Interval State Estimation for a Class of Nonlinear Systems

Tarek Raïssi, Denis Efimov, and Ali Zolghadri

***Abstract*—The goal of this technical note is to design interval observers for a class of nonlinear continuous-time systems. The first part of this work shows that it is usually possible to design an interval observer for linear systems by means of linear time-invariant changes of coordinates even if the system is not cooperative. This result is extended to a class of nonlinear systems using partial exact linearisations. The proposed observers guarantee to enclose the set of system states that is consistent with the model, the disturbances and the measurement noise. Moreover, it is only assumed that the measurement noise and the disturbances are bounded without any additional information such as stationarity, uncorrelation or type of distribution. The proposed observer is illustrated through numerical simulations.**

***Index Terms*—Cooperative systems, interval estimation, nonlinear observer, state transformation.**

I. Introduction

Consider a system described by

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})u \\ y = h(\mathbf{x}) + e \end{cases}, \tag{1}$$

where $\mathbf{x} \in \mathcal{D} \subseteq \mathbb{R}^n$ is the state vector and the initial state value belongs to a compact set $[\mathbf{x}_0] = [\underline{\mathbf{x}}_0, \overline{\mathbf{x}}_0]$; $y \in \mathcal{Y} \subseteq \mathbb{R}$, $u \in \mathcal{U} \subseteq \mathbb{R}$, and $e \in \mathbb{R}$ represent respectively the output, input and measurement noise. In the sequel, only Single Input Single Ouput (SISO) systems are considered. Finally, $\mathbf{f} : \mathcal{D} \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$ and $\mathbf{g} : \mathcal{D} \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$ are two smooth vector fields and $h : \mathcal{D} \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ is a smooth map.

To derive state estimators for (1), the process and measurement noises are often assumed to be Gaussian, and the standard filtering solutions require knowledge of their characteristics (the mean and covariance for a Gaussian distribution). Although these characteristics can be inferred from statistical and calibration procedures of the hardware sensing devices, the task is much more difficult for the process noise, because, in essence, it usually represents modelling errors. Moreover, several observer structures have been proposed for a class of nonlinear systems which can be linearised with a change of coordinates [3], [22], [23], [36]. In addition, some methods that do not require the considered nonlinear systems to be linearisable have been developed (see for instance [9]). They assume the existence of some Lyapunov functions satisfying particular conditions. The methodology presented in this technical note is based on an alternative strategy for estimator design called bounded error or set-membership estimation approach. In this case the measurement and process errors are assumed unknown but bounded, i.e., the domain of the output at t_i is $\left[y^-(t_i), y^+(t_i)\right] = [y(t_i) - \overline{e}, y(t_i) + \overline{e}]$ where $\overline{e}$ is the measurement error bound. This modelling context has been extensively studied, often for discrete-time systems, using several geometrical forms such as parallelotopes, ellipsoids, zonotopes or intervals (for more details, the reader can refer to the non-exhaustive list [1], [11], [17], [24], [30], [31], [35] and the references therein). This approach is basically different from the technique based on "classical observers" since the former one converges asymptotically to the real trajectory of the considered system while interval observers provide certain lower and upper bounds for the estimate at any time. Since applicability conditions of interval observers are typically weaker than for the conventional observers, such relaxation of the estimation objective becomes admissible in several applications.

One can distinguish two main set-membership approaches to perform robust state estimation for nonlinear continuous-time systems. The first one [17], [20], [21], [31] is based on the well-known prediction/correction mechanism as in the Kalman filter. Whereas the second approach addresses closed loop interval observers where the measurements are taken as continuous-time data. The observer gain is chosen such that the observation error is cooperative [7], [15], [28]. In this case, two suitable conventional observers are designed to compute lower and upper bounds for the domain of the state vector. Nevertheless, the issue of existence of such gains is not yet clearly established even for linear systems. The first contribution of this work shows that it is usually possible to design a cooperative observer for linear systems by means of a change of coordinates even if the system is not cooperative. Indeed, the design of the observer gain is formulated as a Sylvester equation that has usually a unique solution. This contribution can be considered as a generalisation of the technique proposed in [25], [26] where a *time-varying* change of coordinates has been constructed for linear exponentially stable time-invariant systems based on the Jordan canonical form. The main advantage of the proposed technique is that there exists a linear time-invariant change of coordinates providing a cooperative representation of linear systems even for the case of complex eigenvalues. The interval observer structure is then easy to implement that is a key feature from a practical point of view.

The second contribution of this technical note concerns the extension of the cooperative observer methodology to nonlinear systems described by (1). The procedure consists, firstly, in a nonlinear transformation $\boldsymbol{\xi} = \Phi(\mathbf{x})$, where $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ is a diffeomorphic function, of the system (1) into an equivalent one described by

$$\dot{\boldsymbol{\xi}} = \mathbf{A}\boldsymbol{\xi} + \psi(\boldsymbol{\xi}, \mathbf{u}) \tag{2}$$

where $\boldsymbol{\xi} \in \mathbb{R}^n$ is the new state vector and ψ is a function representing all nonlinear terms after the state transformation. Through a simple numerical example, it will be shown that the interval methods proposed in [7], [15], [27], [28] could not be directly applied to (2). Thereby, we propose a second linear change of coordinates $\mathbf{z} = \mathbf{P}\boldsymbol{\xi}$ in order to build a cooperative observer.

The technical note is organised as follows. Interval observers for linear systems are considered in Section II with a formulation based on a Sylvester equation. In Section III, it is shown that an interval observer could be constructed for a class of nonlinear systems through two changes of coordinates. The results of this technical note are illustrated through numerical examples in Section IV.

II. Cooperative Transformations for Linear Systems

Given a system described by

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} = \mathbf{C}\mathbf{x} \end{cases}. \tag{3}$$

where $\mathbf{x}, \mathbf{u}, \mathbf{y}$ are the state, the input and the output respectively and $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are some constant matrices. In the context of interval observers, the goal is to derive two trajectories $\underline{\mathbf{x}}(t)$ and $\overline{\mathbf{x}}(t)$ such that

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$\underline{\mathbf{x}}(t) \leq \mathbf{x}(t) \leq \overline{\mathbf{x}}(t)$ at any time starting from some initial conditions satisfying $\underline{\mathbf{x}}_0 \leq \mathbf{x}_0 \leq \overline{\mathbf{x}}_0$. The upper bound could be estimated by a Luenberger-based observer described by

$$\dot{\overline{\mathbf{x}}} = \mathbf{A}\overline{\mathbf{x}} + \mathbf{B}\mathbf{u} + \mathbf{L}(\mathbf{y} - \mathbf{C}\overline{\mathbf{x}}).$$

The upper observation error dynamics of $\tilde{\overline{\mathbf{x}}} = \overline{\mathbf{x}} - \mathbf{x}$ is given by

$$\dot{\tilde{\overline{\mathbf{x}}}} = (\mathbf{A} - \mathbf{LC})\tilde{\overline{\mathbf{x}}}.$$

The observation error $\tilde{\overline{\mathbf{x}}}$ is always positive and bounded if and only if the matrix $(\mathbf{A} - \mathbf{LC})$ is Hurwitz and Metzler (i.e., the off-diagonal terms of $(\mathbf{A} - \mathbf{LC})$ are nonnegative) [2], [34]. The same conditions are used for the positiveness of the lower observation error $\tilde{\underline{\mathbf{x}}} = \mathbf{x} - \underline{\mathbf{x}}$.

The main question to be answered is the following: is it always possible to compute a gain $\mathbf{L}$ such that $\mathbf{A} - \mathbf{LC}$ is Metzler and Hurwitz? The answer is negative for most linear systems. This drawback has been overcome in [25], [26] for systems described by (3) subject to additive disturbances. It has been shown that the stable systems described by (3) could be transformed through a change of coordinates into a cooperative form based on the Jordan representation. Nevertheless, it has been proved that no time-invariant changes of coordinates can transform the system (3) into a Jordan one when the matrix $\mathbf{A}$ has complex eigenvalues [25]. Therefore, because of the time-varying change of coordinates, the interval observers for such systems are also time-varying. In this section we propose an alternative formulation that makes possible a time-invariant change of coordinates even for the case of complex eigenvalues. The proposed technique is not based on the Jordan transformation.

Given a system described by (3), the idea is to find a constant nonsingular transformation matrix $\mathbf{P}$ such that, in the new coordinates $\mathbf{z} = \mathbf{P}\mathbf{x}$, the system

$$\begin{cases} \dot{\mathbf{z}} = \mathbf{PAP}^{-1}\mathbf{z} + \mathbf{PBu} \\ \mathbf{y} = \mathbf{CP}^{-1}\mathbf{z} \end{cases} \tag{4}$$

has an observer

$$\begin{aligned} \dot{\hat{\mathbf{z}}} &= \mathbf{PAP}^{-1}\hat{\mathbf{z}} + \mathbf{PBu} + \mathbf{PL}(\mathbf{y} - \mathbf{CP}^{-1}\hat{\mathbf{z}}) \\ &= \mathbf{R}\hat{\mathbf{z}} + \mathbf{PBu} + \mathbf{PLy} \end{aligned} \tag{5}$$

where $\mathbf{R} = \mathbf{PAP}^{-1} - \mathbf{PLCP}^{-1}$ is a Hurwitz and Metzler matrix. Thus, since $\mathbf{P}$ is a nonsingular matrix, we get

$$\mathbf{PA} - \mathbf{RP} = \mathbf{QC}, \quad \mathbf{Q} = \mathbf{PL}. \tag{6}$$

The (6) is as a Sylvester equation with respect to the unknown transformation matrix $\mathbf{P}$ (see for instance [2]). Thus, if the matrices $\mathbf{A}$ and $\mathbf{R}$ have no common eigenvalues, then this equation has always a unique solution for any $\mathbf{Q}$. Several methods are proposed in the literature to solve the Sylvester (6). The most popular ones, classified as direct methods, are based on a transfomation to the Schur form and solving the corrsponding linear system of equations by a backsubstitution process [4], [12], [14]. The lemma 1 gives a simple numerical procedure for the matrices $\mathbf{L}, \mathbf{P}$ computation.

Lemma 1: Let a matrix $\mathbf{A} - \mathbf{LC}$ *and a Metzler matrix* $\mathbf{R}$ *have the same eigenvalues for some* $\mathbf{L}$. *If there exist two vectors* $\mathbf{e}_1$ *and* $\mathbf{e}_2$ *such that the pairs* $(\mathbf{A} - \mathbf{LC}, \mathbf{e}_1)$ *and* $(\mathbf{R}, \mathbf{e}_2)$ *are observable, then*

$$\mathbf{P} = \mathbf{O}_2^{-1}\mathbf{O}_1 \quad \text{and} \quad \mathbf{Q} = \mathbf{PL}$$

satisfy (6) where

$$\mathbf{O}_1 = \begin{bmatrix} \mathbf{e}_1 \\ \vdots \\ \mathbf{e}_1(\mathbf{A} - \mathbf{LC})^{n-1} \end{bmatrix}; \ \mathbf{O}_2 = \begin{bmatrix} \mathbf{e}_2 \\ \vdots \\ \mathbf{e}_2\mathbf{R}^{n-1} \end{bmatrix}.$$

Proof: Since the pairs $(\mathbf{A} - \mathbf{LC}, \mathbf{e}_1)$ and $(\mathbf{R}, \mathbf{e}_2)$ are observable, the matrices $\mathbf{O}_1$ and $\mathbf{O}_2$ are nonsingular. Moreover, the similarity transformations $\mathbf{O}_1(\mathbf{A} - \mathbf{LC})\mathbf{O}_1^{-1}$ and $\mathbf{O}_2\mathbf{R}\mathbf{O}_2^{-1}$ transform the matrices $\mathbf{A} - \mathbf{LC}$ and $\mathbf{R}$ to their canonical observability forms. Then

$$\mathbf{O}_1(\mathbf{A} - \mathbf{LC})\mathbf{O}_1^{-1} = \mathbf{O}_2\mathbf{R}\mathbf{O}_2^{-1}$$

and straightforward computations yield the equality (6). Finally, note that if $(\mathbf{A}, \mathbf{C})$ is observable, then the vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ always exist. Note that the choice of $\mathbf{e}_1$ and $\mathbf{e}_2$ can be seen as a sensor placement ensuring the observability property of the pairs $(\mathbf{A} - \mathbf{LC}, \mathbf{e}_1)$ and $(\mathbf{R}, \mathbf{e}_2)$ [13]. ■

A simple procedure for choosing the matrix $\mathbf{R}$ could be proposed for the case of real eigenvalues (not necessarily distinct). For instance, assume that $(\mathbf{A}, \mathbf{C})$ is observable and compute a matrix $\mathbf{L}$ such that $\mathbf{A} - \mathbf{LC}$ has real eigenvalues. Then, take $\mathbf{R}$ as a lower triangular matrix with the eigenvalues of $\mathbf{A} - \mathbf{LC}$ on the main diagonal and positive elements below the main diagonal. Thus, the lemma conditions are satisfied.

III. Cooperative Observers for Nonlinear Systems

The methodology proposed in the sequel is based on the observer canonical form (OCF) which has been extensively investigated during the last three decades [3], [22], [23], [33], [36]. The idea is that a nonlinear state transformation, based on the Lie derivatives, yields a partial linear error dynamics in the new state coordinates.

A. Preliminaries

Recall that the Lie derivative of a scalar field h along a vector field $\mathbf{f}$ is defined by the scalar product $L_\mathbf{f}h(\mathbf{x}) = \langle \mathrm{d}h(\mathbf{x}), \mathbf{f}(\mathbf{x})\rangle$, where $\mathrm{d}h$ denotes the gradient of h. Iterated Lie derivatives are defined by $L_\mathbf{f}^k h(\mathbf{x}) = L_\mathbf{f}\left(L_\mathbf{f}^{k-1}h(\mathbf{x})\right)$ with $L_\mathbf{f}^0 h(\mathbf{x}) = h(\mathbf{x})$.

Definition 2: [16] The system described by (1) has a relative degree $r \leq n$ in $\mathcal{D}$ if:

1) $L_\mathbf{g}L_\mathbf{f}^k h(\mathbf{x}) = 0, k = 0, \ldots, r-2, \forall \mathbf{x} \in \mathcal{D}$;
2) $L_\mathbf{g}L_\mathbf{f}^{r-1}h(\mathbf{x}) \neq 0$.

Definition 3: Given the system (1), the matrix $\mathcal{Q}$ defined by

$$\mathcal{Q}(\mathbf{x}) = \begin{pmatrix} \mathrm{d}h(\mathbf{x}) \\ \mathrm{d}L_\mathbf{f}h(\mathbf{x}) \\ \vdots \\ \mathrm{d}L_\mathbf{f}^{n-1}h(\mathbf{x}) \end{pmatrix} \tag{7}$$

is called the observability matrix. The system (1) is locally observable at $\mathbf{x} \in \mathcal{D}$ if the matrix $\mathcal{Q}$ has a rank n at $\mathbf{x}$.

According to [16], if the system (1) is locally observable, the row vectors $\mathrm{d}h, \mathrm{d}L_\mathbf{f}h, \ldots, \mathrm{d}L_\mathbf{f}^{n-1}h$ are linearly independent and the vectors $h(\mathbf{x}), L_\mathbf{f}h(\mathbf{x}), \ldots L_\mathbf{f}^{n-1}h(\mathbf{x})$ are new coordinate functions on a neighborhood $\mathcal{U}$ of $\mathbf{x}$ defined by

$$\begin{cases} \phi_1(\mathbf{x}) = h(\mathbf{x}) \\ \phi_2(\mathbf{x}) = L_\mathbf{f}h(\mathbf{x}) \\ \quad\vdots \\ \phi_n(\mathbf{x}) = L_\mathbf{f}^{n-1}h(\mathbf{x}) \end{cases}. \tag{8}$$

The map $\Phi(\mathbf{x}) = (\xi_1, \xi_2, \ldots, \xi_n)^T = (\phi_1(\mathbf{x}), \phi_2(\mathbf{x}), \ldots, \phi_n(\mathbf{x}))^T$ is a local diffeomorphism and transforms the system (1) into the form

$$\begin{cases} \dot{\boldsymbol{\xi}} = \mathbf{A}\boldsymbol{\xi} + \mathbf{B}\left(\mathbf{a}(\boldsymbol{\xi}) + \mathbf{b}(\boldsymbol{\xi})u\right) \\ y = \mathbf{C}\boldsymbol{\xi} + e \end{cases} \tag{9}$$

with

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_r & \mathbf{0}_{(r-1)\times(n-r)} \\ \mathbf{0}_{(n-r+1)\times r} & \mathbf{0}_{(n-r+1)\times(n-r)} \end{pmatrix}$$

$$\text{where} \quad \mathbf{A}_r = \begin{pmatrix} 0 & 1 & \ldots & 0 \\ 0 & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & 1 \end{pmatrix} \in \mathbb{R}^{r-1\times r}$$

$$\text{and} \quad \mathbf{B} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \mathbf{B}_1 \end{pmatrix} \in \mathbb{R}^n$$

$$\mathbf{C} = (1 \quad 0 \quad \ldots \quad 0) \in \mathbb{R}^n$$

where $\mathbf{B}_1$ is a vector of $(n-r+1)$ elements. Furthermore, note that if the relative degree of the system (1) is n, then $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{B}_1$ are scalar. For more details, the reader can refer for instance to [16].

B. Cooperative Observer Design

A necessary condition to design an interval observer for (1), based on (9), is that *there exists a gain* $\mathbf{L}$ *such that* $(\mathbf{A} - \mathbf{LC})$ *is Hurwitz and Metzler*[15]. Nevertheless, the latter condition is usually unfeasible for observability-based forms as shown in the following example. Given a double integrator system described by (9) with:

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \ \mathbf{B} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \ \mathbf{C} = (1 \qquad 0) \tag{10}$$

and assume that the nonlinear residual functions $\mathbf{a}$ and $\mathbf{b}$ are zero. An interval observer could be designed for such a system if the matrix

$$\mathbf{A}_L = \mathbf{A} - \mathbf{LC} = \begin{pmatrix} -l_1 & 1 \\ -l_2 & 0 \end{pmatrix}$$

is stable and Metzler. Nevertheless, $\mathbf{A}_L$ is Metzler if $l_2 \leq 0$ and Hurwitz if $l_2 > 0$ which means that no interval observer could be built in the $(\xi_1, \ldots, \xi_n)$ coordinates. In such a case, the linear time-invariant change of coordinates proposed in Section II is used to overcome this drawback.

Theorem 4 gives sufficient conditions to build an interval observer for systems with a relative degree $r \leq n$.

Theorem 4: Consider a system described by (1) with a partial linear representation (9). Let $\mathbf{R}, \mathbf{L}, \mathbf{P}$ *be a solution of the Sylvester equation (6) for (9) and* $\mathbf{L}$ *is a gain chosen such that the matrix* $\mathbf{A} - \mathbf{LC}$ *is Hurwitz. Define a function* ψ *as:*

$$\psi(\mathbf{z}, y, u, e) = \mathbf{B}\left(\mathbf{a}(\mathbf{P}^{-1}\mathbf{z}) + \mathbf{b}(\mathbf{P}^{-1}\mathbf{z})u\right) + \mathbf{L}y - \mathbf{L}e. \tag{11}$$

Assume there exist two functions $\underline{\psi}$ *and* $\overline{\psi}$ *satisfying*[1]*:*

$$\begin{cases} \underline{\psi}(\underline{\mathbf{z}}, \overline{\mathbf{z}}, y, u, \overline{e}) \leq \psi(\mathbf{z}, y, u, e) \leq \overline{\psi}(\underline{\mathbf{z}}, \overline{\mathbf{z}}, y, u, \overline{e}) \\ \left(\overline{\mathbf{P}} + \underline{\mathbf{P}}\right)\left(\overline{\psi}(\underline{\mathbf{z}}, \overline{\mathbf{z}}, y, u, \overline{e}) - \underline{\psi}(\underline{\mathbf{z}}, \overline{\mathbf{z}}, y, u, \overline{e})\right) \leq \psi_0 \\ \underline{\mathbf{z}} \leq \mathbf{z} \leq \overline{\mathbf{z}} \ \text{and} \ \forall \mathbf{u} \in \mathcal{U}, y \in \mathcal{Y}, \end{cases} \tag{12}$$

[1]The inequality operators involving vectors should be understood componentwise

then

$$\begin{cases} \dot{\underline{\mathbf{z}}} = \mathbf{R}\underline{\mathbf{z}} + \overline{\mathbf{P}}\underline{\psi}(\underline{\mathbf{z}}, \overline{\mathbf{z}}, y, u, \overline{e}) - \underline{\mathbf{P}}\overline{\psi}(\underline{\mathbf{z}}, \overline{\mathbf{z}}, y, u, \overline{e}) \\ \dot{\overline{\mathbf{z}}} = \mathbf{R}\overline{\mathbf{z}} + \overline{\mathbf{P}}\overline{\psi}(\underline{\mathbf{z}}, \overline{\mathbf{z}}, y, u, \overline{e}) - \underline{\mathbf{P}}\underline{\psi}(\underline{\mathbf{z}}, \overline{\mathbf{z}}, y, u, \overline{e}) \\ \mathbf{z}_0 \in \left[\overline{\mathbf{P}}\underline{\boldsymbol{\xi}}_0 - \underline{\mathbf{P}}\overline{\boldsymbol{\xi}}_0, \overline{\mathbf{P}}\overline{\boldsymbol{\xi}}_0 - \underline{\mathbf{P}}\underline{\boldsymbol{\xi}}_0\right] \end{cases} \tag{13}$$

$$\begin{cases} \underline{\boldsymbol{\xi}} = \overline{\mathbf{M}}\underline{\mathbf{z}} - \underline{\mathbf{M}}\overline{\mathbf{z}} \\ \overline{\boldsymbol{\xi}} = \overline{\mathbf{M}}\overline{\mathbf{z}} - \underline{\mathbf{M}}\underline{\mathbf{z}} \\ \boldsymbol{\xi}_0 \in \left[\underline{\boldsymbol{\xi}}_0, \overline{\boldsymbol{\xi}}_0\right] \end{cases} \tag{14}$$

$$\begin{cases} \underline{\mathbf{x}} = \inf\left(\Phi^{-1}\left([\underline{\boldsymbol{\xi}}, \overline{\boldsymbol{\xi}}]\right)\right) \\ \overline{\mathbf{x}} = \sup\left(\Phi^{-1}\left([\underline{\boldsymbol{\xi}}, \overline{\boldsymbol{\xi}}]\right)\right) \\ \mathbf{x}_0 \in [\underline{\mathbf{x}}_0, \overline{\mathbf{x}}_0] \end{cases} \tag{15}$$

is an interval observer for (1) with $\overline{\mathbf{P}} = \max(0, \mathbf{P})$, $\underline{\mathbf{P}} = \overline{\mathbf{P}} - \mathbf{P}$, $\mathbf{M} = \mathbf{P}^{-1}$, $\overline{\mathbf{M}} = \max(0, \mathbf{M})$ *and* $\underline{\mathbf{M}} = \overline{\mathbf{M}} - \mathbf{M}$.

Proof: Since the system (1) is assumed to be observable, then (1) could be transformed into (9) via a diffeomorphism Φ given by (8) [16]. Furthermore, $y = \mathbf{C}\xi + e$ and the model (9) can be rewritten as

$$\dot{\boldsymbol{\xi}} = \mathbf{A}\boldsymbol{\xi} + \mathbf{B}\left(a(\boldsymbol{\xi}) + b(\boldsymbol{\xi})u\right) + \mathbf{L}\left(y - e - \mathbf{C}\boldsymbol{\xi}\right). \tag{16}$$

The model (16) depends on the unknown measurement noise e which is assumed to be bounded with a prior known bound $\overline{e}$. From (16) we have

$$\dot{\boldsymbol{\xi}} = \left(\mathbf{A} - \mathbf{LC}\right)\boldsymbol{\xi} + \mathbf{B}\left(a(\boldsymbol{\xi}) + b(\boldsymbol{\xi})u\right) + \mathbf{L}\left(y - e\right). \tag{17}$$

Moreover, as the pair $(\mathbf{A}, \mathbf{C})$ is observable, the eigenvalues of $\mathbf{A} - \mathbf{LC}$ could be arbitrarily assigned (with negative real parts). In particular, choosing real eigenvalues makes the resolution of the Sylvester (6) trivial.

Now, given a Metzler and Hurwitz matrix $\mathbf{R}$ (chosen by the user) and the solution $\mathbf{P}$ of the Sylvester (6), then the system (17) can be transformed into

$$\begin{aligned} \dot{\mathbf{z}} &= \mathbf{R}\mathbf{z} + \mathbf{P}\left[\mathbf{B}\left(\mathbf{a}(\mathbf{P}^{-1}\mathbf{z}) + \mathbf{b}(\mathbf{P}^{-1}\mathbf{z})u\right) + \mathbf{L}\left(y - e\right)\right] \\ &= \mathbf{R}\mathbf{z} + \mathbf{P}\psi(\mathbf{z}, y, u, e) \end{aligned} \tag{18}$$

with $\mathbf{z} = \mathbf{P}\boldsymbol{\xi}$.

Consider the observation error $\tilde{\overline{\mathbf{z}}} = \overline{\mathbf{z}} - \mathbf{z}$, then from (13) and (18), we have

$$\dot{\tilde{\overline{\mathbf{z}}}} = \mathbf{R}\tilde{\overline{\mathbf{z}}} + \overline{\mathbf{P}}\overline{\psi}(\underline{\mathbf{z}}, \overline{\mathbf{z}}, y, u, \overline{e}) - \underline{\mathbf{P}}\underline{\psi}(\underline{\mathbf{z}}, \overline{\mathbf{z}}, y, u, \overline{e}) - \mathbf{P}\psi(\mathbf{z}, u, y, e).$$

Moreover, $\overline{\mathbf{P}}\overline{\psi}(\underline{\mathbf{z}}, \overline{\mathbf{z}}, y, u, \overline{e}) - \underline{\mathbf{P}}\underline{\psi}(\underline{\mathbf{z}}, \overline{\mathbf{z}}, y, u, \overline{e}) - \mathbf{P}\psi(\mathbf{z}, u, y, e) \geq 0$ and $\mathbf{R}$ is Metzler then, it follows from theorem 1 in [15] that the error $\tilde{\overline{\mathbf{z}}}$ is always positive. In the same way, we can prove that the observation error $\tilde{\underline{\mathbf{z}}} = \mathbf{z} - \underline{\mathbf{z}}$ is also positive, thus

$$\underline{\mathbf{z}}(t) \leq \mathbf{z}(t) \leq \overline{\mathbf{z}}(t), \quad \forall t \geq 0. \tag{19}$$

In addition, the dynamics of the total observation error $\tilde{\mathbf{Z}} = \overline{\mathbf{z}}(t) - \underline{\mathbf{z}}(t)$ is described by

$$\dot{\tilde{\mathbf{Z}}} = \mathbf{R}\tilde{\mathbf{Z}} + \left(\overline{\mathbf{P}} + \underline{\mathbf{P}}\right)\left(\overline{\psi}(\underline{\mathbf{z}}, \overline{\mathbf{z}}, y, u, \overline{e}) - \underline{\psi}(\underline{\mathbf{z}}, \overline{\mathbf{z}}, y, u, \overline{e})\right). \tag{20}$$

Hence, if $\left(\overline{\mathbf{P}} + \underline{\mathbf{P}}\right)\left(\overline{\psi}(\underline{\mathbf{z}}, \overline{\mathbf{z}}, y, u, \overline{e}) - \underline{\psi}(\underline{\mathbf{z}}, \overline{\mathbf{z}}, y, u, \overline{e})\right)$ is upper bounded componentwise by a positive vector ψ_0, it follows from theorem 2 in [15] that $\tilde{\mathbf{Z}}(t)$ is bounded, that in conjunction with (19) implies boundedness of $\underline{\mathbf{z}}$ and $\overline{\mathbf{z}}$. Moreover, let $\mathbf{M} = \mathbf{P}^{-1}$, $\overline{\mathbf{M}} = \max(0, \mathbf{M})$ and $\underline{\mathbf{M}} = \overline{\mathbf{M}} - \mathbf{M}$, then

$$\overline{\mathbf{M}}\underline{\mathbf{z}} \leq \overline{\mathbf{M}}\mathbf{z} \leq \overline{\mathbf{M}}\overline{\mathbf{z}}, \qquad \underline{\mathbf{M}}\underline{\mathbf{z}} \leq \underline{\mathbf{M}}\mathbf{z} \leq \underline{\mathbf{M}}\overline{\mathbf{z}}.$$

Therefore, it follows that:

$$\overline{\mathbf{M}}\underline{\mathbf{z}} - \underline{\mathbf{M}}\overline{\mathbf{z}} \le \boldsymbol{\xi} \le \overline{\mathbf{M}}\overline{\mathbf{z}} - \underline{\mathbf{M}}\underline{\mathbf{z}}.$$

Consequently, (14) is an interval estimation for $\boldsymbol{\xi}$.

Finally, the state domain in the $(x_1, \ldots, x_n)$ coordinates defined by (15) is a solution of

$$\mathbf{x} = \Phi^{-1}(\boldsymbol{\xi}), \quad \boldsymbol{\xi} \in [\underline{\boldsymbol{\xi}}, \overline{\boldsymbol{\xi}}]. \tag{21}$$

The solution of (21) is straigthforward in the case of monotone Φ. Otherwise, constraint propagation techniques based on interval analysis [18], [29] can be used to solve the problem (21) in a reliable way by generating a conservative set $\overline{S}$ of boxes that are guaranteed to contain the exact solution set of the state values. As only outer approximations are computed in this technical note, the domain $[\underline{\mathbf{x}}, \overline{\mathbf{x}}]$ can be taken as the hull of $\overline{S}$ (i.e., the smallest interval containing $\overline{S}$). For more details about interval tools, the reader could refer to [18], [29]. ∎

Remark 5: If the transition matrix $\mathbf{P}$ and its inverse $\mathbf{P}^{-1}$ are positive elementwise, then the observer (13)–(15) could be simplified and $\underline{\mathbf{P}}$ and $\overline{\mathbf{P}}$ would not be used. Nevertheless, computing a positive transition matrix with a positive inverse remains a hard problem [6]. Furthermore, using a time-invariant change of coordinates is computationally more interesting than a time-varying one since in the former case, the transition matrices $\mathbf{P}$, $\underline{\mathbf{P}}$, $\overline{\mathbf{P}}$ and their inverse are computed only once offline contrary to the methodology proposed in [25], [26].

Remark 6: The full rank of the observability matrix (7) is necessary to ensure the existence of the observability canonical form (9). Nevertheless, such a condition does not always hold globally or even on a sufficiently large set to avoid a singular observer gain. Recently, several works have addressed this drawback by using a partial nonlinear observability canonical form which requires weaker conditions (see for instance [19], [32]). These works are based on the following formulation:

$$\begin{cases} \dot{\boldsymbol{\xi}}_o = \mathbf{A}_o\boldsymbol{\xi}_o + \mathbf{B}_o(a(\boldsymbol{\xi}_o) + b(\boldsymbol{\xi}_o)u) \\ \dot{\boldsymbol{\xi}}_{\bar{o}} = q(\boldsymbol{\xi}_o, \boldsymbol{\xi}_{\bar{o}}) \\ y = \mathbf{C}\boldsymbol{\xi}_o \end{cases}, \tag{22}$$

where $\mathbf{A}_o \in \mathbb{R}^{r\times r}$ is similar to the matrix $\mathbf{A}$ defined in (9), $\mathbf{C} = (1\ 0 \ldots 0) \in \mathbb{R}^{1\times r}$ and $\mathbf{B}_o = (0 \ldots 0\ 1)^T \in \mathbb{R}^r$. Using this subdivision, an interval observer similar to that of theorem 4 could be built if the non-observable part $\boldsymbol{\xi}_{\bar{o}}$ is stable.

IV. Examples

A. Partial Linear System

Consider an uncertain nonlinear system described by

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}(p_1, p_2)\mathbf{f}(\mathbf{x})u(t), \ y = \mathbf{C}\mathbf{x}$$

where $\mathbf{x} \in \mathbb{R}^3$ is the state, $\mathbf{f}(\mathbf{x}) = \mathbf{x}_1\mathbf{x}_2$ is a nonlinear function, $u \in \mathbb{R}_+$ is a positive control input, $\underline{p}_i \le p_i \le \bar{p}_i, i = 1, 2$ are some uncertain parameters

$$\mathbf{A} = \begin{bmatrix} 2 & 0 & 0 \\ 1 & -4 & \sqrt{3} \\ -1 & -\sqrt{3} & -4 \end{bmatrix}, \ B(p_1, p_2) = \begin{bmatrix} -2p_1 \\ 0 \\ p_2 \end{bmatrix},$$
$$C = [1 \quad 0 \quad 0]$$

with $\underline{p}_1 = 4.48$, $\bar{p}_1 = 6.12$, $\underline{p}_2 = 3.2$, $\bar{p}_2 = 3.6$, $u(t) = 1 + \sin(2t)$. This system has bounded solutions. The pair $(\mathbf{A}, \mathbf{C})$ is not observable and there exists no gain $\mathbf{L}$ such that the matrix $\mathbf{A} - \mathbf{LC}$ is Metzler, which makes the techniques of [15], [27] inappropriate. For the observer gain $L = [3 \quad 0 \quad 0]^T$ the matrix $\mathbf{A} - \mathbf{LC}$ is Hurwitz and has the eigenvalues $-1, -4 \pm \sqrt{3}i$. Only the first real eigenvalue can be assigned by the gain $\mathbf{L}$. The matrix $\mathbf{A} - \mathbf{LC}$ is not Metzler for any $\mathbf{L}$. Thus, to find a transformation of coordinates, it is necessary to construct a Metzler and Hurwitz matrix $\mathbf{R}$ with the eigenvalues of $\mathbf{A} - \mathbf{LC}$. The matrix

$$\mathbf{R} = \begin{bmatrix} -a & b & 0 \\ 0 & -a & b \\ b & 0 & -a \end{bmatrix}$$

has eigenvalues $b - a$, $-a - 0.5b \pm 0.5b\sqrt{3}i$, therefore in our example we have to choose $b = 2$ and $a = 3$. The pairs $(\mathbf{A} - \mathbf{LC}, e_1)$ and $(\mathbf{R}, e_2)$ are observable for

$$e_1 = [1 \quad 0 \quad 1], \ e_2 = [1 \quad 1 \quad 0]$$

then

$$\mathbf{P} = \mathbf{O}_2^{-1}\mathbf{O}_1 = \begin{bmatrix} 0.158 & 0.866 & 0.5 \\ 0.842 & -0.866 & 0.5 \\ 0.658 & 0 & -1 \end{bmatrix}.$$

B. Application to a Biological System

Consider a bioreactor system described by the following equations [5], [8], [10]:

$$\begin{cases} \dot{X} = X \cdot \mu_m(X, S) - X \cdot D \\ \dot{S} = -\frac{1}{Y} \cdot X \cdot \mu_m(X, S) + (S_f - S) \cdot D \end{cases}, \tag{23}$$

where X represents the cell mass concentration that can be measured, S the substrate concentration that should be estimated, S_f the substrate concentration in the feed stream, Y the yield of cell mass, and $D > 0$ is the dilution rate and the plant's control input. In the following, we assume that S_f and Y are some known constants. The growth rate function is described by the Contois model [10]

$$\mu_m(X, S) = \frac{\mu_{m0}S}{k_cX + S} \tag{24}$$

where $\mu_{m0} > 0$ and $k_c > 0$ denote, respectively, the maximum growth and the saturation constant. In the following, we assume that [10]:

1) the dilution rate is strictly positive, i.e., $D(t) \ge D_{\min} > 0$;
2) the feed rate S_f is bounded;
3) each reaction involves at least one reactant which is neither a catalyst nor an autocatalyst. According to [5], the state variable X and the control D are bounded for all time.

The cell mass concentration X is measured by a biosensor with a sampling period $T_s = 1$ and assume that the measurement error is bounded with a bound $e = 0.1$. The initial state $(X_0, S_0)^T$ is not exactly known and belongs to the box $([9, 11], [0, 1.3])^T$.

It is straightforward to show that the relative degree of (23) is 1 and that the system is observable. Therefore, a change of coordinates could be constructed with

$$\Phi = \begin{pmatrix} X & \frac{\mu_{m0}XS}{k_cX+S} \end{pmatrix}^T.$$

The observer gain is chosen as $\mathbf{L} = (1.2567, -0.2561)^T$; it permits to ensure the stability of the observer in the (ξ_1, ξ_2). The cooperativity of the observation error is obtained in the (z_1, z_2) basis by using the transition matrix

$$P = \begin{pmatrix} -0.4583 & 0.3997 \\ -0.2039 & -0.9131 \end{pmatrix}.$$

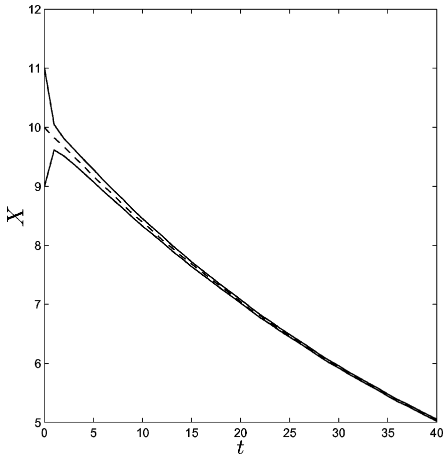

Fig. 1. Bounds of the state X.

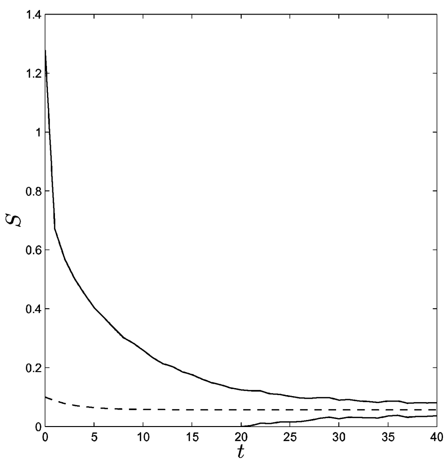

Fig. 2. Bounds of the state S.

The inverse of the diffeomorphism Φ is given by

$$\Phi^{-1} = \begin{pmatrix} \xi_1 & \frac{k_c \xi_1 \xi_2}{\mu_{m0}\xi_1 - \xi_2} \end{pmatrix}^T.$$

Note that Φ^{-1} is an increasing map with respect to ξ_1 and ξ_2. Therefore, given an interval estimate $\left(\left[\underline{\xi}_1, \overline{\xi}_1\right], \left[\underline{\xi}_2, \overline{\xi}_2\right]\right)^T$, the estimation in the basis (X, S) is given by

$$([\underline{X}, \overline{X}], [\underline{S}, \overline{S}])^T = \begin{pmatrix} [\underline{\xi}_1, \overline{\xi}_1] \\ \left[\frac{k_c \underline{\xi}_1 \underline{\xi}_2}{\mu_{m0}\underline{\xi}_1 - \underline{\xi}_2}, \frac{k_c \overline{\xi}_1 \overline{\xi}_2}{\mu_{m0}\overline{\xi}_1 - \overline{\xi}_2}\right] \end{pmatrix}.$$

The bounds of the state vector (X, S) are displayed in Figs. 1 and 2. The results show that the measurements are always inside the estimated bounds and the interval estimate converges to a box with a width depending on the measurements error bound.

V. Conclusion

Interval state estimation for a class of nonlinear systems has been considered in this technical note. The goal is to compute two reliable bounds that are consistent with the error bounds for the actual value of the state vector at each time. It has been proven that a stable interval observer could be constructed for systems for which a partial linearisation is available. The procedure is based on a first nonlinear change of coordinates leading to a partial linear model. A second time-invariant change of coordinates based on the resolution of a Sylvester equation permits to obtain an interval observer in a third basis. The advantage of this approach is its robustness with respect to the measurements and disturbances errors. Moreover, the observer convergence is ensured. In a future work, nonlinear systems with parameter uncertainties will be considered.

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On the Dubins Traveling Salesman Problem

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***Abstract*—We study the traveling salesman problem for a Dubins vehicle. We prove that this problem is NP-hard, and provide lower bounds on the approximation ratio achievable by some recently proposed heuristics. We also describe new algorithms for this problem based on heading discretization, and evaluate their performance numerically.**

***Index Terms*—Algorithms, motion planning, traveling salesman problem (TSP), unmanned aerial vehicles (UAVs).**

I. Introduction

In an instance of the Traveling Salesman Problem (TSP), we are given the distances between any pair of n points. The problem is to find the shortest closed path (tour) visiting every point exactly once. We also call this problem the tour-TSP to distinguish it from the path-TSP, where the requirement that the vehicle must start and end at the same point is removed. This famously intractable problem is often encountered in robotics, and has traditionally been solved in two steps within the common layered controller architectures for mobile robots. At the higher decision-making level, the dynamics of the robot are usually not taken into account and the mission planner might typically chose to solve the TSP for the Euclidean metric (ETSP), i.e., using the Euclidean distances between waypoints. For this purpose, one can directly exploit many existing results on the ETSP (or more general TSPs on graphs), see e.g., [1], [2]. This first step determines the order in which the waypoints should be visited by the robot. At the lower level, a path planner takes as an input this waypoint ordering, and designs feasible trajectories between the waypoints respecting the dynamics of the robot. In this technical note, we assume that the robot has a limited turning radius and can be modeled as a Dubins vehicle [3], [4]. Consequently, the path planner could solve a sequence of Dubins Shortest Path Problems (DSPP) between the successive waypoints. DSPPs have also been extensively studied since the work of Dubins [3], most of the literature concentrating on designing shortest paths between an initial and final configuration for a Dubins vehicle moving among obstacles, see e.g., [4]–[9]. In fact, one should also consider shortest Dubins paths through a sequence of ordered waypoints of length greater than two, since the vehicle configuration at an intermediate waypoint influences the length of the shortest Dubins path between the next two waypoints. This problem was studied by [10], [11] for an environment without obstacles.

Even if each problem is solved optimally however, the ad-hoc separation into two successive steps can be inefficient, since the sequence of points chosen by the TSP algorithm is often hard to follow for the physical system. In order to improve the performance of unmanned aerial systems in particular, researchers are now working on integrating the mission planning and path planning stages [12], [14]. In this note we consider the TSP for the Dubins vehicle (DTSP), in a *planar environment without obstacles*, a problem introduced by Savla *et al.* in [17]. The Dubins model provides a good kinematic model for fixed wing aircraft. At the same time, we can quickly compute the length of the shortest path between any two configurations of the Dubins vehicle, a necessary building block to design algorithms with good performance for the DTSP.

A stochastic version of the DTSP for which the points are distributed randomly and uniformly in the plane was considered in [13]–[16]. Here however, we focus on algorithms and worst-case bounds for the more standard problem where no probability distribution is given for the input. In that case, most of the recently proposed algorithms seem to build on a preliminary solution obtained for the ETSP [11], [17]–[19]. More detailed references on the DTSP can be found in [14].

1) Contributions of This Work: In this note, we first prove that the DTSP is NP-hard, thus justifying the work on heuristics and algorithms that approximate the optimal solution. Recall that an α-approximation algorithm (with *approximation ratio* $\alpha \geq 1$) for a minimization problem is an algorithm that produces *on any instance of the problem* with optimum OPT, a feasible solution whose value Z is within a factor α of the optimum, i.e., such that $OPT \leq Z \leq \alpha\, OPT$. In general, α is allowed to depend on the input parameters of the problem, such as the number of points in the DTSP. This definition of approximation ratio, used throughout the technical note, corresponds to the worst-case performance of the algorithm [20]. On the negative side, we give some lower bounds on the approximation ratio achievable by recently proposed heuristics. Following a tour based on the ETSP or-

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