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Incremental passivity and output regulation for switched nonlinear systems

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ABSTRACT

This paper studies incremental passivity and global output regulation for switched nonlinear systems, whose subsystems are not required to be incrementally passive. A concept of incremental passivity for switched systems is put forward. First, a switched system is rendered incrementally passive by the design of a state-dependent switching law. Second, the feedback incremental passification is achieved by the design of a state-dependent switching law and a set of state feedback controllers. Finally, we show that once the incremental passivity for switched nonlinear systems is assured, the output regulation problem is solved by the design of global nonlinear regulator controllers comprising two components: the steady-state control and the linear output feedback stabilising controllers, even though the problem for none of subsystems is solvable. Two examples are presented to illustrate the effectiveness of the proposed approach.

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Incremental passivity; multiple storage functions; switched nonlinear systems; output regulation; feedback incremental passification

1. Introduction

Switched systems have gained a great amount of attention due to the theoretical developments as well as the widespread applications (Kang, Zhai, Liu, & Zhao, 2015; Lu, Wu, & Kim, 2006; Niu, Zhao, Fan, & Cheng, 2015; Yang, Cocquempot, & Jiang, 2008). A dynamical system which consists of a finite number of subsystems and a switching signal that governs the switching among them is called a switched system. It is a special hybrid system (Kang, Zhai, Liu, Zhao, & Zhao, 2014). The output regulation for switched systems is one of the most important problems in control theory. It is much more difficult and interesting than that for non-switched systems due to the interactions of continuous dynamics and discrete dynamics. Several methods which have been used to study stability were developed to deal with the output regulation problem, such as the common Lyapunov function technique (Niu & Zhao, 2013), the multiple Lyapunov function method (Dong & Zhao, 2012a), the average dwell time approach (Dong & Zhao, 2012a, 2013; Long & Zhao, 2014) and so on.

On the other hand, passivity theory can date back to the beginning of the 1970s (Willems, 1972). Passivity means that the energy dissipated inside a dynamic system do not exceed the energy supplied from outside. A storage function of a passive system is usually selected as a natural candidate for a Lyapunov function. Therefore, passivity theory was used to solve nonlinear output regulation problem (Jayawardhana & Weiss, 2005, 2008; Travieso-Torres, Duarte-Mermoud, & Sepu′ lveda, 2007). As an extension of the conventional passivity property, incremental passivity was originally proposed from an operator point of view in Desoer and Vidyasagar (1975), and Zames (1966). A incremental passivity definition in state space form was given and some preliminary properties of incrementally passive systems were investigated (Bürger & Persis, 2015; Pavlov & Marconi, 2008). It can describe a more extensive class of physical systems which have an equilibrium pointor not. Incremental passivity offers an approach for constructing incremental Lyapunov functions for incremental stability analysis (Hamadeh, Stan, Sepulchre, & Gonçalves, 2012; Stan & Sepulchre, 2007) and convergent system (Pavlov, van de Wouw, & Nijmeijer, 2005). The trajectories of an incrementally passive nonlinear system can be driven to converge to one another by the design of an incrementally passive feedback controller. As such, it is useful for solving the output regulation problem (Bürger & Persis, 2015; Pavlov & Marconi, 2008). A key issue in the output regulation problem is to design a stabiliser which guarantees that all solutions of the closed-loop system converge to a zero-error steady-state trajectory. The stabiliser can be designed using incremental passivity theory.

Passivity property is still expected to be useful for switched systems. The passivity concepts of switched nonlinear systems were proposed and the corresponding feedback passification, passivity-based stabilisation

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problems were studied (Liu, Stojanovski, Stankovski, Dimirovski, & Zhao, 2011; Pang & Zhao, 2015; Zefran, Bullo, & Stein, 2001; Zhao & Hill, 2008a, 2008b). Incremental passivity theory and the incremental passivity-based output tracking for switched nonlinear systems were set up by using weak-storage functions and multiple supply rates (Dong & Zhao 2012b). But the adjacent storage functions are required to be connected at the switching time, which is a strong requirement. However, there have been no results on incremental passivity-based output regulation problem for switched nonlinear systems so far.

In this paper, we will study incremental passivity and global output regulation problem for switched nonlinear systems. The contributions are in three aspects. First, a generalisation of the state-dependent switching law designed by Zhao and Hill (2008c) is presented to render the switched nonlinear system incrementally passive. This gives more design freedom of switching law. Second, the incremental feedback passification which has not been investigated is achieved by the design of a state-dependent switching law and state feedback controllers without incremental minimum-phase condition. Finally, we solve the output regulation problem by the design of a state-dependent switching law and state feedback controllers for switched nonlinear systems, even though the problem for none of the subsystems is solvable. Compared with convention regulators (Dong & Zhao, 2012a, 2013; Long & Zhao, 2014), the regulators designed using incremental passivity property comprise of two components: the steady-state control and the linear output feedback stabilising controllers. In some particular cases, this paper does not need to verify that all the solutions of the switched nonlinear system converge to the bounded steady-state solution, while we only have to verify the regulated output converge to zero directly.

2. Problem formulation and preliminaries

Consider a switched nonlinear system of the form,

$$\dot{x} = F_\sigma\left(x, u_\sigma, \omega\right), \tag{1a}$$

$$e = h_\sigma\left(x, \omega\right) \tag{1b}$$

with state $x \in R^n$, inputs $u_i \in R^m$, the regulated output $e \in R^m$ and the switching signal $\sigma(t)$: $[0, \infty) \to I = \{1, 2, \ldots, M\}$, which is assumed to be a piecewise constant function and has a finite number of switchings on any finite time interval (Liberzon, 2003). The exogenous signal $\omega(t)$ including a disturbance in Equation (1a) and a reference signal in Equation (1b) are generated by the exosystem

$$\dot{\omega} = s\left(\omega\right), \quad \omega\left(t_0\right) \in W, \tag{2}$$

where $W \subset R^s$ is a given positively invariant set of initial conditions. It is assumed that any solution starting from $\omega(t_0) \in W$ is bounded for all $t \geq t_0$. F_i, h_i and s are C^1 functions.

Corresponding to the switching signal, the switching sequence is defined as follows:

$$\Sigma = \left\{x_0; \left(i_0, t_0\right), \left(i_1, t_1\right), \ldots \left(i_k, t_k\right), \ldots | i_k \in I, k \in N\right\}, \tag{3}$$

where t_0 is the initial time, x_0 is the initial state and N is the set of nonnegative integers. When $t \in [t_k, t_{k+1})$, $\sigma(t) = i_k$, that is, the i_kth subsystem is active. For any $j \in I$, let

$$\Sigma_j = \left\{ t_{j_1},\ t_{j_2},\ \ldots,\ t_{j_n} \ldots;\ i_{j_q} = j,\ q \in N\right\} \tag{4}$$

be the sequence of switching times when the jth subsystem is switched on, and thus

$$\left\{t_{j_1+1},\ t_{j_2+1}, \ldots t_{j_n+1} \ldots;\ i_{j_q} = j,\ q \in N\right\} \tag{5}$$

is the sequence of switching times when the jth subsystem is switched off.

The global output regulation problem for system (1) is formulated as follows:

For a given switching signal $\sigma(t)$, design a set of feedback controllers of the form $u_i = \alpha_i(x, e, \omega) = \eta_i(x, \omega) + \phi_i(e, \omega)$, where η_i and ϕ_i are smooth mappings, such that for all $\omega(t_0) \in W$ and $x_0 \in R^n$, the solutions of the system

$$\begin{aligned} \dot{x} &= F_{\sigma(t)}\left(x, \omega, \alpha_\sigma\left(x, e, \omega\right)\right), \\ \dot{\omega} &= s\left(\omega\right) \end{aligned} \tag{6}$$

are bounded for $t \geq t_0$ and $\lim_{t\to\infty} e(t) = 0$.

Remark 2.1: The term $\eta_i(x, \omega)$ plays a role in rendering system (1) incrementally passive. For each $i \in I$, $\phi_i(e, \omega)$ consists of steady control and output feedback controller.

To solve the output regulation problem, we need the following assumption:

Assumption 2.1: *For any solution of the exosystem starting from $\omega(t_0) \in W$ and a given switching signal $\sigma(t)$, there exist $x^*_\omega(t)$ and $\bar{u}_{i\omega}(t)$ that are bounded on R_+ and satisfy*

$$\begin{aligned} \dot{x}^*_\omega\left(t\right) &= F_\sigma\left(x^*_\omega\left(t\right), \bar{u}_{\sigma\omega}\left(t\right), \omega\left(t\right)\right), \forall t \geq t_0 \\ 0 &= h_\sigma\left(x^*_\omega\left(t\right), \omega\left(t\right)\right). \end{aligned} \tag{7}$$

Remark 2.2: Assumption 2.1 is only a necessary condition to solve the problem of output regulation for system (1) and has been adopted for non-switched systems (Pavlov & Marconi, 2008). Equation (7) is a switched regulator equation and should be satisfied for a given switching signal $\sigma(t)$, not for any subsystem, i.e.$\sigma = i, \forall i$. In fact, Equation (7) may be satisfied even the regulator equation of each subsystem of system (1) is not solvable. Assumption 2.1 is a less restrictive counterpart of the common assumption on the solvability of the regulator equations (Dong & Zhao, 2012a). The conventional regulator equations are formulated as

$$\frac{\partial \pi(\omega)}{\partial \omega} s(\omega) = F_\sigma(\pi(\omega), c_\sigma(\omega), \omega),$$
$$0 = h_\sigma(\pi(\omega), \omega). \quad (8)$$

If there exist differentiable maps $\pi(\omega)$ and $c_i(\omega)$ defined on a set Wsatisfying the regulator Equation (8), then Assumption 2.1 holds with $x^*_\omega(t) = \pi(\omega(t))$and $\bar{u}_{i\omega}(t) = c_i(\omega(t))$. On the other hand, if $x^*_\omega(t)$ and $\bar{u}_{i\omega}(t)$, $i \in I$ is a common solution of the regulator equations of all subsystems, namely, Equation (8) is satisfied, when $\sigma = i, i \in I$ (Dong & Zhao, 2012a, 2013; Long & Zhao, 2014) then the solvability of the regulator Equation (7) for the given switching signal $\sigma(t)$ is automatically achieved.

We first introduce the definition of $\mathcal{GK}$ function that will be used in the sequel.

Definition 2.1 (Zhao & Hill, 2008c)**:** A function $\alpha: R_+ \to R_+$is called a class $\mathcal{GK}$function if it is increasing and right continuous at the origin with $\alpha(0) = 0$.

Now, we give the incremental passivity definition for switched nonlinear systems.

Definition 2.2: System (1) is said to be incrementally passive under a given switching signal $\sigma(t)$, if there exists a nonnegative function $S(\sigma(t), t, x, \hat{x})$: $I \times R^+ \times R^{2n} \to R^+$, called a storage function, and class $\mathcal{GK}$ function α such that for any bounded signal $\omega(t)$, any two inputs u_σ and $\hat{u}_\sigma$, and any two solutions of system (1) $x(t)$ and $\hat{x}(t)$ corresponding to these inputs, the respective outputs $e = h_\sigma(x, \omega)$ and $\hat{e} = h_\sigma(\hat{x}, \omega)$ satisfy the inequality

$$S(\sigma(t), t, x(t), \hat{x}(t)) - S(\sigma(t_0), t_0, x(t_0), \hat{x}(t_0))$$
$$\leq \int_{t_0}^{t} (e(\tau) - \hat{e}(\tau))^T (u_{\sigma(\tau)}(\tau) - \hat{u}_{\sigma(\tau)}(\tau))\, d\tau$$
$$+ \alpha(\|x_0 - \hat{x}_0\|), \quad (9)$$

where x_0 and $\hat{x}_0$ are the initial states. If, in addition, there exist positive definite continuous functions $Q_i(\cdot)$ such that

$$S(\sigma(t), t, x(t), \hat{x}(t)) - S(\sigma(t_0), t_0, x(t_0), \hat{x}(t_0))$$
$$\leq \int_{t_0}^{t} (e(\tau) - \hat{e}(\tau))^T (u_{\sigma(\tau)}(\tau) - \hat{u}_{\sigma(\tau)}(\tau))\, \mathrm{d}\tau$$
$$- \int_{t_0}^{t} Q_{\sigma(\tau)}(x(\tau) - \hat{x}(\tau))\, \mathrm{d}\tau + \alpha(\|x_0 - \hat{x}_0\|), \quad (10)$$

then, system (1) is said to be strictly incrementally passive.

Remark 2.3: In Definition2.2, the storage function is not required to be connected and may increase at the switching time. Thus, Definition2.2 is more general than the passivity definition (Dong & Zhao, 2012b). The item $\alpha(\|x_0 - \hat{x}_0\|)$ is used to measure the total change of 'energy' at the switching times. When system (1) has only one subsystem and $\alpha \equiv 0$, Definition2.2 degenerates to incremental passivity definition (Pavlov & Marconi, 2008).

Definition 2.3 (Pavlov & Marconi, 2008)**:** A storage function $S(t, x, \hat{x})$ is called regular if for any sequence $(t_k, x_k(t_k), \hat{x}_k(t_k))$, $k = 1, 2, \ldots,$ such that $\hat{x}_k$ is bounded, t_k tends to infinity and $\|x_k\| \to +\infty$, it holds that $S(t_k, x_k, \hat{x}_k) \to +\infty$, as $k \to +\infty$.

Next, we extend the notion of convergent system (Pavlov et al., 2005) to switched nonlinear systems.

Definition 2.4: System $\dot{x} = f'_\sigma(x, \omega(t))$ with a piecewise continuous external signal $\omega(t) \in R^s$ that are bounded on R^+ and a given switching signal $\sigma(t)$ is called globally uniformly convergent if there exists an unique bounded globally asymptotically stable solution $x^*_\omega(t)$ on R, i.e. there exists a function such that for all initial condition $\|x(t, x_0) - x^*_\omega(t)\| \leq \beta(\|x_0 - x^*_\omega(t_0)\|, t - t_0)$ holds. The solution $x^*_\omega(t)$ is called a steady-state solution.

In this paper, we will investigate incremental passivity, feedback incremental passification for switched nonlinear systems and solve the output regulation problem using the developed incremental passivity theory of switched nonlinear systems.

3. Incremental passivity

In this section, we will present a generalisation of the state-dependent switching law designed by Zhao and Hill (2008c) to render switched nonlinear systems incrementally passive.

Consider a switched system described by

$$\dot{x} = f_\sigma(x, \omega(t)) + g_\sigma(x, \omega(t)) u_\sigma, \quad e = h_\sigma(x, \omega(t)), \tag{11}$$

where $\omega(t)$ is generated by the exosystem (2) and f_i, g_i and h_i are continuous in ω and C^1 in x.

Theorem 3.1: *Suppose that there exist nonnegative smooth functions $S_i(t, x, \hat{x})$, continuous functions $V_i(t, x, \hat{x})$, $\lambda_{ij}(t, x, \hat{x})$, $\beta_{ij}(t, x, \hat{x}) \le 0$, $\delta_{ij}(t, x, \hat{x}) \le 0$, smooth functions $\mu_{ij}(t, x - \hat{x})$, $\nu_{ij}(t, x - \hat{x})$ with $\mu_{ii}(t, x - \hat{x}) = 0$, $\nu_{ij}(t, 0) = 0$ and $\nu_{ii}(t, x - \hat{x}) = 0$ and nonnegative continuous functions $\tilde{\mu}_{ij}(x - \hat{x})$ satisfying $|\mu_{ij}(t, x - \hat{x})| \le \tilde{\mu}_{ij}(x - \hat{x})$ for $i, j \in I$, such that*

$$\frac{\partial S_i}{\partial t} + \frac{\partial S_i}{\partial x} f_i(x, \omega) + \frac{\partial S_i}{\partial \hat{x}} f_i(\hat{x}, \omega) + \sum_{j=1}^{M} \beta_{ij}(t, x, \hat{x}) \big(V_i(t, x, \hat{x}) - V_j(t, x, \hat{x}) + \nu_{ij}(t, x - \hat{x})\big) \le 0, \tag{12}$$

$$\left[\frac{\partial S_i}{\partial x} g_i(x, \omega) - \big(h_i(x, \omega) - h_i(\hat{x}, \omega)\big)\right] \min\left\{\max_{j \ne i}\big(V_i(t, x, \hat{x}) - V_j(t, x, \hat{x}) + \nu_{ij}(t, x - \hat{x})\big), 0\right\} = 0, \tag{13}$$

$$\left[\frac{\partial S_i}{\partial \hat{x}} g_i(\hat{x}, \omega) + \big(h_i(x, \omega) - h_i(\hat{x}, \omega)\big)\right] \min\left\{\max_{j \ne i}\big(\big(V_i(t, x, \hat{x}) - V_j(t, x, \hat{x}) + \nu_{ij}(t, x - \hat{x})\big)\big), 0\right\} = 0, \tag{14}$$

$$\frac{\partial \mu_{ij}}{\partial t} + \frac{\partial \mu_{ij}}{\partial x} f_i(x, \omega) + \frac{\partial \mu_{ij}}{\partial \hat{x}} f_i(\hat{x}, \omega) + \sum_{j=1}^{M} \delta_{ij}(t, x, \hat{x}) \big(V_i(t, x, \hat{x}) - V_j(t, x, \hat{x}) + \nu_{ij}(t, x - \hat{x})\big) \le 0, \tag{15}$$

$$\frac{\partial \mu_{ij}}{\partial x} g(x, \omega) = \frac{\partial \mu_{ij}}{\partial \hat{x}} g(\hat{x}, \omega) = 0, \quad \mu_{ij}(t, x - \hat{x}) + \mu_{jk}(t, x - \hat{x}) \le \min\{0, \mu_{ik}(t, x - \hat{x})\}, \quad \forall i, j, k \tag{16}$$

$$\nu_{ij}(t, x - \hat{x}) + \nu_{jk}(t, x - \hat{x}) \le \min\{0, \nu_{ik}(t, x - \hat{x})\}, \forall i, j, k \tag{17}$$

$$S_i(t, x, \hat{x}) - S_j(t, x, \hat{x}) + \mu_{ij}(t, x - \hat{x}) = \lambda_{ij}(t, x, \hat{x}) \big(V_i(t, x, \hat{x}) - V_j(t, x, \hat{x}) + \nu_{ij}(t, x - \hat{x})\big). \tag{18}$$

hold for $\forall \omega \in W$. Design the switching law as

$$\begin{aligned} &\sigma(t) = i, \ if\ \sigma(t^-) = i\ and\ (x(t), \hat{x}(t)) \in \text{int}\Omega_i(t), \\ &\sigma(t) = \min \arg\{\Omega_j(t) \,|\, (x(t), \hat{x}(t)) \in \Omega_j(t)\}, \\ &\quad if\ \sigma(t^-) = i\ and\ (x(t), \hat{x}(t)) \in \tilde{\Omega}_{ij}(t), \end{aligned} \tag{19}$$

where $\Omega_i(t) = \{(x, \hat{x}) | V_i(t, x, \hat{x}) - V_j(t, x, \hat{x}) + \nu_{ij}(t, x - \hat{x}) \le 0, j \in I\}$ *and*

$$\tilde{\Omega}_{ij}(t) = \{(x, \hat{x}) | V_i(t, x, \hat{x}) - V_j(t, x, \hat{x}) + \nu_{ij}(t, x - \hat{x}) = 0, i \ne j\}. \tag{20}$$

Then, system (11) is incrementally passive under the switching law (19).

Proof: Similar to Zhao and Hill (2008c), we can show that $\{\Omega_i(t)|, i \in I\}$ in Equation (20) makes a partition of $R^{2n}$and the sets $\Omega_i(t)$ have the property that for any fixed t, if $(x, \hat{x}) \in \Omega_i(t) \cap \tilde{\Omega}_{ij}(t)$ for some $i, j \in I$ and $(x, \hat{x}) \in R^{2n}$then $(x, \hat{x}) \in \Omega_j(t)$. Fix some function $\omega(t) \in W$. ■

When $(x, \hat{x}) \in \Omega_i(t)$, differentiating $S_i(t, x, \hat{x})$ together with Equations (12)–(14) gives

$$\begin{aligned} \dot{S}_i &= \frac{\partial S_i}{\partial t} + \frac{\partial S_i}{\partial x} f_i(x, \omega) + \frac{\partial S_i}{\partial x} g_i(x, \omega) u_i + \frac{\partial S_i}{\partial \hat{x}} f_i(\hat{x}, \omega) + \frac{\partial S_i}{\partial \hat{x}} g_i(\hat{x}, \omega) \hat{u}_i \\ &\le (e - \hat{e})^T (u_i - \hat{u}_i). \end{aligned}$$

According to the switching law (19), once the trajectory $(x(t), \hat{x}(t))$ enters $\Omega_i(t)$, it will stay in $\Omega_i(t)$ until it hits the boundary in $\tilde{\Omega}_{ij}(t)$ and then enters $\Omega_L(t)$, where $L = \min\{j | \Omega_i(t) \cap \tilde{\Omega}_{ij}(t)\}$. Thus, we obtain the switching sequence (3) and

$$\begin{aligned} &V_{i_{k+1}}(t_{k+1}, x(t_{k+1}), \hat{x}(t_{k+1})) - V_{i_k}(t_{k+1}, x(t_{k+1}), \hat{x}(t_{k+1})) \\ &= \nu_{i_k i_{k+1}}(t_{k+1}, x(t_{k+1}) - \hat{x}(t_{k+1})) \end{aligned}$$

which implies

$$\begin{aligned} &S_{i_{k+1}}(t_{k+1}, x(t_{k+1}), \hat{x}(t_{k+1})) - S_{i_k}(t_{k+1}, x(t_{k+1}), \hat{x}(t_{k+1})) \\ &= \mu_{i_k i_{k+1}}(t_{k+1}, x(t_{k+1}) - \hat{x}(t_{k+1})). \end{aligned} \tag{21}$$

Equations (15) and (16) tell us that $\mu_{i_k j}(t, x(t) - \hat{x}(t))$ are decreasing on $[t_k, t_{k+1})$. Let $S(\sigma(t), t, x, \hat{x}) \triangleq S_{\sigma(t)}(t, x, \hat{x})$. For $t_0 \le t < \infty$, $\forall t \in [t_k, t_{k+1})$, from Equation (21), we have

$$
\begin{aligned}
&S(\sigma(t), x(t), \hat{x}(t)) - S(\sigma(t_0), x(t_0), \hat{x}(t_0)) \\
&= S_{i_k}(t, x(t), \hat{x}(t)) - S_{i_k}(t_k, x(t_k), \hat{x}(t_k)) \\
&\quad + \sum_{p=0}^{k-1} \big(S_{i_p}(t_{p+1}, x(t_{p+1}), \hat{x}(t_{p+1})) \\
&\quad - S_{i_p}(t_p, x(t_p), \hat{x}(t_p))\big) + \sum_{p=1}^{k} \big(S_{i_p}(t_p, x(t_p), \hat{x}(t_p)) \\
&\quad - S_{i_{p-1}}(t_p, x(t_p), \hat{x}(t_p))\big) \\
&\le \int_{t_0}^{t} (e(\tau) - \hat{e}(\tau))^T (u_{\sigma(\tau)}(\tau) - \hat{u}_{\sigma(\tau)}(\tau)) d\tau \\
&\quad + \sum_{p=1}^{k} \mu_{i_{p-1}i_p}(t_p, x(t_p) - \hat{x}(t_p)) \\
&\le \begin{cases} \int_{t_0}^{t} (e(\tau) - \hat{e}(\tau))^T (u_{\sigma(\tau)} - \hat{u}_{\sigma(\tau)}) d\tau & \text{if } k \text{ is even} \\ \int_{t_0}^{t} (e(\tau) - \hat{e}(\tau))^T (u_{\sigma(\tau)} - \hat{u}_{\sigma(\tau)}) d\tau \\ \quad + \mu_{i_0 i_1}(t_0, x_0 - \hat{x}_0) & \text{if } k \text{ is odd} \end{cases} \\
&\le \int_{t_0}^{t} (e(\tau) - \hat{e}(\tau))^T (u_{\sigma(\tau)} - \hat{u}_{\sigma(\tau)}) d\tau + \alpha(\|x_0 - \hat{x}_0\|),
\end{aligned}
$$

where $\alpha(s) = \max_{\|x-\hat{x}\|\le s}\{|\tilde{\mu}_{ij}(x-\hat{x})|, i, j \in I\}$ is class $\mathcal{GK}$ function. Then, system (11) is incrementally passive under the switching law (19).

Remark 3.1: Equations (12)–(14) mean that the incremental passivity inequality holds on $\Omega_i(t)$.

Remark 3.2: When system (11) is time-invariant and all the functions given in Theorem3.1 are also independent of time and $\mu_{ij} \equiv \nu_{ij}$, $S_i \equiv V_i$ the switching law (19) degenerates into the state-dependent switching law designed by Zhao and Hill (2008c). If, in addition, $\mu_{ij} = \nu_{ij} \equiv 0$, the switching law (19) can be reduced to the 'min-switching' law (Dong & Zhao, 2012). The switching law (19) implies that the adjacent storage functions are not necessarily connected at the switching time. This gives us more design freedom of stabilising switched systems.

Next, we will give an incremental passivity condition for system (11) in the following form:

$$
\begin{aligned}
\dot{x} &= f_\sigma(x, \omega(t)) + B_\sigma u_\sigma, \\
e &= C_\sigma x + H_\sigma(\omega(t)),
\end{aligned} \tag{22}
$$

where $B_i, C_i, i \in I$ are constant matrices and $H_i \in C^1$.

Theorem 3.2: *Suppose that there exist $\beta_{ij} \le 0, \delta_{ij} \le 0$ (β_{ij}, δ_{ij}may depend on x), smooth functions $\mu_{ij}(x-\hat{x}) = (x-\hat{x})^T\Gamma_{ij}(x-\hat{x})$, $\nu_{ij}(x-\hat{x}) = (x-\hat{x})^T\Lambda_{ij}(x-\hat{x})$ with $\Gamma_{ii} = 0$ and $\Lambda_{ii} = 0$ for $i, j \in I$, matrices $Q_i = Q_i^T$, positive definite matrices P_i and constants λ_{ij} such that*

$$
\begin{aligned}
&P_i \frac{\partial f_i}{\partial x}(x, \omega) + \frac{\partial f_i^T}{\partial x}(x, \omega) P_i \\
&+ \sum_{j=1}^{M} \beta_{ij}(Q_i - Q_j + \Lambda_{ij}) \le 0, \\
&P_i B_i = C_i^T, \quad i, j \in I, \ \forall x \in R^n,
\end{aligned} \tag{23}
$$

$$
\begin{aligned}
&\Gamma_{ij} \frac{\partial f_i}{\partial x}(x, \omega) + \frac{\partial f_i^T}{\partial x}(x, \omega) \Gamma_{ij} \\
&+ \sum_{j=1}^{M} \delta_{ij}(Q_i - Q_j + \Lambda_{ij}) \le 0, \\
&\Gamma_{ij} B_i = 0, \ \forall j \in I, \ \forall x \in R^n,
\end{aligned} \tag{24}
$$

$$
\Gamma_{ij} + \Gamma_{jk} - \Gamma_{ik} \le 0, \ \Gamma_{ij} + \Gamma_{jk} \le 0, \ \Lambda_{ij} + \Lambda_{jk} - \Lambda_{ik} \le 0, \ \Lambda_{ij} + \Lambda_{jk} \le 0, \forall i, j, k, \tag{25}
$$

$$
P_i - P_j + \Gamma_{ij} = \lambda_{ij}(Q_i - Q_j + \Lambda_{ij}) \ \forall i, j, k \tag{26}
$$

hold for any $\omega \in W$, where Γ_{ij} and Λ_{ij} are symmetric matrices. Then, system (22) with the storage function $S(\sigma(t), x, \hat{x}) = S_{\sigma(t)}(x, \hat{x}) = \frac{1}{2}(x-\hat{x})^T P_{\sigma(t)}(x-\hat{x})$ is incrementally passive under switching law (19).

Proof: Similar to Pavlov et al. (2005), according to the mean value theorem, we obtain

$$
\begin{aligned}
&\frac{\partial S_i}{\partial x} f_i(x, \omega) + \frac{\partial S_i}{\partial \hat{x}} f_i(\hat{x}, \omega) \\
&= (x-\hat{x})^T P_i (f_i(x, \omega) - f_i(\hat{x}, \omega)) \\
&= \frac{1}{2}(x-\hat{x})^T J_i(\xi, \omega)(x-\hat{x})
\end{aligned}
$$

■

and

$$
\frac{\partial \mu_{ij}}{\partial x} f_i(x, \omega) + \frac{\partial \mu_{ij}}{\partial \hat{x}} f_i(\hat{x}, \omega) = \frac{1}{2}(x-\hat{x})^T (\Gamma_{ij} \frac{\partial f_i}{\partial \xi}(\xi, \omega) + \frac{\partial f_i^T}{\partial \xi}(\xi, \omega)\Gamma_{ij})(x-\hat{x}),
$$

where $J_i(\xi, \omega) = P_i \frac{\partial f_i}{\partial \xi}(\xi, \omega) + \frac{\partial f_i^T}{\partial \xi}(\xi, \omega) P_i$, ξ is some point between x and $\hat{x}$. In addition, since Equations (25) and (26) hold, according to Theorem3.1, Theorem3.2 holds.

4. Feedback incremental passification

In this section, a state-dependent switching law and state feedback controllers are designed simultaneously to render the switched nonlinear systems incrementally passive.

Consider the following system:

$$\begin{aligned}\dot{z} &= q_\sigma (z, e, \omega), \\ \dot{e} &= p_\sigma (z, e, \omega) + a_\sigma (z, e, \omega) u_\sigma,\end{aligned} \tag{27}$$

where $a_i(z, e, \omega)$, $i \in I$ are invertible, e is the output of system (27) and the functions p_i, q_i and a_i are continuous in ω and C^1 in x.

A sufficient condition of feedback incremental passification is given as follows:

Theorem 4.1: *Suppose that there exist nonnegative smooth functions $W_i(z, \hat{z})$, continuous functions $U_i(z, \hat{z})$, $\lambda_{ij}(z, \hat{z})$, $\beta_{ij}(z, \hat{z}) \leq 0$ and $\delta_{ij}(z, \hat{z}) \leq 0$ smooth functions $\mu_{ij}(z - \hat{z})$, $\nu_{ij}(z - \hat{z})$ with $\mu_{ij}(0) = 0$ and $\mu_{ii}(z - \hat{z}) = 0$,$\nu_{ij}(0) = 0$ and $\nu_{ii}(z - \hat{z}) = 0$ for $i, j \in I$ such that*

$$\frac{\partial W_i}{\partial z} q_i (z, e, \omega) + \frac{\partial W_i}{\partial \hat{z}} q_i (\hat{z}, \hat{e}, \omega) + \sum_{j=1}^{M} \beta_{ij} (z, \hat{z}) \left(U_i (z, \hat{z}) - U_j (z, \hat{z}) + \nu_{ij} (z - \hat{z})\right) \leq 0, \tag{28}$$

$$\frac{\partial \mu_{ij}}{\partial z} q_i (z, e, \omega) + \frac{\partial \mu_{ij}}{\partial \hat{z}} q_i (\hat{z}, \hat{e}, \omega) + \sum_{j=1}^{M} \delta_{ij} (z, \hat{z}) \left(U_i (z, \hat{z}) - U_j (z, \hat{z}) + \nu_{ij} (z - \hat{z})\right) \leq 0, \tag{29}$$

$$\begin{aligned}&\mu_{ij} (z - \hat{z}) + \mu_{jk} (z - \hat{z}) \leq \min \left\{0, \mu_{ik} (z - \hat{z})\right\}, \\ &\nu_{ij} (z - \hat{z}) + \nu_{jk} (z - \hat{z}) \leq \min \left\{0, \nu_{ik} (z - \hat{z})\right\}, \forall i, j, k\end{aligned} \tag{30}$$

$$\begin{aligned}&W_i (z, \hat{z}) - W_j (z, \hat{z}) + \mu_{ij} (z - \hat{z}) \\ &= \lambda_{ij} (z, \hat{z}) \left(U_i (z, \hat{z}) - U_j (z, \hat{z}) + \nu_{ij} (z - \hat{z})\right), \\ &\lambda_{ij} (z, \hat{z}) = \lambda_{ji} (z, \hat{z}).\end{aligned} \tag{31}$$

Let $X = (z^T, e^T)^T$, $V_i(X, \hat{X}) = U_i(z, \hat{z}) + \frac{1}{2}(e - \hat{e})^T(e - \hat{e})$ and $\tilde{\nu}_{ij}(X - \hat{X}) = \nu_{ij}(z - \hat{z})$. Design the switching law as

$$\begin{aligned}&\sigma (t) = i \text{ if } \sigma (t^-) = i \text{ and } \left(X (t), \hat{X} (t)\right) \in \text{int}\Omega_i, \\ &\sigma (t) = \min \arg \left\{\Omega_j \middle| \left(X (t), \hat{X} (t)\right) \in \Omega_j\right\}, \\ &\qquad \text{if } \sigma (t^-) = i \text{ and } \left(X (t), \hat{X} (t)\right) \in \tilde{\Omega}_{ij},\end{aligned} \tag{32}$$

where $\Omega_i = \{(X, \hat{X}) | V_i(X, \hat{X}) - V_j(X, \hat{X}) + \tilde{\nu}_{ij}(X - \hat{X}) \leq 0, j \in I\}$ and

$$\tilde{\Omega}_{ij} = \left\{ \left(X, \hat{X}\right) \middle| V_i \left(X, \hat{X}\right) - V_j \left(X, \hat{X}\right) + \tilde{\nu}_{ij} \left(X - \hat{X}\right) = 0, i \neq j \right\}.$$

Then, system (27) with the controllers $u_i = a_i(z, e, \omega)^{-1}(\upsilon_i - p_i(z, e, \omega))$ is incrementally passive under the switching law (32).

Proof: Substituting $u_i = a_i(z, e, \omega)^{-1}(\upsilon_i - p_i(z, e, \omega))$ into Equation (27) gives

$$\begin{aligned}\dot{z} &= q_\sigma (z, e, \omega), \\ \dot{e} &= \upsilon_\sigma\end{aligned} \tag{33}$$

■

We choose $S(\sigma(t), X, \hat{X}) = S_{\sigma(t)}(X, \hat{X}) = W_{\sigma(t)}(z, \hat{z}) + \frac{1}{2}(e - \hat{e})^T(e - \hat{e})$, $i \in I$ as the storage function of Equation (27). Differentiating S_i gives

$$\begin{aligned}\dot{S}_i = &\frac{\partial W_i}{\partial z} q_i (z, e, \omega) + \frac{\partial W_i}{\partial \hat{z}} q_i (\hat{z}, \hat{e}, \omega) \\ &+ (e - \hat{e})^T (\upsilon_i - \hat{\upsilon}_i), \\ \leq &- \sum_{j=1}^{M} \beta_{ij} (z, \hat{z}) \left(U_i (z, \hat{z}) - U_j (z, \hat{z}) + \nu_{ij} (z - \hat{z})\right) \\ &+ (e - \hat{e})^T (\upsilon_i - \hat{\upsilon}_i), \\ \leq &- \sum_{j=1}^{M} \tilde{\beta}_{ij} \left(X, \hat{X}\right) \left(V_i \left(X, \hat{X}\right) - V_j \left(X, \hat{X}\right) \right. \\ &\left. + \tilde{\nu}_{ij} \left(X - \hat{X}\right)\right) + (e - \hat{e})^T (\upsilon_i - \hat{\upsilon}_i),\end{aligned}$$

where $\tilde{\beta}_{ij}(X, \hat{X}) = \beta_{ij}(z, \hat{z})$.

Let $\tilde{\mu}_{ij}(X - \hat{X}) = \mu_{ij}(z - \hat{z})$. Thus, $\dot{\tilde{\mu}}_{ij} = \frac{\partial \mu_{ij}}{\partial z} q_i (z, e, \omega) + \frac{\partial \mu_{ij}}{\partial \hat{z}} q_i(\hat{z}, \hat{e}, \omega) \leq 0$ on Ω_i and $\tilde{\mu}_{ij}(X - \hat{X}) + \tilde{\mu}_{jk}(X - \hat{X}) \leq \min\{0, \tilde{\mu}_{ik}(X - \hat{X})\}, \forall i, j, k$ hold due to Equations (29) and (30).

The rest of proof is similar to that of Theorem3.1.

The next result provides the sufficient condition of feedback incremental passification for system (27) in special case.

Theorem 4.2: *Suppose that there exist$\tilde{\beta}_{ij} \leq 0, \tilde{\delta}_{ij} \leq 0$ ($\tilde{\beta}_{ij}, \tilde{\delta}_{ij}$may depend on z), smooth functions $\mu_{ij}(z - \hat{z}) = (z - \hat{z})^T \tilde{\Gamma}_{ij}(z - \hat{z})$, $\nu_{ij}(z - \hat{z}) = (z - \hat{z})^T \tilde{\Lambda}_{ij}(z - \hat{z})$ with $\tilde{\Gamma}_{ii} = 0$ and $\tilde{\Lambda}_{ii} = 0$ for $i, j \in I$, matrices $\tilde{Q}_i = \tilde{Q}_i^T$, positive definite matrices E_i and constants λ_{ij}, $\rho_i > 0$ such that*

$$E_i \frac{\partial q_i}{\partial z}(z, e, \omega) + \frac{\partial q_i^T}{\partial z}(z, e, \omega) E_i + \sum_{j=1}^{M} \tilde{\beta}_{ij} \left(\tilde{Q}_i - \tilde{Q}_j + \tilde{\Lambda}_{ij} \right) \leq -\rho_i I, \tag{34}$$

$$\tilde{\Gamma}_{ij} \frac{\partial q_i}{\partial z} + \frac{\partial q_i^T}{\partial z} \tilde{\Gamma}_{ij} + \sum_{j=1}^{M} \tilde{\delta}_{ij} \left(\tilde{Q}_i - \tilde{Q}_j + \tilde{\Lambda}_{ij} \right) \leq 0, \tag{35}$$

$$\begin{aligned} &\tilde{\Gamma}_{ij} + \tilde{\Gamma}_{jk} - \tilde{\Gamma}_{ik} \leq 0, \;\; \tilde{\Gamma}_{ij} + \tilde{\Gamma}_{jk} \leq 0, \\ &\tilde{\Lambda}_{ij} + \tilde{\Lambda}_{jk} - \tilde{\Lambda}_{ik} \leq 0, \;\; \tilde{\Lambda}_{ij} + \tilde{\Lambda}_{jk} \leq 0, \forall i, j, k, \end{aligned} \tag{36}$$

$$E_i - E_j + \tilde{\Gamma}_{ij} = \lambda_{ij} \left(\tilde{Q}_i - \tilde{Q}_j + \tilde{\Lambda}_{ij} \right) \;\; \forall i, j, k \tag{37}$$

hold for any $\omega \in W$, where $\tilde{\Gamma}_{ij}$ and $\tilde{\Lambda}_{ij}$ are symmetric matrices. Then, there exist state feedback controllers such that system (27) is incrementally passive under the switching law (32) with $U_i(z, \hat{z}) = (z - \hat{z})^T \tilde{Q}_i (z - \hat{z})$ and $\tilde{v}_{ij}(X - \hat{X}) = (z - \hat{z}) \tilde{\Lambda}_{ij} (z - \hat{z})$.

Proof: Design the feedback controllers as $u_i = a_i(z, e, \omega)^{-1}(v_i - \varphi_i(z, e))$, where function φ_i is to be designed later. Let $X = (z^T, e^T)^T$, $C_i = (0, I_m)$, $B_i = (0, I_m)$, $H_i(\omega) = 0$, $f_i(X, \omega) = (q_i^T(z, e, \omega), p_i^T(z, e, \omega) - \varphi_i^T(z, e))^T$, $P_i = \left[\begin{smallmatrix} E_i & 0 \\ 0 & I_m \end{smallmatrix}\right]$, $\beta_{ij} = \tilde{\beta}_{ij}$, $\Lambda_{ij} = \left[\begin{smallmatrix} \tilde{\Lambda}_{ij} & 0 \\ 0 & 0 \end{smallmatrix}\right]$ and $Q_i = \left[\begin{smallmatrix} \tilde{Q}_i & 0 \\ 0 & I_m \end{smallmatrix}\right]$. We only need to verify that condition (23) holds.

$$\begin{aligned} &J_i(X, \omega) \\ &= P_i \frac{\partial f_i}{\partial X}(X, \omega) + \frac{\partial f_i^T}{\partial X}(X, \omega) P_i + \sum_{j=1}^{M} \beta_{ij} \left(Q_i - Q_j + \Lambda_{ij} \right) \\ &= \begin{bmatrix} E_i \frac{\partial q_i}{\partial z} + \frac{\partial q_i^T}{\partial z} E_i + \sum_{j=1}^{M} \tilde{\beta}_{ij} \left(\tilde{Q}_i - \tilde{Q}_j + \tilde{\Lambda}_{ij} \right) & E_i \frac{\partial q_i}{\partial e} + \frac{\partial p_i^T}{\partial z} - \frac{\partial \varphi_i^T}{\partial z} \\ \frac{\partial q_i^T}{\partial e} E_i + \frac{\partial p_i}{\partial z} - \frac{\partial \varphi_i}{\partial z} & \frac{\partial p_i}{\partial e} + \frac{\partial p_i^T}{\partial e} - \frac{\partial \varphi_i}{\partial e} - \frac{\partial \varphi_i^T}{\partial e} \end{bmatrix} \\ &= -\begin{bmatrix} A_i & M_i \\ M_i^T & N_i \end{bmatrix}. \end{aligned}$$

■

Next, we will show J_i is negative definite for all $(z, e) \in R^n$ and $\omega \in W$. From Equation (35), $A_i > 0$. If

$$\frac{\partial \varphi_i}{\partial e} + \frac{\partial \varphi_i^T}{\partial e} - \frac{2}{\rho_i} \left\| \frac{\partial \varphi_i}{\partial z} \right\|^2 > \left(\left\| \frac{\partial p_i}{\partial e} + \frac{\partial p_i^T}{\partial e} \right\| + \frac{2}{\rho_i} \left\| \frac{\partial q_i^T}{\partial e} E_i + \frac{\partial p_i}{\partial z} \right\| \left\| E_i \frac{\partial q_i}{\partial e} + \frac{\partial p_i^T}{\partial z} \right\| \right) I_m \tag{38}$$

holds, then Equation (23) holds. Since $\omega(t) \in W$, there exists continuous function $\gamma_i(z, e)$ such that the following inequality holds

$$\gamma_i(z, e) > \left\| \frac{\partial p_i}{\partial e} + \frac{\partial p_i^T}{\partial e} \right\| + \frac{2}{\rho_i} \left\| \frac{\partial q_i^T}{\partial e} E_i + \frac{\partial p_i}{\partial z} \right\| \left\| E_i \frac{\partial q_i}{\partial e} + \frac{\partial p_i^T}{\partial z} \right\|.$$

Similar to Pavlov and Marconi (2008), there exists φ_i satisfying

$$\frac{\partial \varphi_i}{\partial e} + \frac{\partial \varphi_i^T}{\partial e} - \frac{2}{\rho_i} \left\| \frac{\partial \varphi_i}{\partial z} \right\|^2 > \gamma_i(z, e) I_m. \tag{39}$$

According to Theorem3.2, Theorem 4.2 holds.

Remark 4.1: Equation (35) implies that $E_i \frac{\partial q_i}{\partial z} + \frac{\partial q_i^T}{\partial z} E_i \leq -\rho_i I$ is only required to hold on Ω_i, which is weaker than incremental minimum-phase condition (Pavlov & Marconi, 2008).

5. Incremental passivity-based output regulation

In this section, the output regulation problem for switched nonlinear systems is solved using the incremental passivity theory.

To get the convergence of solution, we need the finite time detectability.

Definition 5.1: System

$$\dot{x} = F(x, u, \omega), \;\; e = h(x, \omega) \tag{40}$$

is called incrementally asymptotically zero-state detectable if for any $\varepsilon > 0$, any bounded signal $\omega(t)$, any two solutions $x, \hat{x}$ of system (40) corresponding to the inputs u and $\hat{u}$, the respective outputs y, $\hat{y}$, there exists $\delta > 0$, such that when $\|u(t+s) - \hat{u}(t+s)\| < \delta$ and $\| y(t+s) - \hat{y}(t+s) \| < \delta$ hold for some $t \geq t_0$, $\Delta > 0$ and $0 \leq s \leq \Delta$, we have $\|x(t) - \hat{x}(t)\| < \varepsilon$.

Remark 5.1: The incrementally asymptotical zero-state detectability is an incremental version of asymptotical zero-state detectability (Zhao & Hill, 2008a) and weaker than the incremental notion of small-time initial-state observability (Hespanha, Liberzon, Angeli, & Sontag, 2005).

We first solve the output regulation problem under a given switching signal.

Theorem 5.1: *Consider systems (1) and (2) satisfying Assumption 2.1. Suppose that there exist the C^1 feedback controllers $u_i = \eta_i(x, \omega) + v_i$ with $\eta_i(x_\omega^*, \omega) = 0$ such* *that for any solution of the exosystem (2) $\omega(t)$ starting from $\omega(t_0) \in W$, the system*

$$\begin{aligned} \dot{x} &= F_\sigma\left(x, \eta_\sigma\left(x, \omega\right) + \upsilon_\sigma, \omega\right), \\ e &= h_\sigma\left(x, \omega\right) \end{aligned} \tag{41}$$

with a storage function $S(\sigma(t), t, x(t), \hat{x}(t)) = S_{\sigma(t)}(t, x(t), \hat{x}(t))$ is incrementally passive under a given switching signal $\sigma(t)$. If (i) there exist K_∞ functions α_1, α_2 such that $\alpha_1(\|x - \hat{x}\|) \le S_i(t, x, \hat{x}) \le \alpha_2(\|x - \hat{x}\|)$, (ii) there exists at least one j such that $\lim_{k\to\infty}(t_{j_k+1} - t_{j_k}) \ne 0$ and (iii) the corresponding subsystems of the resulting closed-loop system (41) are incrementally asymptotically zero-state detectable, then the output regulation problem is solved by

$$u_i = \eta_i\left(x, \omega\right) + \upsilon_i, \quad \upsilon_i = \bar{u}_{i\omega} - K_i e, \tag{42}$$

where K_i are positive definite matrices.

Proof: For $t \ge t_0$, $\forall t \in [t_k, t_{k+1})$, $k \in N$, since system (41) is incrementally passive, we have

$$\begin{aligned} & S\left(\sigma\left(t\right), t, x\left(t\right), \hat{x}\left(t\right)\right) - S\left(\sigma\left(t_0\right), t_0, x\left(t_0\right), \hat{x}\left(t_0\right)\right) \\ &= S_{i_k}\left(t, x\left(t\right), \hat{x}\left(t\right)\right) - S_{i_0}\left(t_0, x\left(t_0\right), \hat{x}\left(t_0\right)\right) \\ &\le \int_{t_0}^{t} \left(e\left(\tau\right) - \hat{e}\left(\tau\right)\right)^T \left(\upsilon_{\sigma(\tau)}\left(\tau\right) - \hat{\upsilon}_{\sigma(\tau)}\left(\tau\right)\right) \mathrm{d}\tau \\ &\quad + \alpha\left(\left\|x_0 - \hat{x}_0\right\|\right). \end{aligned} \tag{43}$$

■

According to Assumption 2.1, $x^*_\omega(t)$ corresponding to the input $\bar{u}_{\sigma\omega}(t)$ is a bounded solution of closed-loop system (1) and Equation (42) with the output $e = 0$. Substituting $\hat{\upsilon}_\sigma = \bar{u}_{\sigma\omega}$, $x = x^*_\omega$, $\hat{e} = 0$ and $\upsilon_\sigma = \bar{u}_{\sigma\omega} - K_\sigma e$ into Equation (39) gives

$$\begin{aligned} & S_{i_k}\left(t, x\left(t\right), x^*_\omega\left(t\right)\right) - S_{i_0}\left(t_0, x\left(t_0\right), x^*_\omega\left(t_0\right)\right) \\ &\le -\int_{t_0}^{t} e(\tau)^T K_{\sigma(\tau)} e\left(\tau\right) \mathrm{d}\tau + \alpha\left(\left\|x_0 - x^*_\omega\left(t_0\right)\right\|\right) \\ &\le -\lambda \int_{t_0}^{t} e(\tau)^T e\left(\tau\right) \mathrm{d}\tau + \alpha\left(\left\|x_0 - x^*_\omega\left(t_0\right)\right\|\right), \end{aligned} \tag{44}$$

where $\lambda = \min_{i\in I}\{\lambda_{\min}(K_i)\}$, $\lambda_{\min}(K_i) > 0$ is the minimum eigenvalue of K_i.

It follows from Equation (44) and condition (i) that

$$\begin{aligned} & \lambda \int_{t_0}^{t} e(\tau)^T e\left(\tau\right) \mathrm{d}\tau + S_{i_k}\left(t, x\left(t\right), x^*_\omega\left(t\right)\right) \\ &\le \alpha\left(\left\|x_0 - x^*_\omega\left(t_0\right)\right\|\right) + S_{i_0}\left(t_0, x\left(t_0\right), x^*_\omega\left(t_0\right)\right) \end{aligned} \tag{45}$$

and

$$\begin{aligned} & \alpha_1\left(\left\|x\left(t\right) - x^*_\omega\left(t\right)\right\|\right) \le S_{i_k}\left(t, x\left(t\right), x^*_\omega\left(t\right)\right) \\ &\le \alpha\left(\left\|x_0 - x^*_\omega\left(t_0\right)\right\|\right) + S_{i_0}\left(t_0, x\left(t_0\right), x^*_\omega\left(t_0\right)\right) \\ &\le \alpha\left(\left\|x_0 - x^*_\omega\left(t_0\right)\right\|\right) + \alpha_2\left(\left\|x_0 - x^*_\omega\left(t_0\right)\right\|\right). \end{aligned} \tag{46}$$

Therefore, for any given $\varepsilon > 0$, $\delta = \min\{\alpha_2^{-1}(\frac{1}{2}\alpha_1(\varepsilon)),\ \alpha^{-1}(\frac{1}{2}\alpha_1(\varepsilon))\} > 0$, we have $\|x(t) - x^*_\omega(t)\| < \varepsilon$, when $\|x_0 - x^*_\omega(t_0)\| < \delta, t \ge t_0$, the solution x^*_ω of closed-loop system (1) and Equation (42) is stable.

Next, we will show $\lim_{t\to\infty} \|x(t) - x^*_\omega(t)\| = 0$. For the j satisfying $\lim_{k\to\infty}(t_{j_k+1} - t_{j_k}) \ne 0$, there exists $\delta > 0$ such that the set $\Pi = \{k | t_{j_k+1} - t_{j_k} \ge \Delta\}$ is infinite. Let the auxiliary functions

$$\tilde{h}_j\left(t\right) = \begin{cases} h_j\left(x\left(t\right), \omega\left(t\right)\right), t \in \bigcup_{k\in\Pi}\left[t_{j_k}, t_{j_k+1}\right), \\ 0, \qquad \text{otherwise}. \end{cases} \tag{47}$$

Since Equations (45) and (46) hold, we have

$$\begin{aligned} & \int_{t_0}^{t} \tilde{h}_j^T\left(\tau\right)\tilde{h}_j\left(\tau\right) \mathrm{d}\tau \le \int_{t_0}^{t} e(\tau)^T e\left(\tau\right) \mathrm{d}\tau \\ &\le \frac{1}{\lambda}\left(\alpha\left(\left\|x_0 - x^*_\omega\left(t_0\right)\right\|\right) + S_{i_0}\left(t_0, x\left(t_0\right), x^*_\omega\left(t_0\right)\right)\right). \end{aligned} \tag{48}$$

Therefore, $\lim_{t\to\infty} \tilde{h}_j(t) = 0$, which implies $\lim_{t\to\infty}(\upsilon_j(t) - \bar{u}_{j\omega}) = 0$. Namely, for $\forall\delta > 0$, there exists $T_0 > 0$ such that when $t > T_0$, $\|\tilde{h}_j(t)\| < \delta$, $\|\upsilon_j(t) - \bar{u}_{j\omega}(t)\| < \delta$. Suppose this is false, then there exist $\varepsilon_0 > 0$ and a sequence of time $q_1, q_2, \ldots, q_k \to \infty$ such that $\|\tilde{h}_j^T(q_i) - \hat{\tilde{h}}_j(q_i)\| \ge \varepsilon_0, \forall i$. The boundedness of $x^*_\omega(t)$ and Equation (48) imply the boundedness of $x(t)$. Moreover, since $\bar{u}_{i\omega}(t)$ and any solution $\omega(t)$ starting from $\omega(t_0) \in W$ are bounded for all $t \ge t_0$ and the functions F_i, η_i, h_i and s are assumed to be C^1, $\dot{x}(t)$ and $\dot{\omega}(t)$ are bounded. Thus, $x(t)$ and $\omega(t)$ are uniformly continuous and $\tilde{h}_j(t)$ is uniformly continuous over $\bigcup_{k\in\Pi}[t_{j_k}, t_{j_k+1})$. Since $t_{j_k+1} - t_{j_k} \ge \Delta, k \in \Pi$, we have $\int_{t_0}^{\infty} \tilde{h}_j(\tau)^T \tilde{h}_j(\tau)\mathrm{d}\tau = \infty$, which contradicts Equation (48). We can choose $k \in N$ such that $t_{j_k} > T_0$. So $\|\tilde{h}_j(t_{j_k} + s)\| < \delta$, $\|\upsilon_j(t_{j_k}) - \bar{u}_{j\omega}(t_{j_k})\| < \delta$ hold for $k \in \Pi$ and $0 \le s \le \Delta$. $\lim_{k\to\infty} \|x(t_{j_k}) - x^*_\omega(t_{j_k})\| = 0$ follows from incrementally asymptotical zero-state detectability of the jth subsystem. This implies $\lim_{k\to\infty} \|x(t) - x^*_\omega(t)\| = 0$due to uniformly stability. Therefore, $\lim_{t\to\infty} e(t) = 0$. This completes the proof.

Remark 5.2: If there exist regular nonnegative functions $S_i(t, x, \hat{x})$ and one of the following conditions holds:

(a) Equation (1b) is independent of the switching signal $\sigma(t)$, i.e. $e(t) = h(x, \omega)$.
(b) $\lim_{k\to\infty}(t_{j_k+1} - t_{j_k}) \neq 0$ for $j = 1, 2, \dots M$
(c) System (41) is incrementally strictly passive.

Then Theorem 5.1 holds without conditions (i)–(iii).

In fact, the boundedness of $x^*_\omega(t)$ and the regular storage functions $S_i(t, x, x^*_\omega)$ imply the boundedness of $x(t)$ due to Equation (46).

(a) $\lim_{t\to\infty} e(t) = 0$ follows from Barbalat's lemma and Equation (48).
(c) If system (41) is incrementally strictly passive then similar to Equation (41), we can obtain that

$$\int_{t_0}^{t} Q\left(x(\tau) - x^*_\omega(\tau)\right)\mathrm{d}\tau \le \int_{t_0}^{t} Q_{\sigma(\tau)}\left(x(\tau) - x^*_\omega(\tau)\right)\mathrm{d}\tau \le \alpha\left(\left\|x_0 - x^*_\omega(t_0)\right\|\right) + S_{i_0}\left(t_0, x(t_0), x^*_\omega(t_0)\right), \tag{49}$$

where $Q(x - x^*_\omega) = \min_{i\in I}\{Q_i(x - x^*_\omega)\}$ is a continuous positive definite function. $Q(x(t) - x^*_\omega(t))$ is uniformly continuous due to the boundedness of $x(t), x^*_\omega(t), \dot{x}(t)$ and $\dot{x}^*_\omega(t)$. According to Barbalat's lemma and $\int_{t_0}^{\infty} Q(x(\tau) - x^*_\omega(\tau))\mathrm{d}\tau < \infty$ due to Equation (49), we have $Q(x(\tau) - x^*_\omega(\tau)) \to 0, t \to \infty$, which implies $\|x(\tau) - x^*_\omega(\tau)\| \to 0, t \to \infty$. Therefore, $\lim_{t\to\infty} e(t) = 0$.

Remark 5.3: Compared with convention regulators, the regulators designed by using incremental passivity property comprise of two components: the steady-state control and the linear output feedback stabilising controllers. From the proof, we only have to verify the regulated outputs converge to zero directly under the conditions (a) and (b). The incrementally strict passivity condition (c) is strong, so the stabilising controllers can be chosen freely.

Next, we show that the output regulation problem for system (11) is solvable by the design of the switched law. The following assumption on the solvability of the regulator equations is given:

Assumption 5.1: *There exist continuous functions $V_i(t, x, \hat{x})$, smooth functions $\nu_{ij}(t, x - \hat{x})$ with $\nu_{ij}(t, 0) = 0$ and $\nu_{ii}(t, x - \hat{x}) = 0$ and differentiable maps $x^*_\omega = \pi(\omega)$ and $\bar{u}_{i\omega} = c_i(\omega)$ defined on a set W satisfying*

$$\left(\frac{\partial\pi}{\partial\omega}s(\omega) - f_i(\pi(\omega), \omega) - g_i(\pi(\omega), \omega)c_i(\omega)\right) + \max_{j\in I}\left\{V_i\left(t, x, x^*_\omega\right) - V_j\left(t, x, x^*_\omega\right) + \nu_{ij}\left(t, x - x^*_\omega\right)\right\} \doteq 0,$$
$$0 = h_i(\pi(\omega), \omega). \tag{50}$$

Remark 5.4: Define $\Omega_i(t) = \{(x, x^*_\omega)|V_i(t, x, x^*_\omega) - V_j(t, x, x^*_\omega) + \nu_{ij}(t, x - x^*_\omega) \le 0,\ j \in I\}$ and $\tilde{\Omega}_{ij}(t) = \{(x, x^*_\omega)|V_i(t, x, x^*_\omega) - V_j(t, x, x^*_\omega) + \nu_{ij}(t, x - x^*_\omega) = 0,\ i \neq j\}$. Similar to the proof in Zhao and Hill (2008c), $\{\Omega_i(t)|i \in I\}$ makes a partition of R^{2n}. Equation (50) implies that the regulator equation of each subsystem holds on $\Omega_i(t)$. Assumption 5.1 is weaker than the assumption on the solvability of the regulator equations (Dong & Zhao, 2012a).

Theorem 5.2: *Consider systems (11) and (2) satisfying all conditions of Theorem3.1 and Assumption 5.1. Suppose that conditions (i) and (ii) in Theorem 5.1 hold. If, in addition the corresponding subsystems of system (11) are incrementally asymptotically zero-state detectable. Design the switching law*

$$\begin{aligned}
&\sigma(t) = i \text{ if } \sigma\left(t^-\right) = i \text{ and } \left(x(t), x^*_\omega(t)\right) \in \operatorname{int}\Omega_i(t),\\
&\sigma(t) = \min\arg\left\{\Omega_j(t)\left|\left(x(t), x^*_\omega(t)\right) \in \Omega_j(t)\right.\right\},\\
&\qquad \text{if } \sigma\left(t^-\right) = i \text{ and } \left(x(t), x^*_\omega(t)\right) \in \tilde{\Omega}_{ij}(t),
\end{aligned} \tag{51}$$

Then, the output regulation problem is solved by $u_i = \bar{u}_{i\omega} - K_i e$, where K_i are positive definite matrices.

Proof: According to Theorem3.1, system (11) is incrementally passive under the switching law (19). Assumption 2.1 holds for the switching law (51). Therefore, Theorem 5.2 follows Theorem 5.1. ■

Remark 5.5: Since $(x, \hat{x})$ is dependent on $(u(t), \hat{u}(t))$ in Equation (20), the switching law (19) is dependent on $(u(t), \hat{u}(t))$. We can obtain the switching law (51) by setting $u_i = \bar{u}_{i\omega} - K_i e$, $\hat{u}_i = \bar{u}_{i\omega}$.

6. Examples

In this section, we present two examples to demonstrate the effectiveness of our main results.

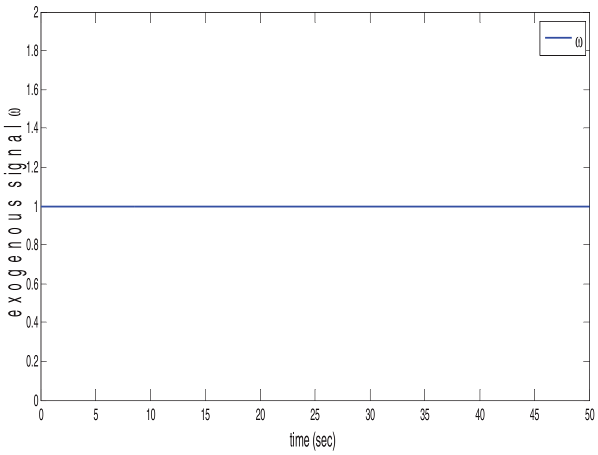

Figure 1. The exogenous signal $\omega(t)$.

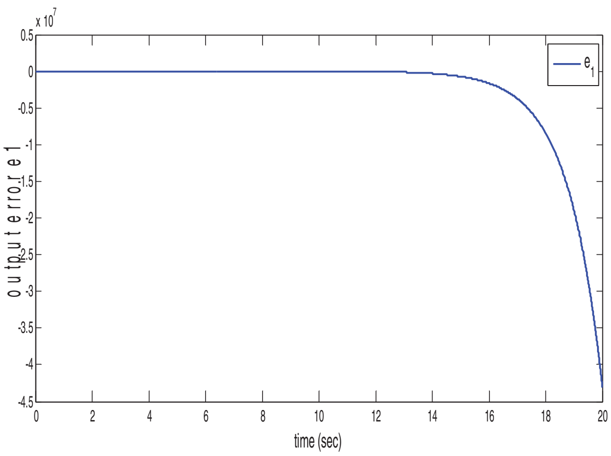

Figure 2. The regulated output of the subsystem (1).

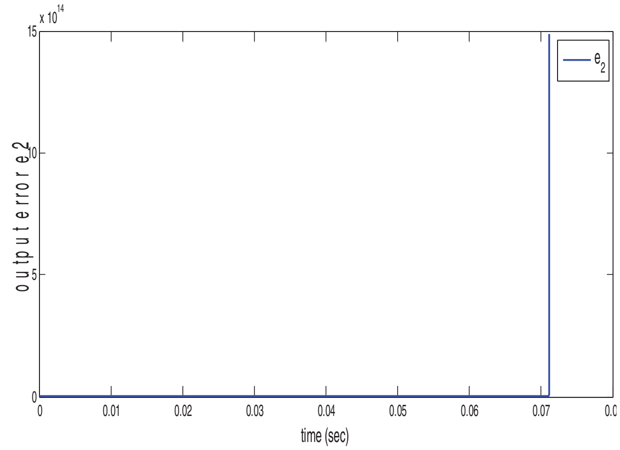

Figure 3. The regulated output of the subsystem (2) .

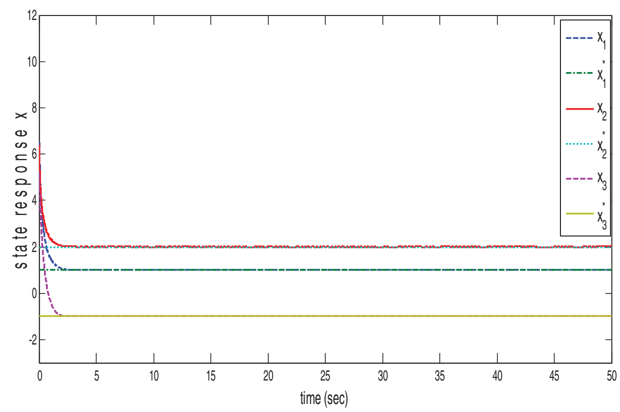

Figure 4. State response of the switched system.

Example 6.1: Consider the system consisting of two subsystems described by

$$f_1(x, u_1, \omega) = \begin{pmatrix} -x_1\left(x_1^2+4\right) + \frac{1}{2}x_2 + 4 + x_3 + \omega^6 \\ \frac{1}{2}x_1 + x_2 + \frac{4}{3} + \frac{1}{3}x_3 - \frac{7}{2}\omega^2 \\ 2 - x_1 - x_2 + u_1 \end{pmatrix},$$

$$f_2(x, u_2, \omega) = \begin{pmatrix} x_1 + x_2 + 2 + \frac{1}{2}x_3 - 4.5\omega^2 \\ 2x_1 - 10x_2 + 4 + x_3 + 15\omega^2 \\ 11 - x_1 - x_2 + u_2 \end{pmatrix}, \quad (52)$$

$e = x_3 - 3\omega^2 + 4$ and the exosystem $\dot{\omega} = 0$. $\bar{u}_{1\omega} = 3\omega^2 - 2$, $\bar{u}_{2\omega} = 3\omega^2 - 11$, $x_\omega^*(t) = (\omega^2, 2\omega^2, 3\omega^2 - 4)^T$ is solution of the regulator equation (7).

We choose the storage functions as

$$S_1(x, \hat{x}) = \frac{1}{2}(x - \hat{x})^T P_1 (x - \hat{x}) \text{ and}$$
$$S_2(x, \hat{x}) = \frac{1}{2}(x - \hat{x})^T P_2 (x - \hat{x}),$$

where $P_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $P_2 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Differentiating S_i gives

$$\dot{S}_1 \le -\beta_{12}(S_1 - S_2) + (u_1 - \hat{u}_1)^T (e - \hat{e}),$$
$$\dot{S}_2 \le -\beta_{21}(S_2 - S_1) + (u_2 - \hat{u}_2)^T (e - \hat{e}),$$

where $\beta_{12} = -3.5$, $\beta_{21} = -7$.

Design the switching law as follows:

$$\sigma(t) = i, \text{ if } \sigma(t^-) = i \text{ and } (x(t), x_\omega^*(t)) \in \operatorname{int} \Omega_i(t), \quad (53)$$

$$\sigma(t) = \min \arg \left\{ \Omega_j(t) \,\middle|\, (x(t), x_\omega^*(t)) \in \Omega_j(t) \right\},$$
$$\text{if } \sigma(t^-) = i \text{ and } (x(t), x_\omega^*(t)) \in \tilde{\Omega}_{ij}(t),$$

where

$$\Omega_i = \left\{ (x, x_\omega^*) \,\middle|\, S_i(x, x_\omega^*) - S_j(x, x_\omega^*) \le 0,\ j = 1, 2 \right\},$$
$$\tilde{\Omega}_{ij} = \left\{ (x, x_\omega^*) \,\middle|\, S_i(x, x_\omega^*) - S_j(x, x_\omega^*) = 0,\ i \ne j \right\}.$$

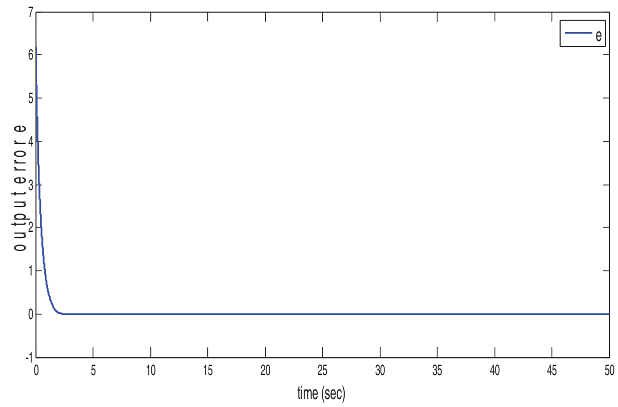

Figure 5. The regulated output of the switched system.

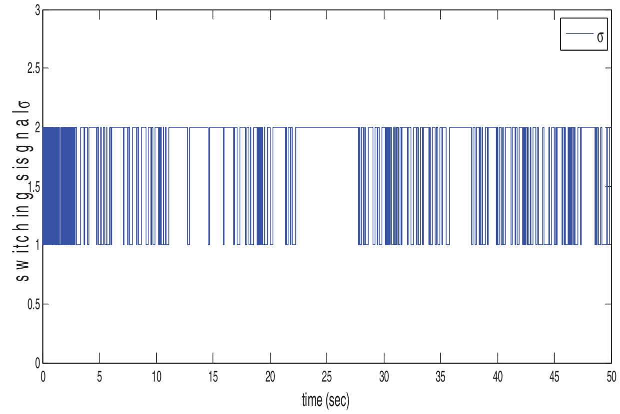

Figure 6. The switching signal of the switched system.

According to Remark 5.2 and Theorem 5.2, the output regulation problem is solved by the feedback controllers $u_i = \bar{u}_{i\omega} - e$, $i = 1, 2$ under the switching law (53).

Let the initial state $x(0) = (6.3, 6.4, 5.2)$, $\omega(0) = 1$, the simulation results are depicted in Figures 1–6. Figures 1–3 indicate that the output regulation problem for none of the subsystems is solvable. It can be seen from Figures 1, 4 and 5 that all the solutions of closed-loop system starting fromx_0 and $\omega(0)$ are bounded and $\lim_{t\to\infty} e(t) = 0$. The switching law is given by Figure 6. Therefore, the global output regulation problem is solvable under the switching law (53).

Example 6.2: Consider a switched Resistance Inductor Capacitance (RLC) circuit (Yang et al., 2008) which consists of N input power sources, N resistances R_i and N capacitors C_i that could be switched between each other. The dynamic equations are given by

$$\begin{aligned} \dot{x}_1 &= \frac{1}{L_i}x_2 - \frac{1}{L_i}v_d, \\ \dot{x}_2 &= -\frac{1}{C_i}x_1 - \frac{R_i}{L_i}x_2 + u_i, \\ e_i &= \frac{1}{L_i}x_2 - \frac{1}{L_i}v_d,\ i = 1, 2, \ldots N, \end{aligned} \tag{54}$$

where the two state variables are the charge in the capacitor and the flux in the inductance $x = [q_c, \varphi_L]^T$, the input is the voltage and $v_d = \omega^T\omega$, $\omega = (\omega_1, \omega_2)^T$ denote the external disturbance or as a reference signal generated by the exosystem

$$\dot{\omega} = S\omega, \quad S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \tag{55}$$

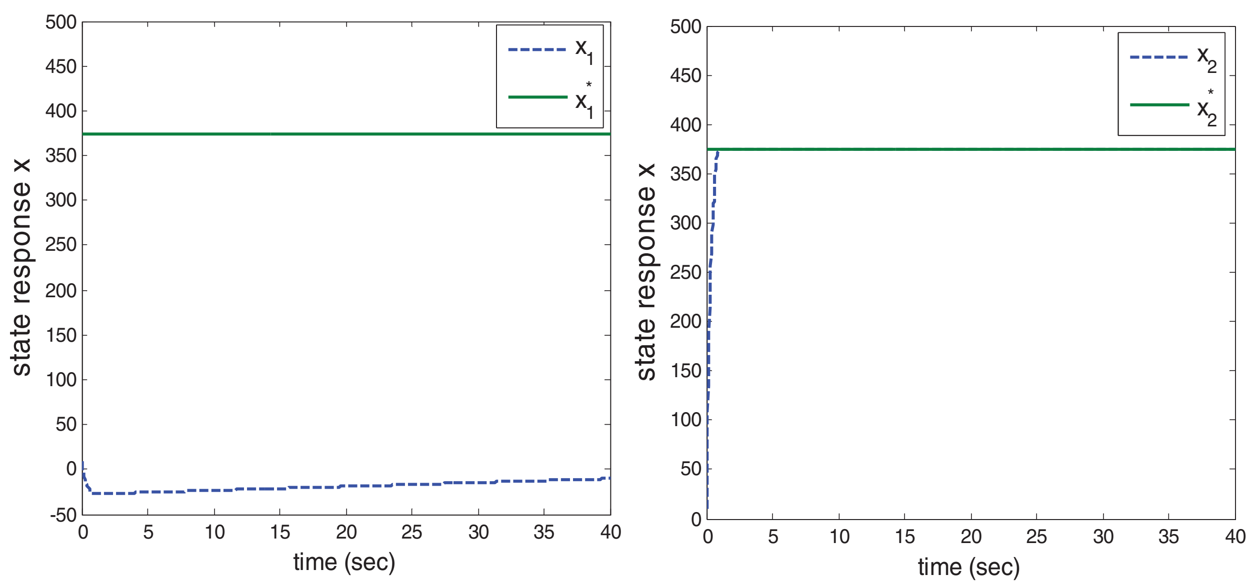

Figure 7. State response of the switched system.

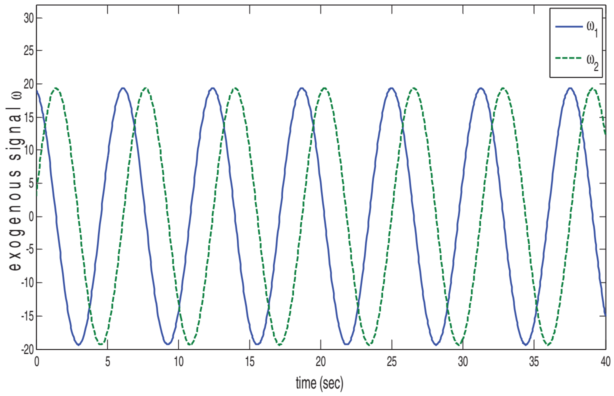

Figure 8. The exogenous signal $\omega(t)$.

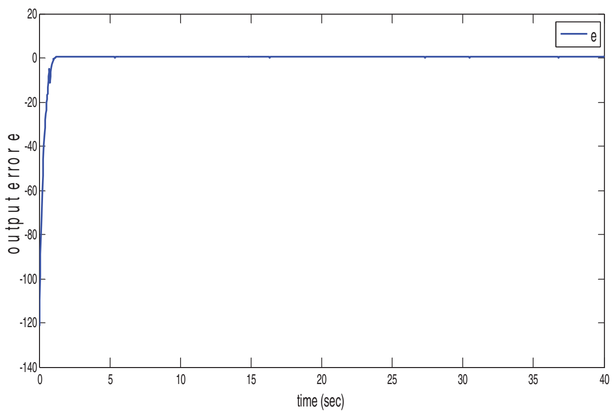

Figure 9. The regulated output of the switched system.

The storage function of each mode is given as

$$S_i = \frac{1}{2C_i}(x_1 - \hat{x}_1)^2 + \frac{1}{2L_i}(x_2 - \hat{x}_2)^2.$$

Consider the case $N = 2$. Set the parameters as $L_1 = 1H$, $L_2 = 3H$, $R_1 = 8\Omega$, $R_2 = 9\Omega$, $C_1 = 100\,\mu$F, $C_2 = 50\,\mu$F. It is easy to verify that two models are incrementally passive, and $\bar{u}_{1\omega} = 8.01(\omega_1^2 + \omega_2^2)$, $\bar{u}_{2\omega} = 3.02(\omega_1^2 + \omega_2^2)$, $x_\omega^*(t) = (\omega_1^2 + \omega_2^2, \omega_1^2 + \omega_2^2)^T$ are solutions of the regulator equations. According to Remark 5.2 and Theorem 5.2, the output regulation problem is solved by the feedback controllers $u_i = \bar{u}_{i\omega} - e_i$, $i = 1, 2$ under the switching law (53).

Let the initial state $x(0) = (9.1, 9.4)$ and $\omega(0) = (19.1, 3.1)$. The simulation results are depicted in Figures 7–10. Seen from Figures 7 and 8, the state of the resulting closed-loop (54) is bounded. The regulated output converges to 0. Therefore, the output regulation is solvable under the switching signal (53). The switching signal is described by Figure 10.

The simulation results well illustrate the effectiveness of the proposed approach. Compared with the other existing methods, the approach presented in this paper has two advantages.

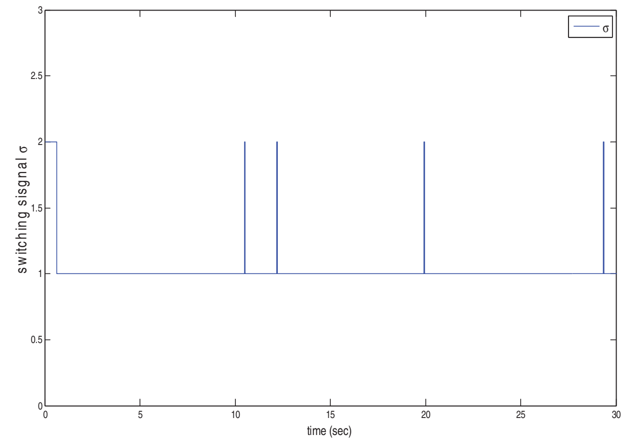

Figure 10. The switching signal of the switched system.

First, the output regulation problem for system (52) can be solved even if the problem for none of the subsystems of system (52) is solvable as shown in Figures 2 and 3, while it is impossible to solve this problem by the common Lyapunov function technique (Niu & Zhao, 2013), the average dwell time approach (Dong & Zhao, 2012a, 2013; Long & Zhao, 2014).

Second, we only have to verify that the regulated outputs of system converge to zero directly by our method, but the multiple Lyapunov function method requires to verify that all the solutions of system converge to the bounded steady-state solution. From Figure 7, the multiple Lyapunov functions method adopted by Dong and Zhao (2012a) is not effective for Example6.2.

7. Conclusions

This paper has investigated incremental passivity and incremental passivity-based output regulation problem for switched nonlinear systems. The designed state-dependent switching law is more general than the well-known min-switching or max-switching. There are many problems that deserve further study. For example, when at least a subsystem is assumed to be incrementally passive, how to design the global regulators is a challenging problem.

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References

Bürger, M., & Persis, C.D. (2015). Dynamic coupling design for nonlinear output agreement and time-varying flow control. *Automatica, 51*, 210–222.

Desoer, C.A., & Vidyasagar, M. (1975). *Feedback systems: Input-output properties*. New York, NY: Academic Press.

Dong, X.X., & Zhao, J. (2012a). Solvability of the output regulation problem for switched non-linear systems. *IET Control Theory and Applications, 6*, 1130–1136.

Dong, X.X., & Zhao, J. (2012b). Incremental passivity and output tracking of switched nonlinear systems. *International Journal of Control, 85*, 1477–1485.

Dong, X.X., & Zhao, J. (2013). Output regulation for a class of switched nonlinear systems: An average dwell-time method. *International Journal of Robust and Nonlinear Control, 23*, 439–449.

Hamadeh, A., Stan, G.B., Sepulchre, R., & Gonçalves, J. (2012). Global state synchronization in networks of cyclic feedback systems. *IEEE Transactions on Automatic Control, 57*, 478–483.

Hespanha, J.P., Liberzon, D., Angeli, D., & Sontag, E.D. (2005). Nonlinear observability notions and stability of switched systems. *IEEE Transactions on Automatic Control, 50*, 154–168.

Jayawardhana, B., & Weiss, G. (2005). Disturbance rejection with LTI internal models for passive nonlinear systems. In *Proceedings of the 16th IFAC World Congress*, 294–299. Prague.

Jayawardhana, B., & Weiss, G. (2008). Tracking and disturbance rejection for fully actuated mechanical systems. *Automatica, 44*, 2863–2868.

Kang, Y., Zhai, D.H., Liu, G.P., & Zhao, Y.B. (2015). On input-to-state stability of switched stochastic nonlinear systems under extended asynchronous switching. *IEEE Transaction on Cybernetics, 46*, 1092–1105.

Kang, Y., Zhai, D.H., Liu, G.P., Zhao, Y.B., & Zhao, P. (2014). Stability analysis of a class of hybrid stochastic retarded systems under asynchronous switching. *IEEE Transaction on Automatic Control, 59*, 1511–1523.

Liberzon, D. (2003). *Switching in systems and control*. Boston, MA: Birkhäuser.

Liu, Y.Y., Stojanovski, G.S., Stankovski, M.J., Dimirovski, G.M., & Zhao, J. (2011). Feedback passification of switched nonlinear systems using storage-like functions. *International Journal of Control Automation and Systems, 9*, 980–986.

Long, L.J., & Zhao, J. (2014). Robust and decentralised output regulation of switched non-linear systems with switched internal model. *IET Control Theory & Applications, 8*, 561–573.

Lu, B., Wu, F., & Kim, S.W. (2006). Switching LPV control of an F-16 aircraft via controller state reset. *IEEE Transaction on Control Systems Technology, 1*, 267–277.

Niu, B., & Zhao, J. (2013). Barrier Lyapunov functions for the output tracking control of constrained nonlinear switched systems. *Systems & Control Letters, 62*, 963–971.

Niu, B., Zhao, X.D., Fan, X.D., & Cheng, Y. (2015). A new control method for state-constrained nonlinear switched systems with application to chemical process. *International Journal of Control, 88*, 1693–1701.

Pang, H.B., & Zhao, J. (2015). Robust passivity, feedback passification and global robust stabilisation for switched nonlinear systems with structural uncertainty. *IET Control Theory & Applications, 9*, 1723–1730.

Pavlov, A., & Marconi, L. (2008). Incremental passivity and output regulation. *Systems and Control Letters, 57*, 400–409.

Pavlov, A., van de Wouw, N., & Nijmeijer, H. (2005). *Uniform output regulation of nonlinear systems: A convergent dynamics approach*. Boston, MA: Birkhauser.

Stan, G.B., & Sepulchre, R. (2007). Analysis of interconnected oscillators by dissipativity theory. *IEEE Transactions on Automatic Control, 52*, 256–270.

Travieso-Torres, J.C., Duarte-Mermoud, M.A., & Sepu′ lveda, D.I. (2007). Passivity-based control for stabilisation, regulation and tracking purposes of a class of nonlinear systems. *International Journal of Adaptive Control and Signal Process, 21*, 582–602.

Willems, J.C. (1972). Dissipative dynamical systems part I: General theory. *Archive for Rational Mechanics and Analysis, 45*, 321–351.

Yang, H., Cocquempot, V., & Jiang, B. (2008). Fault tolerance analysis for switched systems via global passivity. *IEEE Transactions on Circuits and Systems II: Express Briefs, 55*, 1279–1283.

Zames, G. (1966). On the input–output stability of time-varying nonlinear feedback systems, part I: Conditions derived using concepts of loop gain, conicity and positivity. *IEEE Transactions on Automatic Control, 11*, 3–238.

Zefran, M., Bullo, F., & Stein, M. (2001). A notion of passivity for hybrid system. In *Proceedings of the 40th IEEE Conference on Decision and Control*, 768–773. Orlando, FL: IEEE

Zhao, J., & Hill, D. (2008a). Passivity and stability of switched systems: A multiple storage function method. *Systems & Control Letters, 57*, 158–164.

Zhao, J., & Hill, D. (2008b). Dissipativity theory for switched systems. *IEEE Transactions on Automatic Control, 53*, 941–953.

Zhao, J., & Hill, D. (2008c). On stability L_2-gain and H_∞ control for switched systems. *Automatica, 44*, 1220–1232.