial t} = h(t, x),\ \frac{\partial u}{\partial x} = \int_0^t \frac{\partial h}{\partial x}(\tau, x)d\tau \tag{26.5}$$

These are T periodic, zero average, and also bounded for all t and all x in a compact set.

Change of variables: $x = y + \epsilon u(t, y)$, y a new state variable, then

$$\dot{x} = \dot{y} + \epsilon\frac{\partial u}{\partial t}(t, y) + \epsilon\frac{\partial u}{\partial y}(t, y)\dot{y} \tag{26.6}$$

where $\dot{x} = \epsilon f(t, y + \epsilon u, \epsilon)$ and so we get

$$(I + \epsilon\frac{\partial u}{\partial y}(t, y))\dot{y} = \epsilon f(t, y + \epsilon u, \epsilon) - \epsilon\frac{\partial u}{\partial t}(t, y) \tag{26.7}$$

Now plug the definition of h in to get

$$\begin{aligned} (I + \epsilon\tfrac{\partial u}{\partial y}(t, y))\dot{y} &= \epsilon f(t, y + \epsilon u, \epsilon) - \epsilon f(t, y, 0) + \epsilon f_{av}(y) \\ &= \epsilon f_{av}(y) + \epsilon p(t, y, \epsilon) \end{aligned} \tag{26.8}$$

where we define $p(t, y, \epsilon) := f(t, y + \epsilon u, \epsilon) - f(t, y, 0)$ (note that this is T periodic), which we can rewrite as

$$\begin{aligned} p(t, y, \epsilon) &= f(t, y + \epsilon u, \epsilon) - f(t, y, \epsilon) + f(t, y, \epsilon) - f(t, y, 0) \\ &= \tfrac{\partial f}{\partial y}(t, y + \lambda_1 \epsilon u, \epsilon)\epsilon u + \tfrac{\partial f}{\partial \epsilon}(t, y, \lambda_2 \epsilon)\epsilon, \ 0 \leq \lambda_1, \lambda_2 \leq 1 \end{aligned} \tag{26.9}$$

using the mean value theorem in the second line.

Consider the left hand side above:

$$I + \epsilon \frac{\partial u}{\partial y}(t, y) \tag{26.10}$$

is nonsingular for ϵ small enough because $\frac{\partial u}{\partial y}$ is bounded over any compact set of x. So we can consider its inverse $(I + \epsilon \frac{\partial u}{\partial y}(t, y))^{-1} = I + O(\epsilon)$, where $O(\epsilon)$ means "order ϵ", i.e.

$$|O(\epsilon)| \leq c|\epsilon| \tag{26.11}$$

as $\epsilon \to 0$.

$$(I + \epsilon A)^{-1} = I - \epsilon A + \epsilon^2 A^2 - \dots \tag{26.12}$$

We can bring the implicit differential equation for y to the form

$$\dot{y} = \epsilon f_{av}(y) + \epsilon^2 q(t, y, \epsilon) \tag{26.13}$$

where q is T periodic in t and $\epsilon^2 q$ term collects all the terms of order > 1 in ϵ. This is a perturbation of the average system which is simply

$$\dot{y} = \epsilon f_{av}(y) \tag{26.14}$$

We aren't done yet, because we don't want to keep ϵ explicitly on the right hand side. After all, previously we didn't have it. The trick is to rescale the system, divide time by ϵ to get a slow time scale, w.r.t. the ϵ will disappear. Another change of variables: $s = \epsilon t$, where s is a new time, and $t = \frac{1}{\epsilon}s$, and we get a differential equation with $\frac{d}{ds}$:

$$\frac{dy}{ds} = \frac{1}{\epsilon}\dot{y} = f_{av}(y) + \epsilon q(\frac{s}{\epsilon}, y, \epsilon) \tag{26.15}$$

Now q, which was T-periodic in t is now ϵT periodic in s. We bring in three previous results to arrive at the following three claims:

1. If $\overline{y}$ is a solution of the average system

$$\frac{d\overline{y}}{ds} = f_{av}(\overline{y}) \tag{26.16}$$

then by continuity of solutions on finite intervals w.r.t. initial conditions and perturbations, if $y(0, \epsilon)$ and $\overline{y}(0)$ are within $O(\epsilon)$, then for ϵ small enough, we have

$$|y(s, \epsilon) - \overline{y}(s)| = O(\epsilon) \tag{26.17}$$

on some interval, say $s \in [0, b]$.

Remark: in original time t, this interval is for $t \in [0, b/\epsilon]$ which is, though finite, pretty large. Also, y is not the original variable so it may seem that this result is not what we want. But recall the relation $x = y + \epsilon u(t, y)$ where u is T periodic and bounded *and* multiplied by ϵ, the same claim/approximation result holds for x and $\overline{x}$ of original system and its average. To be precise: $d(\overline{x}, \overline{y})$ and $d(x, y)$ are of order ϵ so we can easily pass between the statement for y and the corresponding statement for x. For the following two claims the same holds true.

2. Let's now assume that the average system

$$\frac{\partial \overline{y}}{ds} = f_{av}(\overline{y}) \tag{26.18}$$

has $y = 0$ as an exponentially stable equilibrium, then by continuity on infinite horizon intervals, as long as $\overline{y}(0)$ is in a compact set contained in the region of exponential stability the above $O(\epsilon)$ approximation is valid for all $t \in [0, \infty)$, again for ϵ small enough.

3. Now we can use the fact that q is periodic. From a result in last lecture on the existence of periodic solutions under periodic perturbations, still assuming that $y = 0$ is an exponentially stable equilibrium of the average system, we have that in some neighborhood of that equilibrium, for ϵ small enough, there exists a unique exponentially stable periodic solution of the perturbed system. Period is ϵT in the s scale and therefore T in the original t scale, as expected.

   Note: if moreover $f(t, 0, \epsilon) = 0$ (meaning that 0 is still an equilibrium of perturbed system), then the unique periodic solution in the above statement is this equilibrium, and it is exponentially stable.

26.2 Examples

Consider the LTV system

$$\dot{x} = \epsilon A(t)x \tag{26.19}$$

and average system

$$\dot{x} = \epsilon \overline{A}x \tag{26.20}$$

where $\overline{A} = \frac{1}{T}\int_0^T A(\tau)d\tau$, still has equilibrium 0. Suppose that $\overline{A}$ is Hurwitz, then that guarantees that we can apply each of the above three results, so in particular, the behavior of the system above is close to that of the average system and there is unique equilibrium which is exponentially stable for the original system since $\overline{A}$ LTI has unique 0 equilibrium. Note that this requires ϵ to be small enough (as we don't have the above results for arbitrary ϵ.

Example 10.9 from [1]: Scalar nonlinear system

$$\dot{x} = \epsilon(x \sin^2(t) - \frac{1}{2}x^2) \tag{26.21}$$

where right hand side is periodic with period π; $\sin^2(t) = \frac{1}{2} - \frac{1}{2}\cos(2t)$; also

$$f_{av}(x) = \frac{1}{\pi}\int_0^\pi x \sin^2 t - \frac{1}{2}x^2 dt = \frac{1}{2}(x - x^2) \tag{26.22}$$

So the averaged system is

$$\dot{x} = \epsilon(\frac{1}{2}x - \frac{1}{2}x^2) \tag{26.23}$$

with two equilibria (0 and 1).

Jacobian:

$$\frac{\partial f_{av}}{\partial x} = \frac{1}{2} - x \tag{26.24}$$

which is 1/2 at $x = 0$ and $-1/2$ at $x = 1$. The second is exponentially stable. There is a unique π-periodic soluiton in the vicinity of $x = 1$.

27 Lecture 27

$$\dot{x} = \epsilon f(t, x, \epsilon) \tag{27.1}$$

f is T periodic: $f(t+T, x, \epsilon) = f(t, x, \epsilon)$ and we have average system

$$f_{av}(x) = \frac{1}{T}\int_0^T f(t, x, 0)dt \tag{27.2}$$

If $x(0) - x_{av}(0) = O(\epsilon)$ then for ϵ small enough, $x(t) - x_{av}(t) = O(\epsilon)$, $t \in [0, b/\epsilon]$. Suppose that $x = 0$ is an exponentially stable equilibrium of the average system; then the approximation is valid for all $t \in [0, \infty)$, as long as $x(0), x_{av}(0) \in$ stability region, and for ϵ small enough there is a unique T-periodic solution in the vicinity of $x = 0$ and it is exponentially stable. If $f(t, 0, \epsilon) = 0$, then $x = 0$ is this exponentially stable periodic solution.

Example (continuing from last time): Van der Pol oscillator (in [1] §10.5, also §2.4)

$$\ddot{x} + x = \epsilon\dot{x}(1 - x^2) \tag{27.3}$$

without the $1 - x^2$ this would just be usual linear oscillation with damping. We rewrite as first order differential equation:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + \epsilon x_2(1 - x_1^2) \end{aligned} \tag{27.4}$$

and apply change of coordinates, into polar:

$$\begin{aligned} x_1 &= r\sin(\varphi) \\ x_2 &= r\cos(\varphi) \end{aligned} \tag{27.5}$$

Solving for r and φ, we have

$$\begin{aligned} r &= \sqrt{x_1^2 + x_2^2} \\ \varphi &= \tan^{-1}(\tfrac{x_1}{x_2}) \end{aligned} \tag{27.6}$$

and the resulting differential equation is

$$\begin{aligned} \dot{r} &= \tfrac{x_1\dot{x}_1 + x_2\dot{x}_2}{r} &&= \tfrac{1}{4}(x_1x_2 - x_2x_1 + \epsilon x_2^2(1 - x_1^2)) = \epsilon x_2^2(1 - x_1^2) \\ & &&= \tfrac{1}{r}\epsilon r^2\cos^2(\varphi)(1 - r^2\sin^2(\varphi)) \\ & &&= \epsilon r\cos^2(\varphi)(1 - r^2\sin^2(\varphi)) \\ \dot{\varphi} &= \tfrac{\dot{x}_1x_2 - \dot{x}_2x_1}{x_2^2(1 + \frac{x_1^2}{x_2^2})} &&= \tfrac{1}{r^2}(x_2^2 + x_1^2 - \epsilon x_1x_2(1 - x_1)^2) \\ & &&= 1 - \tfrac{1}{r^2}\epsilon r^2\cos(\varphi)\sin(\varphi)(1 - r^2\sin^2(\varphi)) \\ & &&= 1 - \epsilon\cos(\varphi)\sin(\varphi)(1 - r^2\sin^2(\varphi)) \end{aligned} \tag{27.7}$$

and finally we have

$$\frac{dr}{d\varphi} = \epsilon\frac{r\cos^2(\varphi)(1 - r^2\sin^2(\varphi))}{1 - \epsilon\cos(\varphi)\sin(\varphi)(1 - r^2\sin^2(\varphi))} \tag{27.8}$$

φ is the independent variable and this has period 2π for φ.

Now $f_{av} = \frac{1}{2\pi}\int_0^{2\pi} r\cos^2(\varphi)(1 - r^2\sin^2(\varphi))d\varphi$ andthe average system

$$\frac{dr}{d\varphi} = \epsilon f_{av} = \tag{27.9}$$

$\cos^2(\varphi) = \frac{1+\cos(2\varphi)}{2} \Rightarrow \frac{1}{2\pi}\int_0^{2\pi} r\cos^2(\varphi)d\varphi = \frac{1}{2}r$ and

$$\begin{aligned}\cos^2(\varphi)\sin^2(\varphi) &= \tfrac{1}{4}(1+\cos(2\varphi)(1-\cos(2\varphi)) = \tfrac{1}{4}(1-\cos^2(2\varphi))\\ &= \tfrac{1}{4} - \tfrac{1}{4}(\tfrac{1}{2}+\tfrac{1}{2}\cos(4\varphi)) = \tfrac{1}{8} - \tfrac{1}{8}\cos(4\varphi)\end{aligned} \tag{27.10}$$

and finally we have

$$\frac{1}{2\pi}\int_0^{2\pi} r^3\cos^2(\varphi)\sin^2(\varphi)d\varphi = \frac{1}{8}r^3 \tag{27.11}$$

This system has three equilibria: $r = 0$, $r = \pm 2$, but only two make sense (can't have a negative radius). To see which is stable, we need to look at the jacobian:

$$\frac{df_{av}}{dr} = \frac{1}{2} - \frac{3}{8}r^2 \tag{27.12}$$

and evaluating at our equilibria points, we have $\frac{df_{av}}{dr}|_{r=0} = \frac{1}{2}$ and $\frac{df_{av}}{r}|_{r=2} = -1$; the first is unsetable and the second is exponentially stable. In the original system, in a vicinity (i.e. ϵ small enough) of the circle of radius 2, there is a unique 2π-periodic exponentially stable solution. It is worth noting that in the ordinary linear oscillator case, there are many such periodic stable solutions (they fill the space) but here there is only one.

Can check the following observations by simulation in Matlab (or check [1]). For $\epsilon \approx 0.2$ the limit cycle is pretty much circular.

For $\epsilon = 1$, the cycle shape is distorted but still the same property holds that nearby solutions are attracted to it.

For $\epsilon = 5$, the cycle shape is really distorted, with sharp corners, but still attracting for nearby trajectories.

Remark: these results are for ϵ 'small'. This means: there is an ϵ^* such that for every $\epsilon < \epsilon^*$ the results hold, but generally ϵ^* won't be arbitrarily large.

We think of $\dot{x} = \epsilon f(t,x) + g(t,x)$ as being composed of 'fast' dynamics (f) and slow dynamics (g). This can be studied using the theory of singular perturbations (advertisement: this is covered in 517).

27.1 General Averaging

$$\dot{x} = \epsilon f(t,x,\epsilon) \tag{27.13}$$

but we no longer have that f is periodic in t. We can nevertheless still define

$$f_{av} = \int_{t\to\infty} \frac{1}{t}\int_0^t f(\tau,x,0)d\tau \tag{27.14}$$

Now let's see if this limit exists; if it does, then it gives us the average system, the same as before

$$\dot{x} = \epsilon f_{av}(x) \tag{27.15}$$

which is also still going to be autonomous.

Examples: $f(t, x) = \frac{g(x)}{t+1}$ As a function of time, $f(t, x)$- for x fixed- decays. The average is zero (trivial calculus computation). This example shows that we can use a function which is not necessarily periodic but whose nonperiodic part decays in the average, like this one.

If the average exists, it can be used to approximate the original system, in a way similar to what we did in the particular case, but the details are a little more involved (see §10.6[1]).

27.2 Things not Covered [i.e. summer bedtime reading]

1. General Averaging
2. Singular Perturbations
3. Center Manifold Theory
4. More advanced control design (e.g. backstepping)

References

[1] H. Khalil. *Nonlinear Systems*. Prentice Hall, 2002.