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# The zero dynamics of a nonlinear system: From the origin to the latest progresses of a long successful story ☆

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ABSTRACT

The concept of zero dynamics of a nonlinear system was introduced about thirty years ago as nonlinear analogue of the concept of transmission zero of the transfer function of a system. This paper reviews a few historical landmarks in the development of this concept and then addresses some current challenging issues, such as the development of global and coordinate free version of a standard output-feedback design paradigm, the analysis of problems of stabilization and tracking in the presence of unstable zero dynamics, and extensions to multivariable systems.



## 1. Introduction

The concept of zero dynamics of a nonlinear system was introduced about thirty years ago as nonlinear analogue of the concept of transmission zero of a system. The main original motivation for introducing this concept was the ambition to develop systematic methods for asymptotic stabilization, with guaranteed region of attraction, when the dynamics in question are globally asymptotically stable. But soon after a variety of other applications showed up, in the context of feedback linearization, feedback equivalence to passive systems, non-interacting control with stability, output regulation. Essentially, all applications consider SISO systems (or, at most, "square" MIMO systems), require the system to be preliminarily reduced to a special normal form by means of appropriate change of coordinates, and assume the dynamics in question to be globally asymptotically stable. The analysis of systems having more inputs than outputs, of systems in which normal forms cannot be defined, and of systems in which the zero dynamics are unstable is still a substantially unexplored and open area of research. This paper, after having reviewed the highlights of the historical development of this concept, will describe some current challenging issues, such as the development of a coordinate free version of the standard output-feedback design paradigm, the analysis of problems of stabilization and tracking in the presence of unstable zero dynamics, and extensions to multivariable "non-square" systems.

## 2. Historical background

The decade of the 1960s saw enormous progresses in linear system theory: begun with the milestone contributions of Kalman, who introduced and developed the concepts of controllability and observability, the theory quickly evolved, by the end of this decade, toward the development of sophisticated (and mostly geometric) methods of design. In a nutshell, the aim of these methods was to obtain a full answer to the basic conceptual question: "*What can be achieved by means of feedback*?" With a little delay, the interest in developing similar concepts and answering similar questions for nonlinear control attracted the attention of numerous authors. The 1970s saw a collective effort trying to establish an equivalent corpus of theories and results for nonlinear systems. The analysis of the concepts of controllability and observability begun with the pioneering work of Hermann [16], was expanded in a series of major contributions due to Haynes–Hermes [15], Lobry [37], Sussmann–Jurdjevic [47,46], Brockett [3], Krener [33] and reached a culmination with the milestone paper of Hermann–Krener [17]. The impact of this paper, in particular, on the subsequent development of the theory was enormous. In fact this paper not only refined and unified a number of earlier results, but also – above all – introduced a framework and a "language" that made it possible to begin a systematic study of problems of *feedback design* for nonlinear systems.

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The immediate fallout of [17] was in fact the paper [21], in which the basic geometric tools for feedback design were extended from linear to nonlinear systems.

One of the cornerstones of the geometric theory (for linear as well as for nonlinear systems) is the study of how the observability property can be influenced by feedback. This study was originally conceived in the context of the problem of disturbance decoupling, but it become immediately clear that this study had far reaching consequences in a number of other domains. One of these consequences was the possibility of characterizing in "geometric terms" the notion of "zero" of the transfer function of a system. In fact, in a (single-input single-output and minimal) linear system the "open-loop" notion of zero of the transfer function (that is: $z$ is a zero if, for a suitable initial state, the output response to the input $u(t)=\exp(zt)$ is identically zero) has an appealing "closed-loop" characterization: all such $z$'s coincide with the eigenvalues of the unobservable part of the system, once the latter has been rendered maximally unobservable by means of feedback. Based on this correspondence, [34] in 1980 introduced the concept of *null observable distribution*, defined as a distribution that can be rendered invariant via state feedback (controlled invariant) and is contained in the kernel of the differential of the output map. Given any of such distributions, there is a partition of the state space (into integral manifolds) with the property that any pairs of states that belong to the same integral manifold produce identical outputs, under any input. The fact that the output is constant over these integral manifolds is the nonlinear generalization of the output being zero and the feedback that renders the distribution invariant is a feedback that induces loss of observability.

A few years later, in 1984, independently, [5,39] considered the nonlinear analogue of another feature of the notion of zero: the fact that if a linear single-input single-output system having $m$ zeros is subject to high-gain output feedback, $m$ of its poles approach the zeros. The basis for this analogy (which will be discussed later) was the possibility, offered by the geometric analysis of [21], of representing the system in a special *normal form*, in which the integral manifolds of the (maximal) null observable distribution correspond to subsets characterized by a fixed value of certain coordinates. In the (elementary) case of an input-affine system

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned} \tag{1}$$

having relative degree 1, i.e. such that $L_g h(x) \neq 0$ for all $x$, the form in question, obtained under a diffeomorphism that is globally defined if the vector field $\tilde{g}(x) = g(x)[L_g h(x)]^{-1}$ is complete, is

$$\begin{aligned} \dot{z} &= f(z,\xi) \\ \dot{\xi} &= q(z,\xi) + b(z,\xi)u \\ y &= \xi. \end{aligned} \tag{2}$$

In these coordinates, integral manifolds of the maximal null observable distribution are the subsets of those states having the same $\xi$ coordinate. A state produces zero output under any input if and only if the $\xi$ coordinate is zero. If this is the case, $z(t)$ is constrained to obey

$$\dot{z} = f(z,0). \tag{3}$$

If the system is linear, so it is the map $f(z,0)$ and, as it is well-known, its eigenvalues coincide with the zeros of the transfer function.

This observation leads to the concept of *zero dynamics*, introduced in [5]: a dynamical system that characterizes the internal behavior of a system once initial conditions and inputs are chosen in such a way as to constrain the output to be identically zero. Note also, as it was the case for linear systems, the appealing equivalence between an "open-loop definition" (the one just given) and a "closed-loop" one. In fact, the constraint $y(t)=0$ is forced by the uniquely defined "feedback" input

$$u^*(t) = -\frac{q(z(t),0)}{b(z(t),0)}$$

in which $z(t)$ is any solution of (3). In view of the fact that, in linear systems whose zeros have negative real part, the dynamics of (3) are asymptotically stable, in [5] it was also proposed, with a forgivable abuse of terminology, to say that a nonlinear system is *globally minimum phase* if the dynamics of (3) have a globally asymptotically stable equilibrium at $z=0$.

There is also a third feature of the notion of zero of the transfer function of a (single-input single-output) linear system, namely the fact that the zeros characterize the dynamics of the inverse system. In the elementary example presented above, the inverse is a system modeled by

$$\begin{aligned} \dot{z} &= f(z,y) \\ u &= \frac{1}{b(z,y)}[y^{(1)} - q(z,y)]. \end{aligned} \tag{4}$$

In this case, the unforced internal dynamics of the inverse system coincide with the zero dynamics defined as in [5]. The coincidence, though, is limited to the case of single-input single-output systems. Moreover, it should be stressed that while in a linear system the asymptotic stability of the zero-dynamics (3) implies the property of input-to-state stability for the dynamics of the inverse system (4), with $y$ viewed as input, this implication is no longer true for a nonlinear system. This fact has important consequences on the design of (globally) stabilizing feedback, that have been highlighted only recently in [35] and will be reviewed later.

For a multi-input multi-output nonlinear system, the link between zero dynamics and the dynamics of the inverse system is more subtle. This is essentially due to the fact that while the concept of zero dynamics only seeks to determine the dynamics compatible with the constraint that the output is identically zero, the inverse system must describe all the dynamics resulting in *any* admissible output function. As a consequence, computation of the zero dynamics and computation of the inverse system (whenever this is possible) are not equivalent and the latter is possible only under substantially stronger assumptions. The computation of the zero dynamics is based on an extension (see e.g. [23, pp. 293–311]) of the classical algorithm of Wonham for the computation of the largest controlled invariant subspace in the kernel of the output map, while the computation of the inverse system is based on extensions, due to Hirschorn [18] and Singh [43] of the so-called structure algorithm introduced by Silverman for the computation of inverses and zero structure at the infinity. For a comparison of such assumptions and of their influence on the outcome of the associated algorithms, see [29].

## 3. A few success stories

### 3.1. Zero dynamics and high-gain feedback

Systems having a globally asymptotically stable zero dynamics (or, in short, *globally minimum phase* systems) owe their importance to the fact that, under appropriate assumptions, they lend themselves to the implementation of stabilization strategies based on high-gain output feedback. This property was initially recognized in [5,39]. In this context, [5] introduced the idea of "stabilization with guaranteed region of attraction" (a property that later known as *semiglobal stabilizability*) which – in crude terms – can be defined as follows: given any arbitrary compact set $\mathcal{C}$ of initial conditions, find a (static) feedback law – which in general depends on $\mathcal{C}$ – that preserves a given equilibrium and is such that in the associated closed-loop system the equilibrium is asymptotically stable, with a domain of attraction that contains the set $\ell$. Then, paper [5] claimed that a system of the form (2) which is globally minimum phase is semiglobally stabilizable by means of a feedback of the form $u=ky$ (the sign of $k$ being opposite to that of $b(z,\xi)$). This claim was actually incomplete, because the law in question is sufficient to keep trajectories bounded and to steer them in an arbitrarily small neighborhood of the origin, but it is not sufficient for asymptotic stability unless the equilibrium $z=0$ of (3) is also locally exponentially stable. This and similar pathologies have been pointed out later in [45]. In [39], on the other hand, it was shown that classical approximation results (on a finite time horizon) in singular perturbation theory could be used to prove that, whenever high-gain output feedback is used, the slow reduced subsystem of the resulting singularly perturbed closed-loop system coincides with the dynamics of the inverse system driven by zero input (that is of the zero dynamics).

The viewpoint of the paper [5] was further pursued in a subsequent paper [6] for systems having relative degree $r>1$. This paper claimed that a "purely derivative" feedback of the form $u=k[\sum_{i=0}^{r-2}c_i y^{(i)}+y^{(r-1)}]$ is capable of semiglobally stabilizing a globally minimum phase system (note that the latter can be interpreted as a "partial state" feedback $u=k[\sum_{i=0}^{r-2}c_i L_f^i h(x)+L_f^{r-1}h(x)]$). This claim too was incomplete, as it was pointed out by [32]. The correct version of the claim was provided later in the journal paper [7], where it was shown that the stability result holds under the assumptions that the dynamics of the inverse system are driven only by $y$, and not by its higher order derivatives, and that $L_f^r h(x)=0$.

In the meanwhile, the newly introduced concept of input-to-state stability [44] and the associated version of the small gain theorem [30] made it possible to prove that a system having a globally defined normal form

$$\begin{aligned}
\dot z &= f(z,\xi_1,\dots,\xi_r)\\
\dot\xi_i &= \xi_{i+1}, \quad i=1,\dots,r-1\\
\dot\xi_r &= q(z,\xi_1,\dots,\xi_r)+b(z,\xi_1,\dots,\xi_r)u\\
y &= \xi_1
\end{aligned} \tag{5}$$

in which $b(z,\xi_1,\dots,\xi_r)\neq 0$, if $q(0,0,\dots,0)=0$ and if the dynamics of the inverse system are input-to-state stable (with respect to $\xi_1,\dots,\xi_r$ viewed as inputs), that is – in the terminology introduced later in [35] – if the system is *strongly minimum phase*, it can be globally stabilized by means of a feedback law of the form

$$u=\kappa\left(\sum_{i=0}^{r-2}c_i\xi_{i+1}+\xi_r\right)$$

in which $\kappa(\cdot)$ is a class $\mathcal{K}_\infty$ function. If, in addition, the equilibrium $z=0$ of the inverse dynamics is locally exponentially stable, semiglobal stability can be achieved via dynamic feedback from $y$, with the partial state $(\xi_1,\dots,\xi_r)$ being approximated, as originally suggested in [31], by means of a high-gain robust observer, as proven in detail in the fundamental paper [48].

### 3.2. Zero dynamics and feedback linearization

As initially observed in [21], the normal form (5) reveals that a single-input single-output system possessing a trivial zero dynamics, i.e. in which $\dim(z)=0$, can be rendered linear via (static) state feedback and changes of coordinates. This result, in fact, is valid in a much broader context, as it was shown a little later in [22]. This paper considers systems having $m$ inputs and $m$ outputs, that satisfy the invertibility assumptions proposed by Singh (see also [36] for a global version of these assumptions). These systems, as shown in [11], can always be turned, by means of a suitable dynamic extension, into systems possessing a multi-input multi-output version of the normal form (5). As a consequence, if the normal form in question is such that $\dim(z)=0$, the original system can be rendered linear, via dynamic extension, state feedback and changes of coordinates. The crucial observation of [22], in this respect, was the property that $\dim(z)=0$ in the dynamically extended system is not affected by the extension process (as one might in principle guess) and can in fact be easily identified as a property of the original system. The property in question is simply that the zero dynamics of the original system are trivial. The consequence of this is that any system with $m$ inputs and $m$ outputs that satisfies the invertibility assumptions of Singh and whose zero dynamics are trivial (a property whose fulfillment can be determined as a byproduct of the inversion algorithm) can always be rendered linear via feedback and changes of coordinates. This result has been recently enhanced in [36], where it is shown that a linear behavior can be achieved, to some extent, by pure static state feedback. Note that systems whose zero dynamics are trivial are such that their state can be expressed as a function of the output variables and (a finite number of) their higher derivatives. If no specific output functions are given, this simple observation can obviously be exploited to the purpose of designing *dummy* outputs, for a multi-input system, in order to obtain a system that is invertible and possesses a trivial zero dynamics. The dummy outputs in question have been called *linearizing outputs* (see also [23, pp. 262–263]).

### 3.3. Zero dynamics and stable non-interacting control

One of the earlier results in nonlinear feedback design was the solution of the problem of non-interacting control via static, or dynamic, state feedback. Initially, though, the problem of achieving non-interacting control *with internal stability* was not specifically addressed. Conditions to obtain also internal stability when non-interaction is achieved via static state feedback have been analyzed in [27], which extends an earlier result of Gilbert. This paper shows that there exists a well-defined internal dynamics, a sub-dynamics of the zero dynamics of the system, which is fixed with respect to any decoupling regular static state feedback. Thus, non-interaction with stability via regular static state feedback can only be obtained if the dynamics in question is asymptotically stable. In the case of linear systems, this obstruction can be destroyed if dynamic state feedback is used, as shown earlier by Wonham and Morse. Thus, the question arouse of whether or not a similar result could be obtained for nonlinear systems. A counter-example in [27] showed that this is not possible, in general. A thorough analysis of the necessary conditions for non-interaction with stability via dynamic state feedback was carried out later in [49], where it is shown that there exists a subdynamics of the fixed dynamics identified in [27] which cannot be eliminated by any regular dynamic feedback which makes the system non-interactive. These dynamics evolve on an integral manifold of a distribution spanned by Lie brackets of vector fields in which the vector fields that weigh the input channels appear repeatedly and hence is non-existent in the case of linear systems. Thus, the obstruction identified in [49] is a genuine nonlinear obstruction to the possibility of achieving, in general, non-interacting control with stability. If the fixed dynamics in question is asymptotically stable, though, non-interacting control with (at least local) internal stability can be achieved, as it was shown later in [2], which extends to a differential geometric setting the geometric approach used earlier by Wonham and Morse for linear systems.

### 3.4. Zero dynamics and output regulation

The concept of zero dynamics plays a fundamental role in the problem of output regulation. The problem in question considers a controlled plant modeled by

$$\begin{aligned} \dot{w} &= s(w) \\ \dot{x} &= f(w,x,u) \\ e &= h(w,x) \\ y &= k(w,x), \end{aligned} \tag{6}$$

in which $u$ is the control input, $w$ is a set of exogenous variables (commands and disturbances), $e$ is a set of regulated variables and $y$ is a set of *additional* measured variables, and seeks a (possibly dynamic) controller, driven by $e$ and $y$, such that in the resulting closed-loop system all trajectories are ultimately bounded and $\lim_{t\to\infty} e(t) = 0$. The problem in question has been the object of intensive research in the past years and the reader is referred, for an extensive presentation of various relevant findings, to the monographs [24,19,41]. In what follows we limit ourselves to signal the role of the concept of zero dynamics in this problem.

Assume that the set $W$ where the exosystem evolves is compact and invariant for $\dot{w} = s(w)$, that $y=e$ and $\dim(u) = \dim(e) = 1$. Suppose that the problem of output regulation is solved by an *error driven* controller

$$\begin{aligned} \dot{x}_c &= f_c(x_c, e) \\ u &= h_c(x_c, e). \end{aligned} \tag{7}$$

Then, the associated closed-loop has a steady-state locus (see [26]), the graph of a possibly set-valued map defined on $W$. Suppose the map in question is single-valued, which means that for each given exogenous input function $w(t)$, there exists a *unique* steady-state response, expressed as $x(t) = \pi(w(t))$ and $x_c(t) = \pi_c(w(t))$. If, in addition, $\pi(w)$ and $\pi_c(w)$ are continuously differentiable, it is readily seen that

$$\left.\begin{aligned} L_s\pi(w) &= f(w, \pi(w), \psi(w)) \\ 0 &= h(w, \pi(w)) \\ L_s\pi_c(w) &= f_c(\pi_c(w), 0) \\ \psi(w) &= h_c(\pi_c(w), 0) \end{aligned}\right. \quad \forall w \in W. \tag{8}$$

The first two equations of (8), introduced in [25], are known as the *nonlinear regulator equations*. They clearly show that the graph of the map $\pi(w)$ is a manifold contained in the zero set of the output map $e$, rendered invariant by the control $u = \psi(w)$. In particular, the steady-state trajectories of the closed-loop system (6)–(7) are trajectories of the *zero dynamics* of (6). The second two equations of (8), on the other hand, interpret the ability, of the controller, to generate the feedforward input necessary to keep $e(t) = 0$ in steady-state. This is a nonlinear version of the well-known *internal model principle* of Francis and Wonham.

It is seen from the above that, in order to be able to solve a problem of output regulation, one should at least be able to find an integer $d$, a pair of maps $\varphi : \mathbb{R}^d \to \mathbb{R}^d$ and $\gamma : \mathbb{R}^d \to \mathbb{R}$, and a map $\tau : W \to \mathbb{R}^d$ satisfying

$$\left.\begin{aligned} \psi(w) &= \gamma(\tau(w)) \\ L_s\tau(w) &= \varphi(\tau(w)) \end{aligned}\right. \quad \forall w \in W. \tag{9}$$

The existence of such maps was recently showed in [38]. In particular, it was shown that if the dimension of $d$ is large enough, there exists a controllable pair $(F,G)$, in which $F$ is a $d \times d$ Hurwitz matrix and $G$ is a $d \times 1$ vector, a continuous map $\gamma : \mathbb{R}^d \to \mathbb{R}$ and a continuously differentiable map $\tau : W \to \mathbb{R}^d$ satisfying

$$\begin{aligned} L_s\tau(w) &= F\tau(w) + G\psi(w) \\ \psi(w) &= \gamma(\tau(w)) \end{aligned}$$

from which it is seen that (9) can be fulfilled with

$$\varphi(\eta) = F\eta + G\gamma(\eta).$$

With this in mind, assume now that the zero dynamics of (6) are globally asymptotically stable to a compact invariant set $\mathcal{A}_0$. It can be shown that the zero dynamics of the augmented system obtained controlling (6) via

$$\begin{aligned} \dot{\eta} &= F\eta + G(\gamma(\eta) + v) \\ u &= \gamma(\eta) + v \end{aligned}$$

are globally asymptotically stable to a compact invariant set entirely contained in the set where $e=0$. As a consequence, the augmented system in question is semiglobally stabilizable, by means of a control of the form $v = \kappa(e)$ and the problem of (semiglobal) output regulation can be solved (see [38] for details).

### 3.5. Zero dynamics and passivity

It is well known that the notion of passivity plays an important role in system analysis and that the theory of passive systems leads to powerful methodologies for the design of feedback laws for nonlinear systems. In view of this, and of the role played by the concept of passivity also in the analysis of the stability of interconnected systems, the following question was analyzed in [4]: When can a finite-dimensional nonlinear system be rendered passive via state feedback? As summarized below, the answer to this question is yet another manifestation of the importance of the concept of zero dynamics.

Consider a nonlinear input-affine system having the same number $m$ of inputs and outputs and recall that this system is said to be *passive* if there exists a continuous nonnegative real-valued function $W(x)$, with $W(0) = 0$, that satisfies

$$W(x(t)) - W(x(0)) \le \int_0^t y^{\mathrm{T}}(s)u(s)\, ds$$

along the trajectories. The function $W(x)$ is the so-called *storage function* of the system.

Suppose now that $L_g h(x)$ is nonsingular and that the $m$ columns of the matrix $\tilde{g}(x) = g(x)[L_g h(x)]^{-1}$ are complete and commuting vector fields. In this case, the system is globally diffeomorphic to a system in normal form as in (2), where now $\xi, u, y$ are $m$-dimensional vectors. The system is said to be *globally weakly minimum phase* if there exists a $C^1$ positive definite function $V(z)$ that satisfies $L_{f(z,0)}V(z) \le 0$ for all $z$. The answer to the question posed at the beginning, namely when a system is feedback equivalent to a passive system, has the following answer: a system is *globally feedback equivalent to a passive system* with a $C^2$ storage function $W(x)$, which is positive definite, if and only if it is globally weakly minimum phase.

### 3.6. Zero dynamics and limits of performance

It is well-known that linear systems having zeros in the left-half plane are difficult to control, and obstructions exist to the fulfillment of certain control specifications. One of these is found in the analysis of the so-called *cheap control problem*, namely the problem of finding a stabilizing feedback control that minimizes the functional

$$J_\varepsilon = \frac{1}{2}\int_0^\infty [y^{\mathrm{T}}(t)y(t) + \varepsilon u^{\mathrm{T}}(t)u(t)]\, dt$$

when $\varepsilon > 0$ is small. As $\varepsilon \to 0$, the optimal value $J_\varepsilon^*$ tends to $J_0^*$, the *ideal performance*. It is well-known that, in a linear system, $J_0^* = 0$ if and only if the system is minimum phase and right invertible and, in case the system has zeros with positive real part, it is possible to express explicitly $J_0^*$ in terms of the zeros in question. If the (linear) system is expressed in normal form as

$$\begin{aligned} \dot{z} &= Fz + G\xi \\ \dot{\xi} &= Hz + K\xi + bu \\ y &= \xi \end{aligned}$$

with $b \neq 0$, and the zero dynamics are antistable (that is *all* the eigenvalues of $F$ have positive real part), it can be shown that $J_0^*$ coincides with the minimal value of the energy

$$J = \frac{1}{2}\int_0^\infty \xi^{\mathrm{T}}(t)\xi(t)\, dt$$

required to stabilize the (antistable) system $\dot{z} = Fz + G\xi$. In other words, the limit as $\varepsilon \to 0$ of the optimal value of $J_\varepsilon$ is equal to the least amount of energy required to stabilize the dynamics of the inverse system.

As shown in [42], this result has an appealing nonlinear counterpart. In fact, for a nonlinear input-affine system having the same number $m$ of inputs and outputs in normal form as in (2), with $f(z,\xi)$ of the form $f(z,\xi) = f_0(z) + g_0(z)\xi$ and $\dot{z} = f_0(z)$ antistable, under appropriate technical assumptions (mostly related to the existence of the solution of associated optimal control problems), the same result holds: the lowest attainable value of the $L_2$ norm of the output coincides with the least amount of energy required to stabilize the dynamics of $z$.

## 4. Current advances and open problems

### 4.1. Strongly minimum-phase systems

As observed before, systems whose zero dynamics have a globally asymptotically stable equilibrium can, under appropriate assumptions, be *stabilized* with a guaranteed region of attraction using (possibly dynamic) output feedback. One of the problems of current interest is to enhance the theory so as to include: (i) the case of systems having relative degree $r > 1$ and possessing a normal form in which the dynamics of the inverse system are driven not just by $y$ but *also* by $y^{(1)}, \ldots, y^{(r-1)}$, (ii) the case of systems that *do not possess* a globally defined normal form and (iii) the case in which the region of attraction coincides with the entire state space, if so is possible. Instrumental, in this context, is the concept of a *strongly minimum phase system*, introduced in [35] to the purpose of providing a formal, and coordinate-free, characterization of the class of systems in which the state is bounded by a function of the outputs and its first $r-1$ derivatives ($r$ being the relative degree), modulo a decaying term depending on the initial conditions.

Consider the case of an *input-affine* system

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned} \tag{10}$$

with $x \in \mathbb{R}^n$, $u \in \mathbb{R}$, $y \in \mathbb{R}$, possessing a well-defined relative degree $r$, which is the integer satisfying (if such integer exists)

$$L_g h(x) = \cdots = L_g L_f^{r-2} h(x) = 0, \quad \forall x \in \mathbb{R}^n$$

$$L_g L_f^{r-1} h(x) \neq 0, \quad \forall x \in \mathbb{R}^n.$$

Recall that $y^{(i)} = L_f^i h(x)$ for $i = 0, 1, \ldots, r-1$ and let $\mathbf{y}^{r-1}(t)$ denote the $\mathbb{R}^r$−valued function of $t$

$$\mathbf{y}^{r-1}(t) \triangleq \mathrm{col}(y(t), y^{(1)}(t), \ldots, y^{(r-1)}(t)).$$

Moreover, let $Z$ denote the set

$$Z \triangleq \{x \in \mathbb{R}^n : h(x) = L_f h(x) = \cdots = L_f^{r-1} h(x) = 0\}.$$

The definition that follows expresses the property that the distance $d_{\mathcal{A}}(x(t))$ of $x(t)$ from a compact set $\mathcal{A}$ is bounded by a suitable function of the output and its first $r-1$ derivatives, modulo a decaying term depending on the initial conditions (as in [35], we let $\|\mathbf{f}\|_{[a,b]}$ denote the supremum norm, of a function $\mathbf{f} : \mathbb{R} \to \mathbb{R}^r$, restricted to an interval $[a,b]$).

**Definition.** Consider a relative degree $r$ system and let $\mathcal{A}$ be a compact subset of $Z$. This system is *strongly minimum phase with respect to* $\mathcal{A}$ if there exist a class $\mathcal{KL}$ function $\beta$ and a class $\mathcal{K}_\infty$ function $\gamma$ such that for every initial state $x(0) \in \mathbb{R}^n$ and every admissible input $u(\cdot)$ the corresponding solution $x(t)$ satisfies

$$d_{\mathcal{A}}(x(t)) \leq \max\{\beta(d_{\mathcal{A}}(x(0)), t), \gamma(\|\mathbf{y}^{r-1}\|_{[0,t]})\} \tag{11}$$

as long as it exists.

Note that in the previous definition the existence of a globally defined normal form *is not assumed*. This assumption, as it is known, requires certain vector fields to be complete. If a system possesses a globally defined normal form, the property in question is simply the property that the inverse dynamics are input-to-state stable (viewing the output and its first $r-1$ derivatives as inputs) to the set $\mathcal{A}$. The property thus defined is useful in various instances, as shown below.

### 4.2. Robust stabilization via dynamic output feedback

We summarize in this section a recent approach, pursued in [14,12] (see also [13,51]), to the problem of (semiglobal) asymptotic stabilization – via dynamic output feedback – of a globally minimum phase system having relative degree $r > 1$. This approach proposes an effective and straightforward design procedure.

Consider a single-input single-output system having relative degree $r > 1$, and possessing a globally defined normal form, which can be written as

$$\begin{aligned} \dot{z} &= f(z,\xi) \\ \dot{\xi} &= \hat{A}\xi + \hat{B}[q(z,\xi) + b(z,\xi)u] \\ y &= \hat{C}\xi, \end{aligned} \tag{12}$$

where $z \in \mathbb{R}^{n-r}$, $\xi \in \mathbb{R}^r$ and

$$\hat{A} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad \hat{B} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \quad \hat{C} = (1 \ 0 \ 0 \ \cdots \ 0).$$

Assume that

$$q(0,0) = 0 \tag{13}$$

and that the coefficient $b(z,\xi)$ satisfies

$$0 < b_{\min} \leq b(z,\xi) \leq b_{\max} \quad \text{for all } (z,\xi) \tag{14}$$

for some $b_{\min}, b_{\max}$. Finally, assume that

$$\dot{z} = f(z,\xi), \tag{15}$$

viewed as a system with input $\xi$ and state $z$, is *input-to-state stable*. In view of the definition given in the previous section, this is equivalent to assume that system (12) is strongly minimum phase with respect to the set $\mathcal{A} = \{0\}$.

It is well-known that the feedback law

$$u = \frac{1}{b(z,\xi)}(K\xi - q(z,\xi)), \tag{16}$$

if $K$ is such that $(\hat{A} + \hat{B}K)$ is a Hurwitz matrix, globally asymptotically stabilizes the equilibrium $(z,\xi) = (0,0)$ of the resulting closed-loop system. However, the implementation of this law requires accurate knowledge of $b(z,\xi)$ and $q(z,\xi)$ and availability of the full state $(z,\xi)$. It was shown in [12] that a suitable "asymptotic proxy" of this law can be designed, which does not suffer such limitations.

The idea is to use the measured output $y$ to drive an appropriate dynamical system to the purpose of estimating the components of $\xi$ as well as to overcome the necessity of knowing the functions $b(z,\xi)$ and $q(z,\xi)$. To this end, let $\psi(\xi,\sigma)$ be the function defined as

$$\psi(\xi,\sigma)=\frac{1}{b_0}[K\xi-\sigma],$$

in which $\xi\in\mathbb{R}^r$, $\sigma\in\mathbb{R}$, $b_0$ is a design parameter and $K$ a vector with the properties indicated above (i.e. such that $(\hat{A}+\hat{B}K)$ is a Hurwitz matrix), and let $g:\mathbb{R}\to\mathbb{R}$ be a smooth "saturation" function, characterized as follows: $g(s)=s$ if $|s|\le L$, $g(s)$ is odd and monotonically increasing, with $0<g'(s)\le 1$, and $\lim_{s\to\infty}g(s)=L(1+c)$ with $0<c\ll 1$. The "saturation level" $L$, which in order to simplify the notation is not explicitly indicated in the symbol used to denote the function in question, is a design parameter that will be determined later.

System (12) is controlled by a feedback law of the form

$$u=g(\psi(\hat{\xi},\sigma)),\tag{17}$$

in which $\hat{\xi}\in\mathbb{R}^r$ and $\sigma$ are states of the dynamical system

$$\begin{aligned}
\dot{\hat{\xi}}_1&=\hat{\xi}_2+\kappa\alpha_1(y-\hat{\xi}_1)\\
\dot{\hat{\xi}}_2&=\hat{\xi}_3+\kappa^2\alpha_2(y-\hat{\xi}_1)\\
&\vdots\\
\dot{\hat{\xi}}_{r-1}&=\hat{\xi}_r+\kappa^{r-1}\alpha_{r-1}(y-\hat{\xi}_1)\\
\dot{\hat{\xi}}_r&=\sigma+b_0g(\psi(\hat{\xi},\sigma))+\kappa^r\alpha_r(y-\hat{\xi}_1)\\
\dot{\sigma}&=\kappa^{r+1}\alpha_{r+1}(y-\hat{\xi}_1).
\end{aligned}\tag{18}$$

The coefficients $\kappa$ and $\alpha_1,\alpha_2,\dots,\alpha_{r+1}$ are design parameters.

The dynamical system thus defined has the typical structure of an "observer". In the analysis of the asymptotic properties of the resulting closed-loop system, it is convenient to replace $\hat{\xi}_1,\dots,\hat{\xi}_r,\sigma$ by means of (scaled) "error" variables, defined as follows:

$$\begin{aligned}
e_1&=\kappa^r(\xi_1-\hat{\xi}_1)\\
e_2&=\kappa^{r-1}(\xi_2-\hat{\xi}_2)\\
&\vdots\\
e_r&=\kappa(\xi_r-\hat{\xi}_r)\\
e_{r+1}&=q(z,\xi)+[b(z,\xi)-b_0]g(\psi(\xi,\sigma))-\sigma.
\end{aligned}\tag{19}$$

The first $r$ of these relations can be trivially inverted, to recover each $\hat{\xi}_i$, as a function of $e_i$ and $\xi_i$. To recover $\sigma$ from the latter, $b_0$ needs to be chosen appropriately. To this end, bearing in mind the expression of $\psi(\xi,\sigma)$, observe that the relation in question is equivalent to the following one:

$$\frac{K\xi-q(z,\xi)+e_{r+1}}{b(z,\xi)}=\frac{b_0}{b(z,\xi)}\left[\left(\frac{b(z,\xi)-b_0}{b_0}\right)g(\psi(\xi,\sigma))+\psi(\xi,\sigma)\right].\tag{20}$$

If one sets

$$\psi^*(z,\xi,e_{r+1})=\frac{K\xi-q(z,\xi)+e_{r+1}}{b(z,\xi)}$$

and defines a function $F:\mathbb{R}\to\mathbb{R}$ as

$$F(s)=\frac{b_0}{b(z,\xi)}\left[\left(\frac{b(z,\xi)-b_0}{b_0}\right)g(s)+s\right]\tag{21}$$

the relation (20) can be simply rewritten as

$$\psi^*=F(\psi).$$

Since $b(z,\xi)$, by assumption, is bounded as in (14), it is always possible to pick $b_0$ so as to make

$$\left|\frac{b(z,\xi)-b_0}{b_0}\right|\le\delta_0<1\tag{22}$$

for some $\delta_0$. Thus, since $0<g'(s)\le 1$ by hypothesis, if $b_0$ is chosen in this way, $F'(s)$ is strictly positive, i.e. $F(s)$ is a strictly increasing (odd) function. Moreover, since $\lim_{s\to\infty}g(s)=L(1+c)$, it is seen that $\lim_{s\to\infty}F(s)=\infty$, and consequently $F(\mathbb{R})=\mathbb{R}$. In summary, $F(s)$ is globally invertible. It is also worth noting that, so long as $|s|\le L$, the function $F(s)$ is an identity, i.e. $F(s)=s$.

Hence, if $b_0$ is chosen to satisfy (22), one has

$$\psi=F^{-1}(\psi^*)$$

and this – bearing in mind the expressions of $\psi$ and $\psi^*$ – shows that $\sigma$ can always be recovered, from the last of (19), as a smooth function of $z,\xi,e_{r+1}$. This renders the transformation (19) a diffeomorphism.

Appropriate calculations show that the variable $e=\mathrm{col}(e_1,\dots,e_{r+1})$ thus defined satisfies an equation having the following structure:

$$\dot{e}=\kappa[A-BC\Delta_0(z,\xi,e)]e+B_1\Delta_1(z,\xi,e)+B_2\Delta_2(z,\xi,e),\tag{23}$$

in which

$$A=\begin{pmatrix}-\alpha_1&1&0&\cdots&0\\-\alpha_2&0&1&\cdots&0\\\cdot&\cdot&\cdot&\cdots&0\\-\alpha_r&\cdot&\cdot&\cdots&1\\-\alpha_{r+1}&\cdot&\cdot&\cdots&0\end{pmatrix},\quad B=B_2=\begin{pmatrix}0\\0\\\vdots\\0\\1\end{pmatrix},\quad B_1=\begin{pmatrix}0\\0\\\vdots\\1\\0\end{pmatrix},$$

$$C=(\alpha_{r+1}\ \ 0\ \cdots\ 0\ \ 0)$$

and $\Delta_0(z,\xi,e),\Delta_1(z,\xi,e),\Delta_2(z,\xi,e)$ are suitable real-valued functions with the following properties. If $b_0$ is chosen so as to satisfy (22), then $|\Delta_0(z,\xi,e)|\le\delta_0<1$, for some $\delta_0$. $|\Delta_1(z,\xi,e)|<\delta_1|e|$ for some $\delta_1$. For any compact set $\mathcal{S}$ there is a number $M_{\mathcal{S}}$ such that $|\Delta_2(z,\xi,e)|\le M_{\mathcal{S}}$ for all $(z,\xi)\in\mathcal{S}$ and all $e\in\mathbb{R}^{r+1}$. This number $M_{\mathcal{S}}$ is independent of $\kappa$.

On the other hand, in the new variables, the (controlled) system is described by equations of the form

$$\begin{aligned}
\dot{z}&=f(z,\xi)\\
\dot{\xi}&=(\hat{A}+\hat{B}K)\xi+\hat{B}\left[q(z,\xi)+b(z,\xi)g\left(\frac{K\hat{\xi}-\sigma}{b_0}\right)-K\xi\right],
\end{aligned}\tag{24}$$

in which $\hat{\xi}$ and $\sigma$ are to be seen as functions of $e$.

If the design parameter $\kappa$ is large, the system (24) and (23) has the standard form of a two-time-scale system. Taking advantage of this structure and of the properties of the various functions involved, in [12] it is shown that the following remarkable stabilization result is true.

**Proposition 1.** *Consider system (12), controlled by (17)–(18). Suppose that (13) holds and that $b(x,\xi)$ is bounded as in (14). Suppose (12) is strongly minimum phase (with respect to the set $\mathcal{A}=\{0\}$). Let $K$ be such that $\hat{A}+\hat{B}K$ is Hurwitz. For every choice of a compact set $\mathcal{C}$, there is a choice of the design parameters $b_0$, $L$ and $\alpha_1,\dots,\alpha_{r+1}$ and a number $\kappa^*$ such that, for all $\kappa\ge\kappa^*$, the equilibrium $(z,\xi,\hat{\xi},\sigma)=(0,0,0,0)$ is asymptotically stable, with a domain of attraction that contains the set $\mathcal{C}$.*

### 4.3. A coordinate-free setting

If a system is strongly minimum phase, global stabilization can be achieved in a *coordinate-free* setting, i.e. also in the case of systems that do not possess a globally defined normal form. For the sake of more generality, we discuss in what follows the problem of *output stabilization*, which contains problems of tracking and/or regulation as special cases. The problem in question is defined as follows. Consider a nonlinear smooth system described as in (10). The control law is to be provided by a *continuous* system modeled by equations of the form

$$\begin{aligned}
\dot{x}_c&=f_c(x_c,y)\\
u&=h_c(x_c,y)
\end{aligned}\tag{25}$$

with state $x_c\in\mathbb{R}^{n_c}$.

The controller (25) *solves the problem of global output stabilization*, if the closed-loop system (10)–(25) has the following properties:

(i) its solutions are *ultimately bounded* i.e. there exists a bounded set $B \subset \mathbb{R}^n \times \mathbb{R}^{n_c}$ with the property that, for every pair of compact sets $X^\circ \subset \mathbb{R}^n$ and $X_c^\circ \subset \mathbb{R}^{n_c}$, there is a time $T>0$ such that $(x(t), x_c(t)) \in B$ for all $t \geq T$ and all $(x(0), x_c(0)) \in X^\circ \times X_c^\circ$;
(ii) $\lim_{t\to\infty} h(x(t)) = 0$, $\forall (x(0), x_c(0)) \in \mathbb{R}^n \times \mathbb{R}^{n_c}$.

Strongly minimum phase systems can be globally stabilized using output feedback. We consider first the case $r=1$ for which we have (see [8]):

**Theorem 1.** *Consider a relative degree 1 system and let $\mathcal{A}$ be a compact subset of Z. Suppose the system is strongly minimum phase with respect to $\mathcal{A}$. Suppose that $L_g h(x)=0$ for all $x \in \mathcal{A}$. Then there exists a continuous feedback law $u=\kappa(y)$ such that, in the resulting closed-loop system, any $x(0)\in\mathbb{R}^n$ produces a trajectory that is bounded on $[0,\infty)$ and $\lim_{t\to\infty} d_{\mathcal{A}}(x(t))=0$.*

If $r>1$, one can proceed as follows. Let $\mathrm{col}(c_0, \ldots, c_{r-2}, 1) \in \mathbb{R}^r$ be a vector such that the polynomial

$$s^{r-1} + c_{r-2}s^{r-2} + \cdots + c_1 s + c_0$$

is Hurwitz, and let the output $y$ of system (10) be replaced by the auxiliary "dummy" output

$$\theta = h_\theta(x) = \sum_{i=0}^{r-2} c_i L_f^i h(x) + L_f^{r-1} h(x). \tag{26}$$

It is trivial to check that system (10) with output (26) has relative degree one. Moreover, the following property holds (see [8]).

**Lemma 1.** *Suppose system (10) has relative degree r and is strongly minimum phase with respect to $\mathcal{A}$, a compact subset of Z. Then also system (10) with output (26) is strongly minimum phase with respect to $\mathcal{A}$. In fact, there exist $\beta_\theta \in \mathcal{KL}$ and $\gamma_\theta \in \mathcal{K}_\infty$ such that for every initial state $x(0)\in\mathbb{R}^n$ and every admissible input $u(\cdot)$ the corresponding solution $x(t)$ satisfies*

$$d_{\mathcal{A}}(x(t)) \leq \max\{\beta_\theta(d_{\mathcal{A}}(x(0)), t), \gamma_\theta(\|\theta\|_{[0,t]})\}$$

*as long as it exists.*

This immediately yields the following global stabilization result.

**Theorem 2.** *Suppose system (10) has relative degree r and is strongly minimum phase with respect to $\mathcal{A}$, a compact subset of Z. If $L_f^r h(x)=0$ for all $x\in\mathcal{A}$, there exists a continuous map $\kappa_\theta : \mathbb{R}\to\mathbb{R}$ such that the feedback law $u=\kappa_\theta(h_\theta(x))$ solves the problem of global output stabilization.*

### 4.4. Output redesign for non-minimum phase systems

Prerequisites for the applicability of the stabilization paradigms described in the previous sections are the property that the function $L_f^r h(x)$ vanishes on the invariant set $\mathcal{A}$ and – above all – the property that the system is strongly minimum phase with respect to $\mathcal{A}$. In the problem of stabilizing the equilibrium $(z,\xi)=(0,0)$ of a system of the form (12), the properties in question reduce to property (13) and, respectively, to the property that (15) is input-to-state stable. Systems in which the latter property does not hold (*non-minimum phase* systems) are notoriously more difficult to control and the development of general design methods for these systems, to begin with the basic issue of robust stabilization, is still a widely open area of research.

In what follows, we sketch a (simple-minded) approach that could be useful in some instances. This approach is based on the idea of *reshaping* the zero dynamics of the system by re-designing its output. For the sake of simplicity, we restrict the presentation to the case of an input-affine system having relative degree $r=1$ and in which $L_g h(x)=1$ for all $x\in\mathbb{R}^n$. It is well-known that, if the vector field $g(x)$ is complete, the system is globally diffeomorphic to a system of the form (2) and in such coordinates

$$L_g h(x) = b(z,\xi) = 1$$
$$L_f h(x) = q(z,\xi).$$

Let $\mathcal{A}_0 \in \mathbb{R}^{n-1}$ be a compact set, invariant for the dynamics of

$$\dot z = f(z, 0).$$

If the sub-system $\dot z = f(z,\xi)$, viewed as a system with state $z$ and input $\xi$ is input-to-state stable with respect to $\mathcal{A}_0$, and if $q(z,\xi)=0$ for all $(z,\xi)\in\mathcal{A}$, global output stabilization can be achieved by means of the techniques discussed earlier. If not, one can proceed as follows.

Assume, as it is always possible [38], the existence of an integer $d$, of a pair $(F,G)$ in which $F$ is a $d\times d$ Hurwitz matrix and $G$ is a $d\times 1$ column vector that makes the pair $(F,G)$ controllable, of a continuous map $\gamma : \mathbb{R}^d \to \mathbb{R}$ and a continuously differentiable map $\tau : \mathcal{A}_0 \to \mathbb{R}^d$, satisfying

$$\begin{aligned} &L_{f(z,0)}\tau(z) = F\tau(z) + G\gamma(\tau(z)) \quad \forall z\in\mathcal{A}_0 \\ &-q(z,0) = \gamma(\tau(z)) \quad \forall z\in\mathcal{A}_0. \end{aligned} \tag{27}$$

Then, choose the control as

$$\begin{aligned} u &= \gamma(\eta) + v + \dot N(\xi,\psi) \\ \dot\eta &= F(\eta - G\xi + GN(\psi)) + G(\gamma(\eta) + v) \\ \dot\psi &= L(\psi + M\xi - MN(\psi)) - Mv. \end{aligned} \tag{28}$$

in which $L(\varrho), N(\varrho)$ are smooth maps satisfying $L(0)=0$, $N(0)=0$, with $M$ such

$$\frac{\partial N}{\partial \varrho} M = 0, \tag{29}$$

and $\dot N(\xi,\psi)$ defined as

$$\dot N(\xi,\psi) = \left[\frac{\partial N}{\partial \varrho}\right]_{\varrho=\psi} L(\psi + M\xi - MN(\psi)).$$

Define a *new output* as

$$\theta = \xi - N(\psi).$$

Because of (29), the composed system (2)–(28), with input $v$ and output $\theta$, still has relative degree 1. Change variables as

$$x = \eta - G\theta$$
$$\chi = \psi + M\theta$$

to obtain a system (observe that $N(\chi - M\theta) = N(\chi)$, because of (29))

$$\begin{aligned} \dot z &= f(z, \theta + N(\chi)) \\ \dot x &= Fx - Gq(z, \theta + N(\chi)) \\ \dot\chi &= L(\chi) + M[q(z,\theta+N(\chi)) + \gamma(x + G\theta)] \\ \dot\theta &= q(z,\theta+N(\chi)) + \gamma(x+G\theta) + v. \end{aligned} \tag{30}$$

Observe now that the set

$$\mathcal{A} = \{(z,x,\chi,\theta) : z\in\mathcal{A}_0, x=\tau(z), \chi=0, \theta=0\}$$

is a compact invariant set of (30) if $v=0$. Moreover, by construction,

$$q(z,\theta+N(\chi)) + \gamma(x+G\theta) = 0 \quad \forall (z,x,\chi,\theta)\in\mathcal{A}.$$

All of the above suggest the use of the degrees of freedom in the choice of the parameters of the controller so as to make system (30) strongly minimum phase with respect to the set $\mathcal{A}$, or – what is the same – to make system

$$\dot z = f(z,\theta+N(\chi))$$
$$\dot x = Fx - Gq(z,\theta+N(\chi))$$

$$\dot{\chi} = L(\chi) + M[q(z, \theta + N(\chi)) + \gamma(x + G\theta)], \tag{31}$$

which characterizes the inverse dynamics of the composite system (2)–(28) with respect to the re-designed output $\theta$, input-to-state stable with respect to the set

$$\mathcal{A}_0^{\mathrm{e}} = \{(z, x, \chi) : z \in \mathcal{A}_0, x = \tau(z), \chi = 0\}.$$

If this is the case, in fact, the previous results could be used "off-the-shelf" to obtain output stabilization.

The system in question is not terribly difficult to handle. As a matter of fact, it can be regarded as the interconnection of three simpler subsystems. System (31), in fact, can be seen as the cascade of an "inner loop" consisting of a subsystem, which we call the "auxiliary plant", modeled by equations of the form

$$\begin{aligned} \dot{z} &= f(z, u_{\mathrm{a}} + \theta) \\ y_{\mathrm{a}} &= q(z, u_{\mathrm{a}} + \theta), \end{aligned} \tag{32}$$

controlled by

$$\begin{aligned} \dot{\chi} &= L(\chi) + \mathrm{M}[y_{\mathrm{a}} + \overline{u}] \\ u_{\mathrm{a}} &= N(\chi), \end{aligned} \tag{33}$$

cascaded with a system, which we call the "weighting filter", modeled by equations of the form

$$\begin{aligned} x &= Fx - Gy_{\mathrm{a}} \\ \overline{y} &= \gamma(x + G\theta), \end{aligned} \tag{34}$$

and finally controlled by $\overline{u} = \overline{y}$.

Examples of non-minimum phase systems that can be handled by (a simplified version of) this method have been recently discussed in [9,10,40]. In particular, the former addresses the case in which the controlled plant is a linear system, and output stabilization is sought in spite of exogenous disturbances that are sinusoidal functions of time. In this case, it can be shown that if the controlled plant has an arbitrary number of zeros at the origin while all remaining zeros have negative real part, the problem of output stabilization can always be solved. On the other hand, if the controlled plant has a zero with positive real part, the problem can be solved only if the frequencies which characterize the harmonic components of the exogenous input exceed a minimal value determined by the gain needed to stabilize the inner-loop. This is yet another manifestation of why unstable zero dynamics pose limits to the achievable performances.

### 4.5. Multi-input multi-output systems

The multi-input multi-output version of the stabilization paradigms described in the previous sections is still a largely open domain of research. A recent important advance in this domain has been the contribution of [36]. This paper considers input-affine systems having $m$ inputs and $p \geq m$ outputs, which have the following property: for some integer $N$, there exist a class $\mathcal{KL}$ function $\beta$ and a class $\mathcal{K}_\infty$ function $\gamma$ such that for every initial state $x(0)$ and every admissible input $u(\cdot)$ the corresponding solution $x(t)$ satisfies

$$|x(t)| \leq \max\{\beta(|x(0)|, t), \gamma(\|\mathbf{y}^{N-1}\|_{[0,t]})\}$$

as long as it exists. Comparing with (11), it is seen that a single-input single-output system having relative degree $r$ which is strongly minimum phase with respect to $\{0\}$ has this property, with $N = r$. Thus, the property in question is a possible extension to multivariable systems of the property of being *strongly minimum phase*. Then, paper [36] assumes that the system is *globally left invertible*, in the sense that (the global version of) Singh's inversion algorithm terminates at a stage $k^* \leq m$ in which the input $u(t)$ can be uniquely recovered from the output $y(t)$ and a finite number of its derivatives. Under this (and another technical) assumption it is shown that a static feedback law $u = \alpha(x)$ exists that globally stabilizes the system. The role of this law is essential to guarantee that – in the associated closed-loop system – the individual components of the output *obey linear differential equations* whose characteristic polynomials are Hurwitz.

This result is very promising, and is the more general result available to date dealing with global stabilization of multivariable systems possessing a stable zero dynamics. The feedback law proposed, though, is a static *state feedback* law. The problem of finding a (dynamic) feedback law driven only by the measured output $y$ is still widely open. There are classes of MIMO systems, though, in which the design paradigms based on high-gain feedback (from the output and their higher derivatives) are applicable. The simplest case is the one of an input-affine system having the same number $m$ of input and output components, in which matrix $L_g h(x)$ is nonsingular. In this case, in fact, if there exist a matrix $K$ and a number $b_0 > 0$ such that

$$L_g h(x)K + K^{\mathrm{T}}[L_g h(x)]^{\mathrm{T}} \geq b_0 I$$

and if the property outlined above holds for $N = 1$, the stabilization paradigms described in the previous sections can be trivially extended.

A less trivial scenario is the one in which the property that $L_g h(x)$ is nonsingular does not hold, but there exists a smooth map $\phi : \mathbb{R}^m \times \mathbb{R}^m \times \cdots \times \mathbb{R}^m \to \mathbb{R}^m$ such that, if the output $y$ of

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned}$$

is replaced by

$$\tilde{y} = \phi(y, y^{(1)}, \ldots, y^{(k)}),$$

then $\tilde{y}^{(1)} = A(x) + B(x)u$ for some nonsingular matrix $B(x)$ satisfying $B(x) + B^{\mathrm{T}}(x) \geq b_0 I$. If this is the case, the paradigm in question, supplemented by the robust observer of [48], can still be pursued to obtain semiglobal stability. A special class of systems for which such transformation exists are the systems whose input–output behavior can be rendered linear via static state-feedback (see [23, pp. 277–291] for details).

### 4.6. Internal model design for MIMO systems

In the general multi-input multi-output version of the problem of output regulation, a controlled system of the form (6) is considered, in which $u \in \mathbb{R}^m$, $e \in \mathbb{R}^p$, $y \in \mathbb{R}^q$ and a controller of the form

$$\begin{aligned} \dot{x}_{\mathrm{c}} &= f_{\mathrm{c}}(x_{\mathrm{c}}, e, y) \\ u &= h_{\mathrm{c}}(x_{\mathrm{c}}, e, y) \end{aligned} \tag{35}$$

is sought, so that in the resulting closed-loop system all the trajectories are ultimately bounded and $\lim_{t \to \infty} e(t) = 0$. Considering, without loss of generality, the case in which the state $w$ of the exosystem evolves on a compact invariant set $W$ and assuming that the steady-state locus of the associated closed-loop system is the graph of a single-valued continuously differentiable map defined on $W$, it is readily seen that, if the problem in question is solved, there exist maps $\pi(w)$ and $\pi_{\mathrm{c}}(w)$ satisfying

$$\begin{aligned} L_s\pi(w) &= f(w, \pi(w), \psi(w)) \\ 0 &= h(w, \pi(w)) \\ L_s\pi_{\mathrm{c}}(w) &= f_{\mathrm{c}}(\pi_{\mathrm{c}}(w), 0, k(w, \pi(w))) \\ \psi(w) &= h_{\mathrm{c}}(\pi_{\mathrm{c}}(w), 0, k(w, \pi(w))) \quad \forall w \in W. \end{aligned} \tag{36}$$

The first two are identical to the first two equations of (8), the regulator equations. The last two, though, are different because of the influence of the steady-state measured output $k(w, \pi(w))$, that in general is not expected to vanish. As we will see, this renders the associated design of nonlinear regulators substantially different (and in fact more difficult) than that of linear regulators.

In the case of linear systems, the regulator equations are (robustly) solvable if and only if the system is right-invertible (which in turn implies $m \geq p$) and none of the transmission zeroes are an eigenvalue of the exosystem (non-resonance condition). This being the case, the fulfillment of the extra two conditions is automatically guaranteed if the controller is chosen as the cascade of a "postprocessor" that contains $p$ identical controllable copies of the exosystem

$$\dot{\eta}_i = S\eta_i + G_i e_i, \quad i = 1, \dots, p \tag{37}$$

whose state $\eta = \text{col}(\eta_1, \dots, \eta_p)$ drives, along with the full measured output $(e,y)$, a "stabilizer"

$$\begin{aligned} \dot{\xi} &= F_{11}\xi + F_{12}\eta + B_{c1}e + B_{c2}y \\ u &= H_1\xi + H_2\eta + D_{c1}e + D_{c2}y. \end{aligned} \tag{38}$$

In fact, appealing to the non-resonance condition, it is a simple matter to show that if the controlled plant is stabilizable and detectable so it is the cascade of the controlled plant and of (37) and hence a stabilizer of the form (38) always exists. Then, appealing to Cayley–Hamilton's Theorem, it is not difficult to show that, regardless of what $\psi(w)$ and $\pi(w)$ are, the second two equations of (36) always have a solution $\pi_c(w)$ (even if $k(w, \pi(w))$ is nonzero).

In the case of nonlinear systems having $m > 1$ and $p > 1$, solving the first two equations of (36) is not terribly difficult. This can be achieved, in fact, by means of a suitably enhanced version of the zero dynamics algorithm presented in [23, pp. 293–311] and, if so desired for subsequent stabilization purposes, by bringing the system to the multivariable normal form described in [20, pp. 109–124].

However, the problem of building a controller that also solves the second two equations of (36) is substantially different, because the existence of $\pi_c(w)$ is no longer automatically guaranteed by the fact that the controller is realized as an internal model driven by the error variable $e$ which in turn drives a stabilizer. In fact, to the current state of our knowledge, it is known how to fulfill the equations in question only if the controller is realized as a "preprocessor"

$$\begin{aligned} \dot{\eta} &= \varphi(\eta) + Gv \\ u &= \gamma(\eta) + v, \end{aligned}$$

in which $\varphi(\cdot)$ and $\gamma(\cdot)$ satisfy (9) for some $\tau(\cdot)$, whose input $v$ is provided by a stabilizer only driven by the regulated variable $e$. This substantially limits the generality for at least two reasons: on one side, it does not allow additional measured outputs (which would be otherwise useful for stabilization purposes) because it is not immediate clear how their possibly nontrivial steady-state behavior could be (robustly) handled, on the other side because a scheme in which the internal model is a preprocessor requires (even in the case of linear systems) $m = p$ and this limits the availability of extra inputs (which, again, would be otherwise useful for stabilization purposes).

To the best of our knowledge, the convenience of using "postprocessing" internal models for nonlinear output regulation has been only very recently pointed out in the literature (see [28,1]). In particular, in [28] it was shown how a controller of the form

$$\begin{aligned} \dot{\eta} &= F\eta + G[\gamma(\eta) + e] \\ \dot{\xi} &= \varphi(\xi, e, y) + M_s[\gamma(\eta) + e] \\ u &= \vartheta(\xi, e, y) + N_s[\gamma(\eta) + e] \end{aligned} \tag{39}$$

could be used in order to solve a problem of output regulation. This is the case, in fact, if:

(i) the equations

$$\begin{aligned} \frac{\partial}{\partial w}\begin{bmatrix} \pi \\ \pi_s \end{bmatrix} s(w) &= \begin{bmatrix} f(w, \pi(w), \vartheta(\pi_s(w), 0, k(w, \pi(w)) + N_s\psi(w)) \\ \varphi(\pi_s(w), 0, k(w, \pi(w)) + M_s\psi(w) \end{bmatrix} \\ 0 &= h(w, \pi(w)) \end{aligned} \tag{40}$$

have a solution $\pi(w), \pi_s(w), \psi(w)$;

(ii) $F$, $G$ and $\gamma(\eta)$ are chosen as to fulfill

$$\begin{aligned} \frac{\partial\sigma}{\partial w} s(w) &= F\sigma(w) + G\psi(w) \\ \psi(w) &= \gamma(\sigma(w)), \end{aligned} \tag{41}$$

as it is always possible (see [38]);

(iii) the manifold $x = \pi(w), \eta = \sigma(w), \xi = \pi_s(w)$, that by construction is invariant in the closed-loop system, attracts all its trajectories.

This general design paradigm has then been used in [1] to solve a problem of output regulation for a system having the same number $m$ of control inputs and regulated outputs, a well-defined relative degree, a "high-frequency gain matrix" that satisfies a suitable "positivity condition", and a globally asymptotically stable zero dynamics.

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