Stability Analysis via Averaging Functions

A. Yu Pogromsky and A. S. Matveev

***Abstract*—A new class of Lyapunov functions is proposed for analysis of incremental stability for nonlinear systems. This class of Lyapunov functions allows to establish input-dependent incremental stability criteria. Two substantially different situations are considered: when incremental stability is guaranteed by the inputs of sufficiently small amplitude and when, similar to the excited van der Pol oscillator, the stability is induced by sufficiently large inputs.**

***Index Terms*—Lyapunov methods, nonlinear systems, stability criteria.**

I. Introduction

The technical note proposes and advocates a novel technique of Lyapunov analysis that is based on the concept of averaging functions and Steklov's averaging method. This technique is applicable for analysis of incremental stability [1]–[3] of nonlinear systems and aims at relaxing the quadratic criteria [4], [5]. Its potential is demonstrated via obtaining new stability criteria that rest on non-quadratic Lyapunov functions and unlike many previous results, offer better account for the role of the external excitation by providing input-dependent conditions. In particular, the proposed method works even if the system fails to satisfy the incremental version of the circle criterion and proves stability whenever the uniform root mean square value of the input signal is less than a computable threshold. The method equally copes with the opposite situation where the incremental stability needs inputs with sufficiently large amplitudes. The general theoretical results are illustrated by a number of examples.

II. Averaging Functions and Steklov's Averaging Technique

The objective of this section is to highlight main ideas underlying the proposed techniques of the stability analysis. To this end, we consider a vector LTV differential equation

$$\dot{x} = A(t)x, \quad x(t) \in \mathbb{R}^n. \tag{1}$$

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Stability criteria based on quadratic Lyapunov functions often give overly conservative sufficient stability bounds for the system (1). In this technical note, we advocate more general Lyapunov functions candidates of the following form:

$$V(x,t) = x^\top P x \exp\left[w(t)\right] \tag{2}$$

where P is a positive definite matrix and $w(t) \in \mathbb{R}$ is a smooth function. Formula (2) really gives a Lyapunov function if the time derivative of V along the solutions of (1) is uniformly negative definite, i.e., there exists $\varepsilon > 0$ such that

$$A(t)^\top P + PA(t) + \dot{w}P \le -\varepsilon I \qquad \forall\, t \tag{3}$$

where I is the identity matrix. A smooth and bounded function $w(\cdot)$ that satisfies (3) (with some $\varepsilon > 0$) is said to be *averaging*.

The notion of the averaging function was set forth in [6], [7]. In this technical note, we complement it by a novel systematic technique of finding such functions. The idea is to utilize input-output properties of the system at hands with properly chosen output, along with associated dissipation inequalities. In doing so, we recommend the use of the averaging technique due to Steklov (see, e.g., [8]) and display its benefits in reducing the conservatism of the resultant stability bounds.

This technique uses replacement of a concerned function $r(\cdot)$ by its *Steklov average* with a certain step $T > 0$

$$r^T(t) = T^{-1} \int_t^{t+T} r(\tau) d\tau.$$

In doing so, various other relations between the original and averaged functions appear to be useful. Now we present some of them. They will be used to transform input-output dissipation inequalities into a tractable form via averaging out oscillating inputs and related terms.

Let $r \in [1, \infty)$ and $\Delta \subset \mathbb{R}$ be an interval, with $\Delta := \mathbb{R}$ and $\Delta = [\tau, \infty)$ being of especial interest for us. The symbol $L_r^{\rm loc}[\Delta \to \mathbb{R}]$ stands for the space of all measurable functions $v(\cdot) : \Delta \to \mathbb{R}$ that are L_r integrable on any bounded sub-interval of Δ. Let Δ_T stand for the set of all $t \in \Delta$ such that $t + T \in \Delta$, and let $v(\cdot) \in L_r^{\rm loc}[\Delta \to \mathbb{R}]$ and $t_0 \in \Delta_T$ be given.

In looking for an averaging function, a rather efficient hint is often to build it, either partly or completely, from the following (non-normalized) bias caused by Steklov averaging of $|v(\cdot)|^r$:

$$\sigma_r(t,T) := \int_{t_0}^{t} \left(\frac{1}{T} \int_{\tau}^{\tau+T} |v(s)|^r \, ds - |v(\tau)|^r \right) d\tau \tag{4}$$

where $t \in \Delta_T$. The following two properties will be of especial interest in this technical note:

$$\dot{\sigma}_r(t,T) = T^{-1} \int_t^{t+T} |v(s)|^r \, ds - |v(t)|^r$$

$$\sup_{t \in \Delta_T} |\sigma_r(t,T)| \le T \sup_{t \in \Delta} |v(t)|^r. \tag{5}$$

They are straightforward from (4) and the following lemma (where $p(\cdot) := |v(\cdot)|^r$), which is reminiscent of some results in [9, p. 541] and [8, Lemma 5.4.1 and eq. (5.4.4)] and is presented here for the convenience of the reader.

Lemma 2.1: Let $\Delta \subset \mathbb{R}$ be an interval, and let $T > 0$ and $t_0 \in \Delta_T$ be given. For any $t \in \Delta_T$ and any measurable function $p(\cdot) : \Delta \to \mathbb{R}$ such that $|p(\tau)| \leq M$ for almost all $\tau \in \Delta$, the following inequality holds:

$$|\sigma(t, T)| \leq MT \tag{6}$$

where

$$\sigma(t, T) := \int_{t_0}^{t} \left(\frac{1}{T} \int_{\tau}^{\tau+T} p(s)ds - p(\tau) \right) d\tau.$$

In the sequel, we demonstrate how the concept of averaging function combined with the Steklov's averaging technique may aid in stability analysis of nonlinear systems. A particular attention will be drawn to a systematic procedure of finding an averaging function based on dissipation inequalities.

III. Global Stability of Solutions of Nonlinear Systems: A General Theory

This section is devoted to the respective general theory. Its main result extends that from [6] by omitting a series of restrictive assumptions.

Specifically, we consider nonlinear systems of the form

$$\dot{x} = f(x, u), \qquad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m. \tag{7}$$

Here, $u = u(t)$ is a function of $t \in \mathbb{R}$ taken from a given class $\mathcal{U}$ of piece-wise continuous and bounded functions; elements of $\mathcal{U}$ are called *admissible inputs*. The continuous function $f(\cdot, \cdot)$ is locally Lipschitz continuous in x, i.e., for any $(x_*, u_*) \in \mathbb{R}^n \times \mathbb{R}^m$, there exist two constants $L > 0, \delta > 0$ such that

$$\|f(x_2, u) - f(x_1, u)\| \leq L\|x_2 - x_1\| \quad \text{whenever} \quad \|x_i - x_*\| \leq \delta, \quad \|u - u_*\| \leq \delta.$$

This necessarily holds if $f(\cdot, \cdot)$ is continuous and continuously differentiable with respect to x.

Assumption 3.1: For any admissible input $u(\cdot) \in \mathcal{U}$, the solutions of (7) are globally uniformly ultimately bounded in the sense of [10, Definition 4.6]: for any $u(\cdot) \in \mathcal{U}$ there exist positive constants b_0, b_∞, independent of t_0, and there is $T = T(b_0, b_\infty) \geq 0$, independent of t_0, such that

$$\|x(t_0)\| \leq b_0 \Longrightarrow \|x(t)\| \leq b_\infty, \qquad \forall\, t \geq t_0 + T(b_0, b_\infty).$$

The following definition extends that due to Demidovich [5], [11].

Definition 3.2: The system (7) is said to be *uniformly convergent* for the input $\mathbf{u} = u(\cdot) \in \mathcal{U}$ if the system has a solution $\bar{x}_{\mathbf{u}}(t)$ with the following properties:

i) $\bar{x}_{\mathbf{u}}(t)$ is defined and bounded on the entire time axis $\mathbb{R}$;
ii) $\bar{x}_{\mathbf{u}}(t)$ is uniformly globally asymptotically stable.

Our interest is focused on the situation, rather ubiquitous in applications, where the system is not uniformly convergent for all admissible inputs $u(\cdot)$ though it possesses this property within some non-empty proper subset $\mathcal{U}_{\text{conv}} \subset \mathcal{U}$, like in the motivating examples from [6], [7]. The objective is to obtain an explicit estimate of this subset. A criterion for uniform convergence providing such an estimate in the outlined situation is said to be *input-dependent*.

The following main result of the section generalizes a similar theorem in [6]. For relevant motivating and explanatory remarks, we refer the reader to [6] and [7, Section IV].

Theorem 3.3: Suppose that Assumption 3.1 is fulfilled, $u(\cdot) \in \mathcal{U}$, and there exist a symmetric positive definite $n \times n$-matrix P, a continuous and continuously differentiable function $w : \mathbb{R}^n \times \mathbb{R} \mapsto \mathbb{R}$, and a positive number $\varepsilon > 0$ such that the following statements hold:

i) For any t and almost all[1] points $x \in \mathbb{R}^n$ where the function $f(\cdot, u(t))$ is differentiable, i.e., $\exists f'_x(x, u(t)) =: J(x, t)$, the following matrix inequality is true:

$$PJ(x, t) + J^\top(x, t)P + \dot{w}(x, t)P \leq -\varepsilon P. \tag{8}$$

ii) The function $w(x, t)$ remains bounded as $t \to \pm\infty$ and x ranges over a bounded set

$$\sup_{t \in \mathbb{R}, \|x\| \leq a} |w(x, t)| < \infty \qquad \forall\, a \in [0, \infty).$$

Then the system (7) is uniformly convergent.

The proof of this theorem follows along the lines of the proof of the main result in [6] and is therefore omitted.

According to Rademacher's theorem, the Lipschitz-continuous function $f[\cdot, u(t)]$ is differentiable almost everywhere. So given t, (8) should be checked for almost all $x \in \mathbb{R}^n$.

IV. Stability Analysis of Saturated Systems Stable in the Linear Mode

A. General Theory

We consider the following linear system with saturation:

$$\dot{x} = Ax + B\text{sat}(Fx + u), \qquad x \in \mathbb{R}^n, \quad u \in \mathbb{R} \tag{9}$$

with constant matrices A, B, F of appropriate dimensions. In (9), $\text{sat}(\cdot)$ is the standard saturation function $\text{sat}z = \text{sign}z \min\{1, |z|\}$ and $u = u(t)$ is the excitation input. Our interest is focused on the case where this system has converse stability properties in the linear and saturation modes, which strive against each other in course of time. Specifically, among the two Jacobian matrices $A + BF$ and A of the right-hand side from (9) in the linear and saturation modes, respectively, someone is Hurwitz, whereas the other is permitted to be unstable. The cases where $A + BF$ or A is Hurwitz, respectively, will be considered separately.

We begin with the situation where $A + BF$ is Hurwitz, while A is not necessarily stable.

Assumption 4.1: There exist a positive definite matrix $P = P^\top > 0$ and numbers $\lambda_1 > 0$, $\lambda_2 \geq 0$, such that the following matrix inequalities hold:

$$(A + BF)^\top P + P(A + BF) \leq -\lambda_1 P$$
$$A^\top P + PA \leq \lambda_2 P. \tag{10}$$

The first inequality from (10) implies that the matrix $A + BF$ is Hurwitz indeed. As for the second one, the subsequent results can be extended on the case where $\lambda_2 < 0$. However, it is of little interest for us since this case is "input-independent": the system (9) is quadratically convergent by the standard argument and existence of the desired $\bar{x}_{\mathbf{u}}(\cdot)$ from Definition 3.2 needs no restrictions on the input $\mathbf{u} = u(\cdot)$.

[1]With respect to the Lebesgue measure.

To model, if necessary, spectral properties of this input, we assume that it is generated by a linear stable pre-filter

$$\dot{\zeta} = E\zeta + Gv, \qquad \zeta \in \mathbb{R}^m, \quad v \in \mathbb{R}$$
$$u = H\zeta \tag{11}$$

where the matrix E is Hurwitz and $v = v(t)$ is bounded. The mean power of the filter input $v(\cdot)$ is upper limited by a known constant $\beta > 0$. More precisely, we impose the following.

Assumption 4.2: The input $u(t)$ to the system (9) is the unique bounded on $\mathbb{R}$ output of the filter (11) with a bounded input $v(\cdot)$ satisfying the following averaged constraint:

$$\limsup_{T\to\infty} \sup_{t_0} \frac{1}{T} \int_{t_0}^{t_0+T} v(s)^2 ds \leq \beta^2. \tag{12}$$

We also assume that not only the mean power of the signal $v(\cdot)$ but also its influence on the system (9) can be estimated. Specifically, we impose the following.

Assumption 4.3: The L_2-gain from v to $s := \mathrm{sat}(Fx+u)$ does not exceed a constant γ: there exists a positive definite, radially unbounded storage function $W(x,\zeta)$ that satisfies the following dissipation inequality:

$$\dot{W} \leq -\mathrm{sat}^2(Fx+u) + \gamma^2 v^2. \tag{13}$$

This restricts the class of admissible A's to critically unstable matrices. In a preliminary form, the following result was partly reported in [7] without proof.

Theorem 4.4: The system (9) is uniformly convergent whenever Assumptions 4.1–4.3 are satisfied and

$$\beta^2\gamma^2 < \lambda_1/(\lambda_1+\lambda_2). \tag{14}$$

Proof: Based on (14) and (12), we consecutively pick $\varepsilon_0 > 0$ and then $T > 0$ such that

$$(\beta^2+\varepsilon_0)\gamma^2 < \lambda_1/(\lambda_1+\lambda_2) \tag{15}$$
$$\sup_{t_0} \frac{1}{T} \int_{t_0}^{t_0+T} v^2(s)ds \leq \beta^2 + \varepsilon_0. \tag{16}$$

We also introduce the following averaging function candidate:

$$w(x,\zeta,t) = \mu\left[W(x,\zeta) + \gamma^2\sigma_2(t,T)\right] \tag{17}$$

where $\mu := \lambda_1 + \lambda_2 > 0$ and σ_2 is defined by (4). Due to Lemma 2.1, $w(\cdot)$ satisfies ii) in Theorem 3.3. Meanwhile, i) holds if for some $\varepsilon > 0$, the matrix P from Assumption 4.1 obeys the following matrix inequalities:

$$A^\top P + PA + \dot{w}P \leq -\varepsilon P$$
$$(A+BF)^\top P + P(A+BF) + \dot{w}P \leq -\varepsilon P. \tag{18}$$

To verify them, we observe that by Assumption 4.3 and (16)

$$\dot{w} \leq \mu\left[-\mathrm{sat}^2(Fx+u) + \gamma^2 v^2 + \gamma^2\dot{\sigma}_2(t,T)\right]$$
$$\leq \mu\left[-\mathrm{sat}^2(Fx+u) + \gamma^2\beta^2 + \gamma^2\varepsilon_0\right]$$

or, in other words

$$\begin{array}{ll} \dot{w} \leq \mu\gamma^2(\beta^2+\varepsilon_0) & \text{in linear mode} \\ \dot{w} \leq \mu\left(-1+\gamma^2(\beta^2+\varepsilon_0)\right) & \text{in saturation mode.} \end{array} \tag{19}$$

Hence, by invoking (10), we see that (18) is valid whenever

$$\mu\gamma^2(\beta^2+\varepsilon_0) - \lambda_1 \leq -\varepsilon, \quad \mu\left(-1+\gamma^2(\beta^2+\varepsilon_0)\right) + \lambda_2 \leq -\varepsilon$$

which is really true with $\varepsilon := \lambda_1 - \gamma^2(\lambda_1+\lambda_2)(\beta^2+\varepsilon_0) > 0$, where the last inequality follows from (15). Thus, i) in Theorem 3.3 does hold.

It remains to justify Assumption 3.1, i.e., the uniform ultimate boundedness of $x(t)$. To this end, we introduce the following scalar function:

$$V_{\mathrm{b}}(x,\zeta,t) = x^\top Px \exp\left[w(x,\zeta,t)\right]$$

where P and w are as before. Since P is positive definite and in (17), $W(\cdot)$ is radially unbounded, whereas $\sigma_2(t,T)$ is conversely bounded by Lemma 2.1, the function $V_{\mathrm{b}}(\cdot)$ goes to ∞ uniformly over $t \in (-\infty,\infty)$ as $\|x\|, \|\zeta\| \to \infty$. For the time derivative of $V_b(\cdot)$, we have

$$\dot{V}_{\mathrm{b}} = \left[(x^\top(A^\top P + PA + \dot{w}P)x + 2x^\top PB\mathrm{sat}(Fx+u)\right]e^w$$
$$\overset{(18)}{\leq} \begin{cases} \left(-\varepsilon x^\top Px + 2|x^\top PB|\right)e^w & \text{in saturation mode} \\ \left(-\varepsilon x^\top Px + 2\left|x^\top PBu(t)\right|\right)e^w & \text{in linear mode.} \end{cases}$$

Since $u(t)$ is bounded, the quadratic term $\varepsilon x^\top Px$ dominates over both $2|x^\top PB|$ and $2|x^\top PBu(t)|$ for large enough $\|x\|$, thus making the time derivative of V_{b} strictly negative. This implies global uniform boundedness of the solutions thanks to [10, Theorem 4.18]. ■

B. Example: Double Integrator With Saturated Linear Feedback

As an illustration, we study a benchmark example from [12]

$$\dot{x} = y, \qquad \dot{y} = \mathrm{sat}(z), \quad z = -x - y + u \tag{20}$$

where the input $u = u(t)$ is defined on the entire real line $\mathbb{R}$. The following result can be derived from Theorem 4.4 similar to the arguments from [7].

Proposition 4.5: The system (20) is uniformly convergent for any bounded and differentiable input $u(\cdot)$ whose derivative is bounded and obeys the following averaged bound:

$$\limsup_{T\to\infty} \sup_{t_0} \frac{1}{T} \int_{t_0}^{t_0+T} \dot{u}^2(s)\,ds < 2\sqrt{3} - 3. \tag{21}$$

In the particular case of the harmonic input $u(t) = b\sin\omega t$, (21) shapes into the following sufficient condition for the system (20) to be uniformly convergent:

$$|b|\omega < \sqrt{4\sqrt{3}-6} \approx 0.9634. \tag{22}$$

To assess the level of conservatism in this criterium, we performed computer simulations along the lines similar to those from [6, Section 1]. The results are depicted in Fig. 1 and provide an evidence that the lower component of the true convergence domain is estimated by (22) in the region $\omega \in [0.5, 0.75]$ with no more than ≈30% error.

One of the rare input-dependent criteria for a system to be convergent is elaborated in [13] on the basis of the Zames-Falb integral quadratic constraints. However, the system (20) fails to satisfy this criterium, whereas the proposed novel approach allowed us to derive a non-void and informative result for this system. We believe that the stability analysis performed in this section could be useful for study of the integrator windup phenomenon and anti-windup correction [14], [15].

Although (20) looks like a simple toy example, input-output behaviors of this system and the likes are far from trivial. Despite the fact that the system is globally asymptotically stable in open loop ($u \equiv 0$),

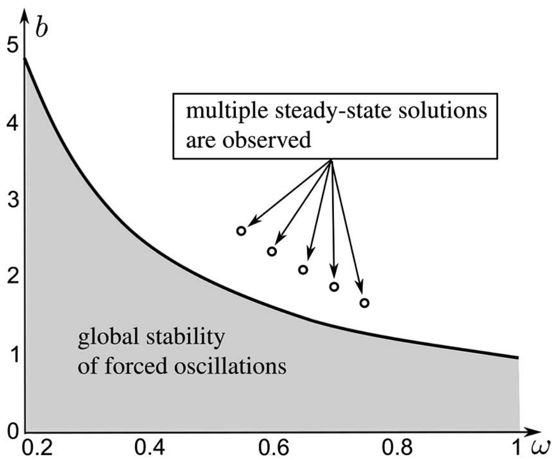

Fig. 1. Estimation of the domain where the system is uniformly convergent.

its L_2-gain from u to x is infinite [16]. So to examine the performance of this and similar systems in terms of the associated L_2-gains, i.e., within the input-output operator framework, it would be useful not only to know whether the gain from u to $\mathrm{sat}(x+u)$ is finite but also to estimate this gain. However even the first question is posed as an open non-trivial problem in [12], which still remains unsolved to the best of our knowledge. From a computer simulation similar to [6], [7], one can find multiple steady-state solutions that correspond to the same input. This fact indicates that the incremental L_2-gain from Δu to $\Delta\mathrm{sat}$ is infinite. That, in turn, exposes severe limitations of the performance analysis of similar systems within input-output operator framework.

V. Stability Analysis of Saturated Systems Stable in the Saturation Mode

A. General Theory

Now we are going to demonstrate that the proposed approach is equally capable of coping with just the opposite situation when stability is enforced by the inputs of large amplitude. Specifically, we will consider the case when A is Hurwitz, whereas $A+BF$ is allowed to be hyperbolically unstable, which means stability in the saturation mode and possible instability in the linear mode. In this case, the system expectedly may fail to have a globally stable solution for an arbitrary input $u(\cdot)$. At the same time, we shall show that under slight technical assumptions, the system is uniformly convergent whenever the input is "large" enough "on average."

Our interest to the outlined situation stems from the phenomenon studied first by van der Pol and van der Mark [17]. Though the example by van der Pol and van der Mark is not concerned with saturated systems, it is underlain, more or less, by features that are shared by the systems to be studied: their Jacobians are stable far from the origin and may be unstable in its vicinity. So it does not come as a surprise that a similar phenomenon can be revealed in these saturated systems.

Now we revert to the general system (9), for simplicity not assuming any longer that its bounded input $u(\cdot)$ is necessarily generated by the pre-filter (11). As in Section IV-A, convergence and divergence in the saturation and linear modes, respectively, as well as their rates, are captured by the following counterpart of Assumption 4.1.

Assumption 5.1: There is a positive definite matrix $P = P^\top > 0$ and numbers $\lambda_1 > 0, \lambda_2 \geq 0$ such that the following inequalities are true:

$$A^\top P + PA \leq -\lambda_1 P$$
$$(A+BF)^\top P + P(A+BF) \leq \lambda_2 P. \tag{23}$$

The case $\lambda_2 < 0$ is excluded only for conciseness of presentation and due to the reasons stated after Assumption 4.1.

The following analog of Assumption 4.3 characterizes power supply and dissipation in both linear and saturation modes.

Assumption 5.2: The system (9) is output passive from the input $s := -\mathrm{sat}(Fx+u)$ to output $y := Fx$. In other words, there is a number $\gamma > 0$ and a quadratic function $W(x)$ such that the following dissipation inequality holds along the solutions of (9):

$$\dot W \leq -\gamma^2 y^2 + y\,\mathrm{sat}\,(y+u). \tag{24}$$

In order not to lose focus of the proposed novel approach, we confine ourselves to study of the case where $\gamma < 1$: if $\gamma > 1$, the system is quadratically convergent for any input by the standard argument, which case is of little interest for us.

Theorem 5.3: Suppose that Assumptions 5.1 and 5.2 hold and there is a number $\chi > 0$ such that

$$\varrho \left\{ \limsup_{T\to\infty} \sup_{t_0} \frac{1}{T} \int_{t_0}^{t_0+T} \left[\chi^2 - \eta^2(t)\right]^+ dt + 1 \right\} - \chi^2 < 0 \tag{25}$$

where $v^+ := \max\{0, v\}\ \forall v \in \mathbb{R}$ and

$$\eta(t) := 2\gamma^2\,|u(t)| - |1-2\gamma^2|, \quad \varrho := 1 + \lambda_2/\lambda_1. \tag{26}$$

Then the system (9) is uniformly convergent.

The key assumption (25) can be always satisfied by "sufficiently large" inputs. Thus Theorem 5.3 does show that the system is uniformly convergent whenever the input is "large enough," as was promised at the beginning of the section.

Proof of Theorem 5.3: Assumption 3.1 of Theorem 3.3 holds since A is Hurwitz due to Assumption 5.1 and the function $\mathrm{sat}(\cdot)$ is bounded. To verify i) and ii) in Theorem 3.3, we are going to examine the following averaging function candidate:

$$w = 4\gamma^2 \mu W + \sigma_1(t,T). \tag{27}$$

Here $\mu > 0, T > 0$ are "free" parameters to be chosen later on, and $\sigma_1(t,T)$ is defined by (4), where

$$v(t) := \left[\lambda_1 + \lambda_2 - \mu\eta^2(t)\right]^+ \geq 0. \tag{28}$$

Then ii) in Theorem 3.3 is true by Lemma 2.1. To justify i), we separately shape the dissipation inequality (24) in a more convenient form for the saturation and linear modes, respectively. To this end, we start with completing the full squares in the right-hand side of (24). This results in

$$\dot W \leq -\left[\gamma y - (\gamma^{-1}/2)\mathrm{sat}(y+u)\right]^2 + (\gamma^{-2}/4)\mathrm{sat}^2(y+u)$$

and assures that in the saturation mode

$$\dot W \leq \gamma^{-2}/4. \tag{29}$$

In the linear mode, $\mathrm{sat}(y+u) = y+u \in [-1,1]$. So by (24), (26), and using $\gamma \in (0,1)$,

$$\dot W \leq (1-\gamma^2)y^2 + yu \leq \frac{1}{4\gamma^2}\left[1 - \eta^2(t)\right]. \tag{30}$$

The other preliminary step concerns the inequality (25), which is rewritten as follows:

$$\limsup_{T\to\infty} \sup_{t_0} \frac{1}{T} \int_{t_0}^{t_0+T} \left[\chi^2 - \eta^2(t)\right]^+ dt < \frac{\lambda_1}{\lambda_1+\lambda_2}\chi^2 - 1.$$

By picking $\mu := (\lambda_1 + \lambda_2)/\chi^2 > 0$ and invoking (28), the last inequality transforms into

$$\limsup_{T\to\infty} \sup_{t_0} \frac{1}{T} \int_{t_0}^{t_0+T} v(t)\,dt < \lambda_1 - \mu.$$

This permits us to pick $\varepsilon > 0$ and $T > 0$ such that

$$\frac{1}{T} \int_{t_0}^{t_0+T} v(t)\,dt < \lambda_1 - \mu - \varepsilon \qquad \forall\, t_0. \tag{31}$$

Now we take P from Assumption 5.1 and start to check i) in Theorem 3.3. In the saturation mode, consecutively employing (4), (23), (27), (29), nonegativity of $v(t)$, and (31) yields that

$$A^\top P + PA + \dot{w}P \le -\varepsilon P.$$

Meanwhile in the linear mode, consecutively employing (4), (23), (30), (28), and (31) results in

$$(A+BF)^\top P + P(A+BF) + \dot{w}P \le -\varepsilon P.$$

Thus, i) in Theorem 3.3 does hold, which theorem completes the proof. ■

Corollary 5.4—Frequency synchronization: Suppose that Assumptions 5.1 and 5.2 are true.

i) Let the input $\mathbf{u} = u(\cdot)$ be τ-periodic and (25) hold with some $\chi > 0$. Then the steady-state process $\bar{x}_{\mathbf{u}}(\cdot)$, which exists by Theorem 5.3 and Definition 3.2, is also τ-periodic. In other words, it oscillates at the frequency of the excitation signal, i.e., entrainment (frequency synchronization) occurs.

ii) Now suppose that the input is harmonic $u(t) = b\sin\omega t$. Then the system (9) is uniformly convergent and frequency synchronization holds if

$$|b| > \frac{\sqrt{\varrho} + |1-2\gamma^2|}{\sqrt{2}\gamma^2\sqrt{1-\alpha^{-1}\sin\alpha}}, \qquad \text{where} \quad \alpha := \frac{\pi}{\varrho} \le \pi. \tag{32}$$

Proof:

i) We note that due to Theorem 5.3 and Definition 3.2, the solution $\bar{x}_{\mathbf{u}}(\cdot)$ is defined on the entire time axis, is bounded, and attracts all other solutions. Since the right-hand side of (9) is invariant with respect to the τ-shift in time due to periodicity of $u(\cdot)$, it is easy to see that the τ-shift of $\bar{x}_{\mathbf{u}}(\cdot)$ is also a solution of (9), which retains all listed properties of the original solution $\bar{x}_{\mathbf{u}}(\cdot)$. However such solution is unique. So the process $\bar{x}_{\mathbf{u}}(\cdot)$ is identical to its τ-shift, i.e., is τ-periodic.

ii) It suffices to show that the assumption (25) of Theorem 5.3 is true provided that the harmonic input satisfies (32). We start by noting that the change $t' = \omega t$ of the time variable reduces the proof to the case $\omega = 1$. Then by (26), the function $\eta(\cdot)$ is π-periodic and so (25) takes the form

$$\mathcal{H} := \varrho\left\{1 + \frac{1}{\pi}\int_0^\pi \left[\chi^2 - \eta^2(t)\right]^+ dt\right\} - \chi^2 < 0. \tag{33}$$

As $\chi > 0$ monotonically decays from $+\infty$ to 0, the Lebesgue measure $\mathbf{mes}$ of the set $\mathfrak{T}_\chi := \{0 \le t \le \pi : \chi^2 - \eta^2(t) \ge 0\}$ continuously reduces from π to 0. Hence, there exists $\chi > 0$ such that $\mathbf{mes}(\mathfrak{T}_\chi) = \pi/\varrho \le \pi$. For this χ, the scaled left hand side of (33) shapes into

$$\varrho^{-1}\mathcal{H} = 1 + \frac{\chi^2}{\pi}\int_{\mathfrak{T}_\chi} dt - \frac{1}{\pi}\int_{\mathfrak{T}_\chi}\eta^2(t)\,dt - \varrho^{-1}\chi^2 = 1 - \frac{1}{\pi}\int_{\mathfrak{T}_\chi}\eta^2(t)dt.$$

Now we employ the following elementary quadratic inequality:

$$-\xi^2 \le -\tau[\xi+\zeta]^2 + \zeta^2\tau/(1-\tau), \qquad \tau \in (0,1) \tag{34}$$

with

$$\xi = \eta(t), \quad \zeta = |1-2\gamma^2|, \qquad \xi + \zeta \overset{(26)}{=\!=} 2\gamma^2|u(t)|.$$

This results in the following estimates:

$$\varrho^{-1}\mathcal{H} \le 1 - \frac{\tau}{\pi}\int_{\mathfrak{T}_\chi} 4\gamma^4 b^2 \sin^2 t\,dt + \frac{1}{\pi}\frac{\tau}{1-\tau}\int_{\mathfrak{T}_\chi}(1-2\gamma^2)^2\,dt$$

$$\le 1 - \tau\frac{2\gamma^4 b^2}{\varrho} + \frac{\tau}{1-\tau}\frac{(1-2\gamma^2)^2}{\varrho} + \tau\frac{2\gamma^4 b^2}{\pi}\sin\left(\frac{\pi}{\varrho}\right).$$

Minimizing the right-hand side over $\tau \in (0,1)$ shows that (32) $\Rightarrow$ (33). The proof is completed by Theorem 5.3 and i). ■

B. Entrainment in Saturated Linear Systems: Example

Now we illustrate Theorem 5.3 via its application to presumably the simplest example of a saturated system that is stable in the saturation mode and unstable in the linear mode

$$\dot{x} = -x + 2\text{sat}\left[x + u(t)\right], \qquad u(t) = b\sin(\omega t) \tag{35}$$

$x = x(t) \in \mathbb{R}, \omega > 0$. In this case, $P = 1$, $\lambda_1 = \lambda_2 = 2$, $\varrho = 2$, $\sin\alpha = 1$, $2\gamma^2 = 1$. Therefore according to Corollary 5.4, there is a unique bounded on $(-\infty, +\infty)$ globally asymptotically stable solution if

$$|b| > 2\sqrt{\pi/(\pi-2)} \approx 3.318. \tag{36}$$

As was remarked, the system (35) can also be treated by the approach set forth in [18]–[20]. Now we briefly outline its main ideas and outcome as applied to (35), omitting many technical details for the sake of brevity and because of the illustrative purpose of this vignette.

We first note that for $u \equiv 0$, the system (35) evidently has an unstable equilibrium $x = 0$ and two asymptotically stable equilibria $x = \pm 2$. By means of the quadratic Lyapunov-like function $V = x^2$, it is easy to see that all solutions converge to the invariant set $[-2, 2]$ regardless of the input $u(t)$. So analysis of the asymptotic behavior can be focused on solutions from this set. For them, $|u(t)| > 3$ implies that the system is in the saturation mode, where the Jacobian equals -1 and so the solutions converge to each other at the unit exponential rate. If conversely $|u(t)| \le 3$, they may diverge at no worse than the unit exponential rate. This gives rise to a guess that convergence dominates over divergence and results in an uniform convergency of the overall system if $|u(t)| > 3$ for more than 50% of time. For the harmonic input $u(t) = b\sin\omega t$, the last requirement evidently means that

$$|b| > 3\sqrt{2} \approx 4.243. \tag{37}$$

In (36), the lower threshold is smaller than in (37). This indicates that the approach based on averaging functions can give less conservative stability criteria than the method developed in [18]–[20].

C. Example: Global Stability of Forced Oscillations in a Piece-Wise Linear Analog of the van der Pol Equation

Now we are going to demonstrate that the approach based on averaging functions is effective in global stability analysis of equations that are similar to van der Pol equation in both structure and generated dynamics after a suitable time-dependent coordinate change. Specifically, we pick an analog from the class of linear saturated systems

$$\ddot{x} + \dot{x} - 2\text{sat}\left[\dot{x} + u(t)\right] + x = 0. \tag{38}$$

This equation can be written in the form (9)

$$\frac{d}{dt}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}0&1\\-1&-1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}+\begin{bmatrix}0\\2\end{bmatrix}\mathrm{sat}(x_2+u).$$

To find an estimate for the stability region, let us employ Corollary 5.4. It is easy to see that Assumption 5.1 holds with

$$P=\begin{bmatrix}1&\frac{1}{2}\\\frac{1}{2}&1\end{bmatrix}>0,\qquad \lambda_1=1,\quad \lambda_2=1+\frac{4}{\sqrt{3}}$$

and Assumption 5.2 is satisfied with

$$W(x_1,x_2)=\frac{(x_1^2+x_2^2)}{4},\quad \gamma^2=\frac{1}{2}.$$

Now, $\eta(t)=|u(t)|$ and $\alpha=\pi/\varrho=\pi[2\sqrt{3}-3]/2$. So Corollary 5.4 yields that the system (38) is uniformly convergent and has a globally attracting solution oscillating with the period of the excitation if $b^2>2\pi/(\alpha-\sin\alpha)$. Thus analysis of global stability of forced oscillations for a piece-wise linear analog of the van der Pol equation becomes a relatively easy exercise with the help of Corollary 5.4.

VI. Conclusion

By a technique based on the so-called averaging functions, novel constructive stability criteria are established that allow to prove stability of forced oscillations in the case when the classical (incremental) circle criterion fails. Benefits from averaging functions are also demonstrated for linear systems with saturation in feedback and an external input by obtaining input dependent stability criteria. Both situations, where stability occurs for relatively small and large inputs, respectively, are discussed.

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