A New Solution Concept and Family of Relaxations for Hybrid Dynamical Systems

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Abstract— **Hybrid dynamical systems have proven to be a powerful modeling abstraction, yet fundamental questions regarding their dynamical properties remain. In this paper, we develop a novel solution concept for a class of hybrid systems, which is a generalization of Filippov's solution concept. In the mathematical theory, these *hybrid Filippov solutions* eliminate the notion of Zeno executions. Building on previous techniques for relaxing hybrid systems, we then introduce a family of smooth control systems that are used to approximate this solution concept. The trajectories of these relaxations vary differentiably with respect to initial conditions and inputs, may be numerically approximated using existing techniques, and are shown to converge to the hybrid Filippov solution in the limit. Finally, we outline how the results of this paper provide a foundation for future work to control hybrid systems using well-established techniques from Control Theory.**

I. Introduction

The hybrid dynamical systems framework has been used as an effective modeling technique for a wide range of engineering systems. However, the flexibility the framework provides does not come without its challenges. Despite considerable efforts to extract classic systems theoretic properties from hybrid systems [1], [2], [3], fundamental questions regarding even the existence and uniqueness of their executions remain, as the interplay between their discrete and continuous dynamics is not fully understood.

Zeno executions [4], executions which undergo an infinite number of discrete transitions in finite time, have proven particularly troublesome to analyze. Numerous frameworks have been proposed to regularize [5], relax [6], or otherwise transform Zeno hybrid systems [7], [8] into approximations or continuations which do not display Zeno phenomena. Yet, a single solution concept which directly describes executions past the Zeno point has remained elusive.

Meanwhile, significant progress has been made towards characterizing the topological structure of hybrid systems [1], [6], [9], [8]. In this paper, we consider a class of hybrid systems similar to those analyzed in [1], which may be endowed with the structure of a smooth manifold, or *hybridfold*. In particular, we find these systems appealing in light of the results from [9], which demonstrated hybrid models naturally reduce to this class of hybrid systems near periodic orbits, and broad efforts to control legged robots on low dimensional hybrid models [10].

As our first contribution, we generalize the solution concept of Filippov [11] to this class of hybrid systems. These *hybrid Filippov solutions* are defined using the solution to a single differential inclusion over a smooth topological manifold. These solutions are not defined using discrete transitions, thus the notion of Zeno executions cannot apply to this solution concept. We then relax the problem, introducing a family of smooth, stiff vector fields over the relaxed topology from [6], which can be used to approximate the hybrid Filippov solution in both continuous and discrete time. These relaxed vector fields are an extension of Teixeira's method for regularizing classical Filippov systems [12], and intuitively these vector fields can be thought of as a generalization of the *regularization in space* from [5] and the *smoothing* techniques from [1] and [9]. Convergence guarantees for these *relaxed hybrid trajectories* to the hybrid Filppov solution are provided, suggesting hybrid dynamics are merely the limit of a family of stiff interactions. Numerous well-established control techniques [13], [14] immediately apply to our relaxations, opening new avenues to control hybrid dynamical systems using the results we establish here.

II. Mathematical Notation

In this section we fix mathematical notation used throughout the paper. We assume a strong background in topology and differential geometry. If the reader is unfamiliar with any of the concepts used throughout the paper, they are refered to [15] or [14, Appendix C] for comprehensive introductions to these topics.

Given a set D, ∂D is the boundary of D and $int(D)$ is the interior of D. For a topological space V, we let $\mathcal{B}(V)$ denote all subsets of V. Given a metric space (X,d), we denote the ball of radius δ centered at $x \in X$ by $B^{\delta}(x)$. The 2-norm is our metric of choice for finite-dimensional real spaces, unless otherwise noted. We use $\overline{co}S$ to denote the convex closure of a set S, which is a subset of some vector space V. Given a collection of sets $\{D_i\}_{i\in\mathcal{I}}$ the *disjoint union* of this collection is $\coprod_{i\in\mathcal{I}} D_i = \bigcup_{i\in\mathcal{I}} D_i \times \{i\}$, which is endowed with the piecewise topology. For a topological space $\mathcal{S}$ and a function $f\colon A \to B$, where $A, B \subset \mathcal{S}$, we define the following equivalence relation: $A \sim B = \{(a,b) \in \mathcal{S}\times\mathcal{S}\colon a \in f^{-1}(b)\}$, and denote the set of equivalence classes of $\mathcal{S}$ under $\sim$ by $\frac{\mathcal{S}}{\Lambda_f}$. There is a natural *quotient projection* $\pi\colon \mathcal{S} \to \frac{S}{\Lambda_f}$ taking each $s \in S$ to its equivalence class $[s] \in \frac{S}{\Lambda_f}$ and we endow $\frac{S}{\Lambda_f}$ with the finest topology that makes π continuous [15, Theorem A.27], the *quotient toplogy*. The reader is referred to [6] or [1] for details on how these concepts will used throughout this paper.

A topological space M is said to be a topological n-manifold if it is covered by an atlast of coordinate charts

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$\{U_\alpha, \varphi_\alpha\}_{\alpha \in \mathcal{A}}$, where each $U_\alpha \subset \mathcal{M}$ is open and $\varphi_\alpha \colon U_\alpha \to \mathbb{R}^n$ is a homeomorhpism onto its range. We say that M is a topological manifold with boundary if we instead remove the requirement that U_α is open and we have $\varphi_\alpha \colon U_\alpha \to H^n$, for each $\alpha \in \mathcal{A}$, where $H^n = \{(x_1, \ldots, x_n) \colon x_n \geq 0\}$. We say that a topological manifold is smooth if its coordninate charts are smooth. In particular, when we say the coordinate charts are smooth we mean that $\varphi_i \circ \varphi_j^{-1}$ is a smooth diffeomorphism over $\varphi_j(U_i \cap U_j)$, for $i, j \in \mathcal{A}$ such that $U_i \cap U_j$ is non-empty. Each point $x \in M$ is endowed with a *tangent space* $T_x M$, which is an n-dimensional vector space, and we denote the *tangent bundle* of M as $TM = \coprod_{x \in M} T_x M$.

Inspired by [14, Definition 4.3.1], we define a *control system* to be a 3-tuple $S = (\mathcal{M}, U, F)$ where $\mathcal{M}$ is a n-dimensional topological manifold, $U \subset \mathbb{R}^m$ is a space of admissible inputs, and $F \colon \mathcal{M} \times U \to T\mathcal{M}$ is a vector field defining the dynamics of the system, recalling that $T\mathcal{M}$ is the *tangent bundle* of $\mathcal{M}$. We say that S is a *smooth control system* if $\mathcal{M}$ is smooth topological manifold and F is a smooth map. Throughout the paper, we will consider input signals in the space of piecewise continuous controls, which we denote with $PC([0,T], U)$. The trajectories of smooth control systems are unique, and vary differentiably with respect to their initial conditions and inputs (see e.g. [14, Chapter 4]). Given two smooth n-manifolds M, N and a mapping $P \colon M \to N$, then at each $x \in M$ there is an associated linear map $DP(x) \colon T_x M \to T_{P(x)} N$, known as the *pushforward*. In coordinates, DP is simply the Jacobian of P. If P is a diffeomorphism, then a smooth vector field $F \colon M \times U \to TM$ pushes forward to a unique smooth vector field $DP \circ F \colon N \times U \to TN$. Throughout the paper we use the term *smooth* to mean infinitely differentiable and it is understood that *diffeomorphisms* are smooth mappings.

III. Filippov Solutions

We now briefly introduce Filippov's solution concept [11] for differential equations with discontinuous right-hand sides. Let $g \colon \mathbb{R}^n \to \mathbb{R}$ be a smooth regular map, and let $D_+ = \{x \in \mathbb{R}^n \colon g(x) > 0\}$, $D_- = \{x \in \mathbb{R}^n \colon g(x) < 0\}$, and let $\Sigma = \{x \in \mathbb{R}^n \colon g(x) = 0\}$ be a smooth $(n-1)$-manifold separating D_+ and D_-. Define $f \colon \mathbb{R}^n \times U \to T\mathbb{R}^n$ by

$$f(x,u) = \begin{cases} f_+(x,u) & \text{if } x \in D_+ \\ f_-(x,u) & \text{if } x \in D_-, \end{cases} \tag{1}$$

where f_+ and f_- are smooth globally Lipschitz continuous functions from $\mathbb{R}^n \times U \to T\mathbb{R}^n$, and U is a space of admissible controls. Note that f is undefined and discontinuous along Σ. The *Filippov Regularization* of f is the set-valued map $\mathcal{F}[f] \colon D \times U \to B(TR^n)$, where

$$\mathcal{F}[f](x,u) = \overline{co} \bigcap_{\delta > 0} \bigcap_{\mu(S)=0} f(B^\delta(x) - S, u), \tag{2}$$

and $\bigcap_{\mu(S)=0}$ denotes the intersection over all sets of non-zero measure. We say that a *Filippov solution* for this system on the time interval $[0,T]$, given data $x_0 \in D$ and $u \in PC([0,T], U)$, is an absolutely continuous curve $x \colon [0,T] \to D$ satisfying the differential inclusion with conditions $x(0) = x_0$ and

$$\dot{x}(t) \in \mathcal{F}[f](x(t), u(t)) \text{ a.e } t \in [0,T]. \tag{3}$$

The following is a sufficient condition for the uniqueness of Filippov solutions from [11, Chapter 2.10, Theorem 2].

Lemma 1: Assume that for each $x, u \in \Sigma \times U$ that either $\nabla g^T(x) \cdot f_-(x,u) > 0$ or $\nabla g^T(x) \cdot f_+(x,u) < 0$. Then the Filippov solutions for the discontinuous system (1) are unique.

IV. Hybrid Dynamical Systems

In this section we introduce the class of hybrid dynamical systems considered in this paper. The following definition is inspired by [6].

Definition 1: A hybrid dynamical system is a seven-tuple

$$\mathcal{H} = (\mathcal{J}, \Gamma, \mathcal{D}, \mathcal{U}, \mathcal{F}, \mathcal{G}, \mathcal{R}), \tag{4}$$

where:

- $\mathcal{J}$ is a finite set indexing the discrete states of $\mathcal{H}$;
- $\Gamma \subset \mathcal{J} \times \mathcal{J}$ is the set of edges, forming a graphical structure over $\mathcal{J}$, where edge $e = (j, j') \in \Gamma$ corresponds to a transition from j to j';
- $\mathcal{D} = \{D_j\}_{j \in \mathcal{J}}$ is the set of domains, where $D_j \subset \mathbb{R}^n$ is a smooth n-dimensional manifold with boundary;
- $\mathcal{F} = \{f_j\}_{j \in \mathcal{J}}$ is the set of vector fields, where each $f_j \colon \mathbb{R}^n \times U \to T\mathbb{R}^n$ is smooth and globally Lipschitz continuous, and defines the continuous dynamics of the system on D_j;
- $\mathcal{G} = \{G_e\}_{e=(j,j') \in \Gamma}$ is the set of guards, where each $G_{(j,j')} \subset \partial D_j$ is a smooth embedded $(n-1)$-manifold;
- $\mathcal{R} = \{R_e\}_{e=(j,j') \in \Gamma}$ is the set of reset maps where, for each $e = (j, j') \in \Gamma$, $R_e \colon \mathbb{R}^n \to \mathbb{R}^n$ is smooth and globally Lipschitz continuous and $R_e(G_e) \subset \partial D_{j'}$.

Taking after [9], let us define $G = \coprod_{e \in \Gamma} G_e$ and $D = \coprod_{j \in \mathcal{J}} D_j$, where we note that $\partial D = \coprod_{j \in \mathcal{J}} \partial D_j$. Next, define the map $R \colon G \to \partial D$ by $R(x) = R_e(x)$ for each $x \in G_e$, and $e \in \Gamma$. We endow our hybrid systems with the quotient topology introduced in [1], but borrow our notation from [6]. We define the *hybrid quotient space* to be $\mathcal{M} = \frac{\coprod_{j \in \mathcal{J}} D_j}{\Lambda_R}$. The construction of the hybrid quotient space for a simple two-mode hybrid system is depicted in Figure 1. Intuitively, the hybrid quotient space is constructed by attaching G_e to $R_e(G_e)$, so that $\mathcal{M}$ is a connected topological space. When interpreted on this space, there are no loner any jumps in hybrid trajectories. In fact, trajectories of hybrid systems, to be defined formally in the sequel, are absolutely continuous on $\mathcal{M}$ with respect to the following metric from [6]. Let $d \colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+$ be a metric, then define $d_\mathcal{M} \colon \mathcal{M} \times \mathcal{M} \to \mathbb{R}_+$ for $x, y \in \mathcal{M}$ by

$$d_\mathcal{M}(x,y) = \inf_{k \in \mathbb{N}} \left\{ \sum_{i=1}^{k} d(p_i, q_i) \colon x = p_1, y = q_k, q_i \sim p_{i+1} \right\}. \tag{5}$$

Though we refer the reader to [6] for further details, intuitively, for two points $x, y \in \mathcal{M}$ the distance $d_\mathcal{M}(x,y)$

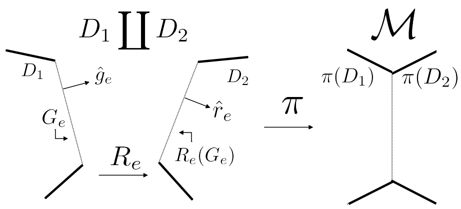

Fig. 1: Construction of the Hybrid Quotient Space (right) from the disjoint union of the continuous domains (left) for a bimodal hybrid system with a single edge.

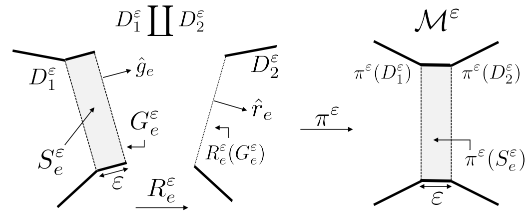

Fig. 2: Construction of the relaxed hybrid quotient space (right) from the disjoint union of the relaxed continuous domains (left) for a bimodal hybrid system with a single edge $e = (1, 2)$.

is simply the shortest curve connecting the two points on $\mathcal{M}$. Throughout the rest of the paper, whenever we refer to trajectories on $\mathcal{M}$ as absoloutely continuous, it is understood to be with respect to this metric. Next, we invoke an assumption inspired by those made in [1] and [9, Section 3] that ensures $\mathcal{M}$ is a smooth manifold [9, Theorem 3].

Assumption 1: The map R is a diffeomorpshism. [1]
As demonstrated by our examples, hybrid models for mechanical systems undergoing elastic impacts satisfy this Assumption. It was also shown in [9] that hybrid systems collapse to (sub)-systems satisfying this hypothesis near periodic orbits. We make one final assumption about our hybrid systems, which is not made in [1] or [9]:

Assumption 2: For each $e = (j, j') \in \Gamma$ there exists unit vectors $\hat{g}_e, \hat{r}_e \in \mathbb{R}^n$ and scalars c_e, d_e such that $G_e \subset \hat{G}_e$ and $R_e(G_e) \subset \hat{R}_e$, where

1) $\hat{G}_e := \{x \in \mathbb{R}^n : g_e(x) := \hat{g}_e^T x - c_e = 0\}$; and,
2) $\hat{R}_e := \{x \in \mathbb{R}^n : r_e(x) := \hat{r}_e^T x - d_e = 0\}$,

and $g_e(x) \leq 0$ for each $x \in D_j$, and $r_e(x) \geq 0$ for each $x \in D_{j'}$.

In other words, each guard set and its image are subsets of $(n-1)$-dimensional planes. As was demonstrated in [16], it is often possible to transform a hybrid system with nonlinear guard sets into an equivalent hybrid systems satisfying Assumption 2 by adding auxiliary continuous states to the hybrid system. Employing this additional Assumptions will enable us to develop a set of techniques which will greatly simplify our analysis of the executions of hybrid systems, as will become apparent in Section VI.

Notation: For the rest of the paper let $\pi : D \to \mathcal{M}$ denote the quotient map induced by R. For each $j \in J$, define $a_j : D_j \to D_{j'} \times \{j\}$ by $a_j(x) = x \times \{j\}$, for each $x \in D_j$. Subsequently, let $\pi_j : D_j \to \pi \circ a_j(D_j)$ be defined by $\pi_j = \pi \circ a_j$, a diffeomorphism which takes each domain from $\mathbb{R}^n$ to the hybrid quotient space. For each $j \in \mathcal{J}$ we define $\mathcal{N}_j = \{e \in \Gamma : \exists j' \in \mathcal{J} \text{ s.t. } e = (j, j')\}$, to be the *neighborhood* of mode j as in [6].

[1]This Assumption has several implications. First, it implies that $G \coprod R(G) = \partial D$. Next, it implies that, for each $e \in \Gamma$, R_e is a diffeomorphism onto its range. Finally, the Assumption implies that for each pair of distinct edges $e, e' \in \Gamma$ that if $G_e \cap G_{e'} \neq \emptyset$, then e and e' can be combined into a single edge $\hat{e}$, where $G_{\hat{e}} = G_e \cup G_{e'}$, and $\hat{R}_{\hat{e}} : G_{\hat{e}} \to \partial D$ is a diffeomorphism onto its range and $R_{\hat{e}}(x) = R_e(x)$ if $x \in G_e$ and $R_{\hat{e}}(x) = R_{e'}(x)$ if $x \in G_{e'}$. Thus, as far as our notation in this paper is concerned, we will proceed as if guards do not overlap.

V. Relaxed Hybrid Topology

We construct our relaxations on the relaxed hybrid topology from [6], which is constructed by attaching an ε-thick strip to each of the guard sets. The novelty of our relaxations arise from the smooth vector fields we impart over this topology, to be defined in Section VII. Here, we simply introduce the relaxed topology from [6] for use later.

Concretely, for each $e \in \Gamma$ we define the *relaxed strip*

$$S_e^\varepsilon := \{p + \hat{g}_e q \in \mathbb{R}^n : p \in G_e \text{ and } q \in [0, \varepsilon]\} \tag{6}$$

and then for each $j \in \mathcal{J}$ define the *relaxed domain* $D_j^\varepsilon = D_j \cup_{e \in \mathcal{N}_j} S_e^\varepsilon$. Next, for each $e = (j, j') \in \Gamma$ we then define the *relaxed guard set*

$$G_e^\varepsilon := \{x \in S_e^\varepsilon : g_e^\varepsilon(x) := \hat{g}_e^T x - (c_e + \varepsilon) = 0\}. \tag{7}$$

We then define $R_e^\varepsilon : \mathbb{R}^n \to \mathbb{R}^n$ by

$$R_e^\varepsilon(x) = R_e(x - \hat{g}_e \varepsilon), \tag{8}$$

and we note that $R_e^\varepsilon(G_e^\varepsilon) = R_e(G_e)$.

Define $D^\varepsilon = \coprod_{j \in \mathcal{J}} D_j^\varepsilon$ and let $G^\varepsilon = \coprod_{e \in \Gamma} G_e^\varepsilon$. Subsequently define $R^\varepsilon : G^\varepsilon \to \partial D^\varepsilon$ by $R^\varepsilon(x) = R_e^\varepsilon(x)$ if $x \in G_e^\varepsilon$, for each $e \in \Gamma$, and then define the *relaxed hybrid quotient space* to be $\mathcal{M}^\varepsilon = \frac{D^\varepsilon}{\Lambda_{R^\varepsilon}}$. The construction of the relaxed hybrid quotient space is depicted in Figure 2.

Proposition 1: Assume $\mathcal{H}$ satisfies Assumption 1. Then, for each $\varepsilon > 0$, $\mathcal{M}^\varepsilon$ is a smooth topological manifold.

Proof: Note that for each $j \in \mathcal{J}$ the domain D_j^ε is a smooth topological manifold with boundary. Furthermore, if Assumption 1 holds for a given hybrid system, then it also holds for its ε-relaxations. Thus, $\mathcal{M}^\varepsilon$ is a smooth topological manifold, by an appeal to [9, Theorem 3]. ■

As shown in [6], a *relaxed distance metric* on $\mathcal{M}^\varepsilon$, which we denote $d_{\mathcal{M}^\varepsilon} : \mathcal{M}^\varepsilon \times \mathcal{M}^\varepsilon \to \mathbb{R}_+$, may be defined analogously to how $d_{\mathcal{M}}$ was defined in (5). We also borrow the following metric between curves (trajectories) on $\mathcal{M}^\varepsilon$ from [6], which will be used to study the convergence of our relaxed trajectories. Let $\gamma_1, \gamma_2 : [0, T] \to \mathcal{M}^\varepsilon$, then define

$$\rho^\varepsilon(\gamma_1, \gamma_2) = \sup\{d_{\mathcal{M}^\varepsilon}(\gamma_1(t), \gamma_2(t)) : t \in [0, T]\}. \tag{9}$$

Notation: Let $\pi^\varepsilon : \coprod D_j^\varepsilon \to \mathcal{M}^\varepsilon$ denote the quotient projection induced by R^ε. For each $j \in \mathcal{J}$, define $a_j^\varepsilon : D_j^\varepsilon \to D_j^\varepsilon \times \{j\}$ by $a_j^\varepsilon(x) = x \times \{j\}$, then let $\pi_j^\varepsilon : D_j^\varepsilon \to \pi^\varepsilon \circ a_j^\varepsilon(D_j^\varepsilon)$ be defined by $\pi_j^\varepsilon = \pi^\varepsilon \circ a_j^\varepsilon$.

VI. HYBRID FILIPPOV SOLUTIONS

In this section, we generalize Filippov's solution concept to our class of hybrid systems. We begin with a number of definitions which will make our intuition for this solution concept clear. First, for each $e = (j, j') \in \Gamma$ we define

$$\Sigma_e \colon = \pi_j(G_e) = \pi_{j'}(R_e(G_e)), \tag{10}$$

and then define

$$D_e \colon = \pi_j(int(D_j)) \bigcup \pi_{j'}(int(D_{j'})) \bigcup \Sigma_e. \tag{11}$$

Note that, as depicted on the right of Figure 4, Σ_e is an $(n-1)$-dimensional manifold, which forms a surface separating the two open sets $\pi_j(int(D_j))$ and $\pi_{j'}(int(D_{j'}))$. We will extend the Filippov regularization to the following vector field over $\mathcal{M}$, the *hybrid vector field*:

Definition 2: Let $\mathcal{H}$ be a hybrid system. We define the *hybrid vector field* to be $F \colon \mathcal{M} \times U \to T\mathcal{M}$ where

$$F(\pi_j(x), u) = D\pi_j \circ f_j(x, u) \text{ if } x \in int(D_j). \tag{12}$$

That is, we simply push forward each vector field f_j onto $\mathcal{M}$ using π_j. Much like the piecewise smooth vector field (1), F is a piecewise smooth vector field, which is discontinuous and undefined along Σ_e, for each $e \in \Gamma$. Thus the tuple $(\mathcal{M}, U, F)$ is a non-smooth control system and we find it natural to extend the Filippov regularization to describe its trajectories.

In order to accomplish this, for each edge $e \in \Gamma$, we will construct a set $\hat{D}_e \subset \mathbb{R}^n$ and a diffeomorphism $\pi_e \colon \hat{D}_e \to D_e$. We will then be able to represent the flow of F on $\hat{D}_e$ using the pushed forward vector field $f_e = D\pi_e^{-1} \circ F|_{D_e \times U}$, and we can subsequently apply the classical Filippov regularization to this local representation of F. In order to define each of these objects, we require some intermediate definitions.

For each $e = (j, j') \in \Gamma$, let $p_e \colon \mathbb{R}^n \to \hat{R}_e$ be defined by $p_e(x) = x - \hat{r}_e r_e(x)$, the Euclidian projection onto $\hat{R}_e$. Next, define the diffeomorphism $P_e \colon \mathbb{R}^n \to \mathbb{R}^n$ by

$$P_e(x) = R_e^{-1} \circ p_e(x) + \hat{g}_e r_e(x), \tag{13}$$

and consider the domain $P_e(D_{j'})$, which we depict in Figure 3, and is the result of smoothly attaching $D_{j'}$ to D_j, by passing $D_{j'}$ through P_e. To understand this action, we first note that, as depicted in Figure 3, each point $x \in D_{j'}$ may be decomposed as $p_e(x) + \hat{r}_e r_e(x)$, where $p_e(x) \in \hat{R}_e$ and $\hat{r}_e r_e(x)$ is a vector of length $r_e(x)$ units in the direction $\hat{r}_e$. The map P_e decomposes x into $p_e(x) + \hat{r}_e r_e(x)$, and sends $p_e(x)$ to $R_e^{-1} \circ p_e(x) \in \hat{G}_e$, while sending $\hat{r}_e r_e(x)$ to $\hat{g}_e r_e(x)$, which is a vector of length $r_e(x)$ units in the direction $\hat{g}_e$. In other words, P_e attaches $\hat{R}_e$ to $\hat{G}_e$ via R_e^{-1}, and rotates the coordinate that is transverse to $\hat{R}_e$ (the direction $\hat{r}_e$) to align with the coordinate that is transverse

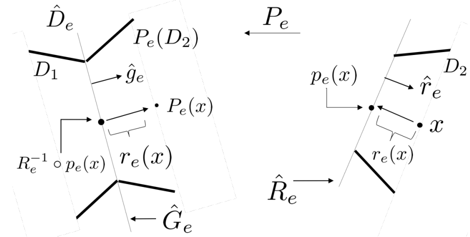

Fig. 3: The domain D_2 is smoothly attached to domain D_1 using the map P_e, resulting in $P_e(D_2)$, where $e = (1, 2)$. The various components of P_e are illustrated.

to $\hat{G}_e$ (the direction $\hat{g}_e$). [2] Finally, for each $e = (j, j') \in \Gamma$, let

$$\hat{D}_e := int(D_j) \cup int(P_e(D_{j'})) \cup G_e. \tag{14}$$

Proposition 2: For each $e = (j, j') \in \Gamma$ the the mapping $\pi_e \colon \hat{D}_e \to D_e$, where

$$\pi_e(x) = \begin{cases} \pi_j(x) & \text{if } x \in int(D_j) \cup G_e \\ \pi_{j'} \circ P_e^{-1}(x) & \text{if } x \in int(P_e(D_{j'})) \end{cases} \tag{15}$$

is a diffeomorphism.

Proof: The argument largely parallels the proof of [15, Theorem 9.29], with a mild refactoring of notation. In particular, it is easily verified that π_e is smooth and full rank on $int(D_j)$ and $int(P_e(D_{j'}))$, since $\pi_j|_{int(D_j)}$, $\pi_{j'}|_{int(P_e(D_{j'}))}$ and P_e are all diffeomorphisms. Continuity of π_e is also easy to establish, as if $x \in \hat{G}_e$ then $\pi_j(x) = \pi_{j'} \circ R_e(x) = \pi_{j'} \circ P_e^{-1}(x)$, which follows since $P_e^{-1}|_{\hat{G}_e} = R_e|_{\hat{G}_e}$. We are now left to determining the smoothness of π_e along Σ_e, which follows by an argument analogous to the one used to establish the smoothness of the map $\tilde{\Phi}$ in [15, Theorem 9.29]. In particular, we note that if V is any open set in D_e, then $(V, \pi_e^{-1}|V)$ is a smooth chart on $\mathcal{M}$. ∎

Other authors [9], [1] have demonstrated that it is theoretically possible to smoothly attach one domain of a hybrid system to another. However, to the best of our knowledge, we provide the first explicit representation of the diffeomorphisms (15) required for this proccess, which we are able to construct largely by virtue of Assumption 2. When either G_e or $R_e(G_e)$ is a general non-linear surface, it may not be possible to write down a closed-form expression for the projections needed to construct P_e or its inverse, and implicit techniques [1, Lemma 2.8], [6, Theorem 3] are required.

Resuming our construction, for each $e = (j, j') \in \Gamma$ define the piecewise smooth vector field $f_e \colon \hat{D}_e \times U \to T\mathbb{R}^n$ by

$$f_e(x, u) = D\pi_e^{-1} \circ F(\pi_e(x), u). \tag{16}$$

[2] It is easy to verify that P_e is a diffeomorphism, since each of its terms are smooth and it has a closed-form inverse $P_e^{-1}(x) = R_e \circ \mu_e(x) + \hat{r}_e g_e(x)$, where $\mu_e(x) = x - \hat{g}_e g_e(x)$ is the Euclidian projection onto $\hat{G}_e$. Indeed, note that $P_e^{-1} \circ P_e(x) = R_e \circ \mu_e(R_e^{-1} \circ p_e(x) + \hat{g}_e r_e(x)) + \hat{r}_e g_e(R_e^{-1} \circ p_e(x) + \hat{g}_e r_e(x))$, but $\mu_e(R_e^{-1} \circ p_e(x) + \hat{g}_e r_e(x)) = \mu_e(R_e^{-1} \circ p_e(x))$ and $g_e(R_e^{-1} \circ p_e(x) + \hat{g}_e r_e(x)) = r_e(x)$, thus $P_e^{-1} \circ P_e(x) = R_e \circ \mu_e(R_e^{-1} \circ p_e(x)) + \hat{r}_e r_e(x) = p_e(x) + \hat{r}_e r_e(x) = x$, as desired.

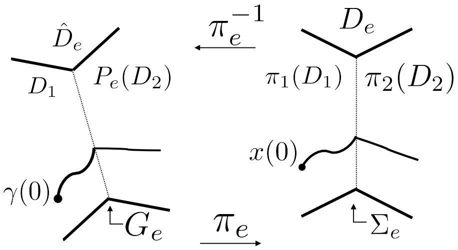

Fig. 4: A hybrid Filippov solution x with initial condition $x(0)$ flows from one domain to another, crossing Σ_e, where $e = (1, 2)$. This flow is diffeomorphic to the curve γ (which is a classical Filippov solution for $\mathcal{F}[f_e]$ with initial condition $\gamma(0) = \pi_e^{-1}(x(0))$) where we have $x = \pi_e \circ \gamma$.

By appropriately evaluating the arguments of $D\pi_e^{-1}$ one can obtain the following explicit representation of f_e:

$$f_e(x, u) = \begin{cases} f_j(x, u) & \text{if } x \in int(D_j) \\ DP_e \circ f_{j'}(P_e^{-1}(x), u) & \text{if } x \in int(P_e(D_{j'})). \end{cases} \tag{17}$$

We now define the *hybrid Filippov regularization*, using the classical Filippov regularizations of the vector fields $\{f_e\}_{e\in\Gamma}$ and the maps $\{\pi_e\}_{e\in\Gamma}$.

Definition 3: Let $\mathcal{H}$ be a hybrid system. The *hybrid Filippov regularization* of F is the set-valued map $\widehat{\mathcal{F}}[F] : \mathcal{M} \times U \to \mathcal{B}(T\mathcal{M})$ where

$$\widehat{\mathcal{F}}[F](\pi_e(x), u) = D\pi_e(\mathcal{F}[f_e](x, u)) \text{ if } x \in \hat{D}_e. \tag{18}$$

In other words, the hybrid Filippov regularization is constructed by taking the classical Filippov regularizations of the vector fields $\{f_e\}_{e\in\Gamma}$, and then pushing each element of the resulting set-valued maps forward to $T\mathcal{M}$.

Definition 4: Let $x_0 \in \mathcal{M}$ and $u \in PC([0, T], U)$. We say that an absolutely continuous curve $x : [0, T] \to \mathcal{M}$ is a *hybrid Filippov solution* for this data if $x(0) = x_0$ and

$$\dot{x}(t) \in \widehat{\mathcal{F}}[F](x(t), u(t)), \text{ a.e. } t \in [0, T]. \tag{19}$$

Example 1: (Bouncing Ball) Consider the following simplified model of a ball that is bouncing vertically and loses a fraction of its energy during each bounce, which we borrow from [1]. The ball has two identical modes, $\mathcal{J}_{bb} = \{1, 2\}$. [3] For $j \in \{1, 2\}$ the continuous dynamics are given by

$$D_j = \{(x_1, x_2) : x_1 \geq 0\} \text{ and } f_j(x_1, x_2) = (x_2, -g)^T,$$

where g is the gravitational constant. Each mode has a single edge leaving it to the other mode:

$$G_{(j,1-j)} = \{(x_1, x_2) : x_1 = 0,\ x_2 \leq 0\} \text{ and}$$
$$R_{(j,1-j)}(x_1, x_2) = (x_1, -cx_2)^T,$$

where $c \in (0, 1]$ is the *coefficient of restitution.*

The hybrid quotient space for the bouncing ball, $\mathcal{M}_{bb}$, as well as a hybrid Filippov solution for this system are depicted on the left in Figure 5, for $c < 1$. The trajectory flows between the two modes an infinite number of times by

[3]As noted in [1], the use of two identical modes ensures the hybrid quotient space for the bouncing ball is a smooth manifold.

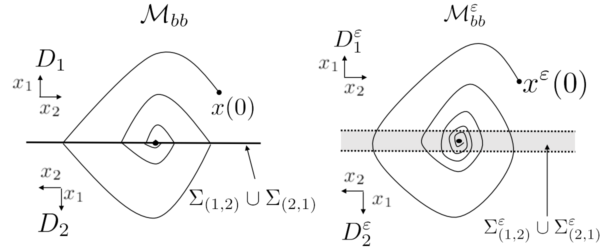

Fig. 5: The hybrid quotient space $\mathcal{M}_{bb}$ (left) and relaxed hybrid quotient space $\mathcal{M}_{bb}^{\varepsilon}$ (right) for the bouncing ball, where $c \in (0, 1)$. A hybrid Filippov Solution and a relaxed hybrid trajectory are depicted. For both figures, the axes denote the orientation of the states in each mode.

some finite time t_∞ (see [5] for details), before coming to rest at $\pi_1((0, 0)) = \pi_2((0, 0))$ for all $t \geq t_\infty$. In other words, previous notions of hybrid executions (e.g. [5], [2], [1], [6], [3]) for the bouncing ball are *Zeno*, as they require an infinite number of reset map evaluations to define. On the other hand, the hybrid Filippov solution defines such trajectories using the solution of a single differential inclusion. [4] Of course, in practice constructing such trajectories poses numerous challenges, as even classical Filippov solutions are non-trivial to simulate [17], motivating the construction of our relaxations in the following section. We conclude this section by presenting conditions under which the Hybrid Filippov solution is unique.

Assumption 3: Let $\mathcal{H}$ be a hybrid system. For each $e = (j, j')$ and $(x, u) \in G_e \times U$ either $\hat{g}_e^T \cdot f_j(x, u) > 0$ or $\hat{r}_e^T \cdot f_{j'}(R_e(x), u) < 0$.

Remark 1: For a given edge $e = (j, j') \in \Gamma$, by carefully inspecting P_e, one observes that if Assumption 3 holds then the hypothesis of Lemma 1 is saticefied for f_e. Intuitively, this follows from the fact that P_e rotates vectors in the direction $\hat{r}_e$ to align with the vector $\hat{g}_e$.

Theorem 1: Let $\mathcal{H}$ be a hybrid system satisfying Assumption 3. Then the hybrid Filippov solutions of $\mathcal{H}$ are unique.

Proof: Let $x : [0, T] \to \mathcal{M}$ be a hybrid Filippov solution for some data $x_0 \in \mathcal{M}$ and $u \in PC([0, T], U)$. By our construction of $\{\pi_e\}_{e\in\Gamma}$ and $\{f_e\}_{e\in\Gamma}$, there is a diffeomorphic correspondence between each segment of the curve x and a segment of a classical Filippov solution for one of the regularizations $\{\mathcal{F}[f_e]\}_{e\in\Gamma}$, as depicted in Figure 4. By Remark 1 we conclude that each of these segments is unique. ■

Note that the conditions for uniqueness in Theorem 1 are sufficient but not necessary. For example, though we do not prove it formally, it is possible to show that the boucing ball admits a unique, infinite hybrid Filippov solution, even though careful inspection reveals it does not satisfy the hypothesis of Theorem 1.

[4]This solution concept should not be conflated with solutions of the "Hybrid Inclusion" [3], which defines solutions using a (possibly infinite) sequence of flows and jumps.

VII. Relaxed Hybrid Vector Fields

In this section we introduce the *relaxed hybrid vector fields* which we use to approximate the hybrid Filippov solution. This family of vector fields will be constructed by extending Teixeira's method [12], which approximates classical Filippov solutions using a family of smooth, stiff vector fields. The result of this relaxation will be a vector field F^ε such that the tuple $(\mathcal{M}^\varepsilon, U, F^\varepsilon)$ is a *smooth control system*. We begin by defining relaxed analogues to some of the objects defined in the previous section. First, for each edge $e = (j, j') \in \Gamma$, we define

$$\Sigma_e^\varepsilon = \pi_j^\varepsilon(S_e^\varepsilon), \tag{20}$$

and then define

$$D_e^\varepsilon \colon = \pi_j^\varepsilon(D_j) \bigcup \pi_{j'}^\varepsilon(D_{j'}) \bigcup \Sigma_e^\varepsilon. \tag{21}$$

As depicted in Figure 6, $\Sigma_{(j,j')}^\varepsilon$ forms a strip separating $\pi_j(D_j)$ and $\pi_{j'}(D_{j'})$. The main idea behind our relaxation technique is to smoothly transition between the dynamics of mode j and the dynamics of mode j' along $\Sigma_{(j,j')}$. For each edge $e = (j, j')$ we define the diffeomorphism $P_e^\varepsilon \colon \mathbb{R}^n \to \mathbb{R}^n$ by

$$P_e^\varepsilon(x) = (R_e^\varepsilon)^{-1} \circ p_e(x) + \hat{g}_e r_e(x), \tag{22}$$

and consider the domain $P_e^\varepsilon(D_{j'})$, the result of attaching $D_{j'}$ to D_j^ε via P_e^ε, which is depicted on the left of Figure 6. [5] We then define

$$\hat{D}_e^\varepsilon := int(D_j) \cup int(P_e^\varepsilon(D_{j'})) \cup S_e^\varepsilon. \tag{23}$$

The proof of the following result is analogous to that of Proposition 2.

Proposition 3: For each $e = (j, j') \in \Gamma$ the mapping $\pi_e^\varepsilon \colon \hat{D}_e^\varepsilon \to D_e^\varepsilon$ where

$$\pi_e^\varepsilon(x) = \begin{cases} \pi_j^\varepsilon(x) & \text{if } x \in int(D_j^\varepsilon) \cup S_e^\varepsilon \\ \pi_{j'}^\varepsilon \circ (P_e^\varepsilon)^{-1}(x) & \text{if } x \in int(P_e^\varepsilon(D_{j'})). \end{cases} \tag{24}$$

is a diffeomorphism.

Next, we will construct a set of smooth vector fields $\{f_e^\varepsilon\}_{e\in\Gamma}$ where $f_e^\varepsilon \colon \hat{D}_e^\varepsilon \times U \to T\mathbb{R}^n$, that will be used with the maps $\{\pi_e^\varepsilon\}_{e\in\Gamma}$ to define F^ε. We use the following set of functions to smoothly transition between the dynamics of neighboring modes along the relaxed strips:

Definition 5: [18] We say that $\varphi \colon \mathbb{R} \to [0, 1]$ is a *transition function* if it is smooth and

1) $\varphi(a) = 0$ if $a \leq 0$;
2) $\varphi(a) = 1$ if $a \geq 1$; and,
3) φ is monotonically increasing on $(0, 1)$.

For the rest of the paper, we assume a single transition function has been chosen. Then, for each $e = (j, j') \in \Gamma$ and $\varepsilon > 0$ let us define $\varphi_e^\varepsilon \colon \mathbb{R}^n \to [0, 1]$ by

$$\varphi_e^\varepsilon(x) = \varphi\left(\frac{g_e(x)}{\varepsilon}\right), \tag{25}$$

and then define $f_e^\varepsilon \colon \hat{D}_e^\varepsilon \times U \to T\mathbb{R}^n$ by

$$f_e^\varepsilon(x, u) = (1 - \varphi_e^\varepsilon(x)) f_j(x, u) + \varphi_e^\varepsilon(x) DP_e \circ f_{j'}\big((P_e^\varepsilon)^{-1}(x), u\big). \tag{26}$$

For edge $e = (j, j')$, note that when $g_e(x) \leq 0$ (and $x \in D_j$) we have $f_e^\varepsilon(x, u) = f_j(x, u)$. When $g_e(x) \geq \varepsilon$ (and $x \in P_e(D_{j'})$) we have that $f_e^\varepsilon(x, u) = DP_e^\varepsilon \circ f_j((P_e^\varepsilon)^{-1}(x), u)$. And finally when $0 \leq g_e(x) \leq \varepsilon$ (and $x \in S_e^\varepsilon$) f_e^ε produces a convex combination of these two vector fields. The following result follows from the main construction in [12].

Lemma 2: For each $e \in \Gamma$ the vector field f_e^ε is smooth.

We are now ready to define the relaxed vector field we impart over $\mathcal{M}^\varepsilon$, for each $\varepsilon > 0$.

Definition 6: Let $\mathcal{H}$ be a hybrid system. We define the *relaxed hybrid vector field* $F^\varepsilon \colon \mathcal{M}^\varepsilon \times U \to T\mathcal{M}^\varepsilon$ by

$$F^\varepsilon(\pi_e^\varepsilon(x), u) = D\pi_e^\varepsilon \circ f_e^\varepsilon(x, u) \text{ if } x \in \hat{D}_e^\varepsilon. \tag{27}$$

In other words, F^ε is constructed by pushing forward the smooth vector fields $\{f_e^\varepsilon\}_{e\in\Gamma}$ to $\mathcal{M}^\varepsilon$. Note that $F^\varepsilon(\pi_j^\varepsilon(x), u) = D\pi_j^\varepsilon \circ f_j(x, u)$ if $x \in D_j$.

Theorem 2: Let $\mathcal{H}$ be a hybrid system. Then the tuple $(\mathcal{M}^\varepsilon, U, F^\varepsilon)$ is a smooth control system.

Proof: The result follows from a straightforward application of Proposition 3 and Lemma 2. ∎

The construction of the relaxed vector field F^ε can be thought of as a generalization of the *regularization in space* introduced in [5]. This vector field also approximates the *smoothing* discussed in [1, Section 5] and [9, Section 3], but we emphasize that we provide a constructive, explicit means to accomplish this smoothing, unlike either of these works. We also consider hybrid systems with continuous control inputs, a necessary development for the Control and Robotics communities.

We define *relaxed hybrid trajectories* as flows of F^ε:

Definition 7: Given $x_0 \in \mathcal{M}$ and $u \in PC([0, T], U)$, we say that the absolutely continuous curve $x^\varepsilon \colon [0, T] \to \mathcal{M}^\varepsilon$ is a *relaxed hybrid trajectory* for this data if $x(0) = x_0$ and

$$\dot{x}^\varepsilon(t) = F^\varepsilon(x^\varepsilon(t), u(t)), \ \forall t \in [0, T]. \tag{28}$$

Relaxed hybrid trajectories are unique and vary differentiably with respect to their initial conditions and inputs, due to $(\mathcal{M}^\varepsilon, U, F^\varepsilon)$ being a smooth control system, as noted in Section II. As depicted in Figure 6, each segment of a relaxed hybrid trajectory may be explicitly constructed using an integral flow of f_e^ε and the map π_e^ε, for some $e \in \Gamma$. The algorithmic technique for integrating vector fields on the relaxed hybrid quotient space presented in [6] can be used to explicitly construct a full relaxed hybrid trajectory. We close this section by studying the convergence of relaxed hybrid trajectories to the hybrid Filippov solution. We leave the proofs of the following results to the Appendix, as they rely on supportive lemmas contained therein.

Theorem 3: Let Assumption 3 holds for $\mathcal{H}$. Let $x_0 \in \mathcal{M}$ and $u \in PC([0, T], U)$, and let $x \colon [0, T] \to \mathcal{M}$ be the corresponding hybrid Filippov solution, guaranteed to be unique by Theorem 1. Let $x^\varepsilon \colon [0, T] \to \mathcal{M}^\varepsilon$ be the relaxed

[5] It is again easy to check that P_e^ε is a diffeomorphism, whose inverse is $(P_e^\varepsilon)^{-1}(x) = R_e^\varepsilon \circ \mu_e^\varepsilon(x) + \hat{r}_e g_e^\varepsilon(x)$, where $\mu_e^\varepsilon(x) = x - \hat{g}_e g_e^\varepsilon(x)$.

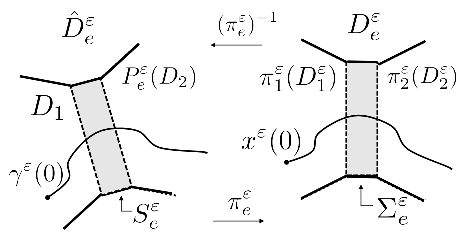

Fig. 6: A relaxed hybrid trajectory x^ε with initial condition $x^\varepsilon(0)$ flows from one domain to another, crossing Σ_e^ε, where $e=(1,2)$. This flow is diffeomorphic to the curve γ^ε (which is a solution of the vector field f_e^ε with initial condition $\gamma^\varepsilon(0)=(\pi_e^\varepsilon)^{-1}(x^\varepsilon(0)))$ where we have $x^\varepsilon=\pi_e^\varepsilon\circ\gamma^\varepsilon$.

hybrid trajectory corresponding to this data. Then $\exists C>0$ such that $\rho^\varepsilon(x,x^\varepsilon)\le C\varepsilon$ for each ε small enough.

In other words, when Assumption 3 is satisfied, and the hybrid Filippov solution is unique, relaxed hybrid trajectories converge to the hybrid Filippov solution at a rate that is linear in ε. Next, we demonstrate that our relaxations always converge to a unique, well-defined limit even when the corresponding hybrid Filippov solution may be non-unique.

Theorem 4: For $x_0\in\mathcal{M}$ and $u\in PC([0,T],U)$, let $x^\varepsilon\colon[0,T]\to\mathcal{M}^\varepsilon$ be the resulting relaxed hybrid trajectory, for each $\varepsilon>0$. Then there exists an absolutely continuous $x^0\colon[0,T]\to\mathcal{M}$ such that $\lim_{\varepsilon\to 0}\rho^\varepsilon\big(x^\varepsilon,x^0\big)=0$.

Due to the uniqueness of this limit, we have found it convenient to think of hybrid systems as the limit of our relaxations, analogously to [8]. In Figure 5 we depict the relaxed hybrid quotient space of the bouncing ball, $\mathcal{M}_{bb}^\varepsilon$, and a relaxed hybrid trajectory evolving on this space. The trajectory spirals towards an equilibrium point on the relaxed strips separating the two domains. Although we do not prove it formally, it can be shown that the relaxed trajectories for the bouncing ball converge to the corresponding hybrid Filippov solution as one takes $\varepsilon\to 0$, even though Theorem 3 does not apply to this system. A similar result holds for our second example presented in the following section.

VIII. Numerical Implementation

In this section we demonstrate how to implement the of numerical approximations for relaxed hybrid systems in [6] for the smooth vector fields we introduced in the previous section. Before defining these integrators, we first require some additional notation.

For each $j\in\mathcal{J}$ we then define

$$\hat{D}_j^\varepsilon = D_j^\varepsilon \bigcup_{e=(j,j')\in\mathcal{N}_j} P_e^\varepsilon(D_{j'}), \tag{29}$$

then define $\hat{\pi}_j^\varepsilon\colon\hat{D}_j\to\cup_{e\in\mathcal{N}_j}D_e$ by

$$\hat{\pi}_j^\varepsilon(x)=\begin{cases}\pi_j^\varepsilon(x) & \text{if } x\in D_j^\varepsilon\\ \pi_{j'}^\varepsilon\circ(P_e^\varepsilon)^{-1}(x) & \text{if } x\in P_e(D_{j'}),\end{cases} \tag{30}$$

and finally define $\hat{f}_j^\varepsilon\colon\hat{D}_j^\varepsilon\times U\to T\mathbb{R}^n$ by

$$\hat{f}_j^\varepsilon(x,u)=f_e^\varepsilon(x,u) \text{ if } x\in D_e^\varepsilon,\ \forall e\in\mathcal{N}_j. \tag{31}$$

These constructions will prove use full since we may not *a priori* which of the edges in $\mathcal{N}_j$ a trajectory might leave mode j through. We will extend the following class of integrators to construct numerical approximations for relaxed hybrid trajectories.

Definition 8: [6] Given a hybrid system $\mathcal{H}$, we say $\mathcal{A}\colon\mathbb{R}^n\times U\times\mathcal{J}\times\mathbb{R}\to\mathbb{R}^n$ is a numerical integrator of order ω, if for each $j\in\mathcal{J}$ and $h=T/N$ (where $N\in\mathbb{N}$), and each $x_0\in D_j$ and $u\in PC([0,T],U)$ we have

$$\sup\big(\big\|x(kh)-z^{\varepsilon,h}(kh)\big\|\colon k\in\{0,1,\dots,N\}\big)=O(h^\omega),$$

where $x(0)=x_0$ and $\frac{\mathrm{d}}{\mathrm{d}t}x=f_j^\varepsilon(x,u)$, and $z^{\varepsilon,h}(0)$ and $z^{\varepsilon,h}((k+1)h)=\mathcal{A}(z(kh),u(kh),j,h)$.

As was noted in [6], this definition of a numerical integrator is compatible with a large class of discretization schemes, including Euler and the Runge-Kutta family.

Definition 9: Given a relaxed hybrid system $\mathcal{H}$, initial condition $\pi_j^\varepsilon(x_0)\in\pi_j^\varepsilon(D_j)$, input $u\in PC([0,T],U)$, step size $h=\frac{T}{N}$ (where $N\in\mathbb{N}$), we construct the discrete approximation $z^{\varepsilon,h}\colon[0,t]\to\mathcal{M}^\varepsilon$ according to the following algorithm.

1) Let $z^{\varepsilon,h}(0)=x_0$, $t=0$, $k=0$ and $j\in\mathcal{J}$.
2) If $k=N$, terminate the execution. Otherwise, let $\gamma^{k+1}=\mathcal{A}\big(z^{\varepsilon,k}(kh),u(kh),j,h\big)$.
3) For each $t\in\big[kh,(k+1)h\big)$ set $z^{\varepsilon,h}(t)=\pi_j^\varepsilon\Big(\frac{(k+1)h-t}{h}\gamma^{k+1}+\frac{t-kh}{h}z^{\varepsilon,h}(kh)\Big)$.
4) If $\gamma^{k+1}\notin\hat{D}_j^\varepsilon$, then let $\bar{t}=\inf\Big\{t\colon z^{\varepsilon,h}(t)\in\hat{D}_j^\varepsilon\Big\}$ and return $z^{\varepsilon,h}|_{[0,\bar{t}]}$. Terminate the execution.
5) If $\exists e=(j,j')\in\gamma$ such that $\gamma^{k+1}\in D_{j',e}^\varepsilon$, set $z^{\varepsilon,h}((k+1)h)=(P_e^\varepsilon)^{-1}(\gamma^{k+1})$, set k = k+1, and set $j=j'$. Go to step 2.
6) Otherwise, set $z^{\varepsilon,h}((k+1)h)$ and k = k+1. Go to step 2.

We also recover the convergence results from [6] for completeness.

Theorem 5: Let $\mathcal{H}$ be a hybrid system. For a given initial condition $x_0\in D_j$ and input $u\in PC([0,T],U)$, let x^ε the corresponding relaxed trajectory, and let $z^{\varepsilon,h}$ be its numerical approximation. Then, $\exists C>0$ such that $\rho\big(x^\varepsilon,z^{\varepsilon,h}\big)\le Ch^\omega$ for each h small enough.

Proof: The proof is analogous to that of Theorem 3, except that instead of continually appealing to Lemma 3, we note that we incur an numerical error that of order h^ω each time we integrate one of the vector fields $\Big\{\hat{f}_j^\varepsilon\Big\}_{j\in\mathcal{J}}$, by the convergence of A on each of the domains $\Big\{\hat{D}_j\Big\}_{j\in\mathcal{J}}$. ■

In other words, numerical approximations of relaxed hybrid systems retain the convergence rate of the integrator $\mathcal{A}$. Furthermore, when Assumption 3 is satisfied, our discrete approximations converge to the hybrid Filippov solution.

Corollary 1: Let $\mathcal{H}$ be a hybrid system that satisfies Assumption 3. For data $x_0\in\mathcal{M}$ and $u\in PC([0,T],U)$, let $x\colon[0,T]\to\mathcal{M}$ be the unique hybrid Filippov solution for this data. For each $\varepsilon>0$ and $h>0$ let $z^{\varepsilon,h}\colon[0,T]\to\mathcal{M}^\varepsilon$ be the numerical approximation of a relaxed execution corresponding to this data. Then $\lim_{\varepsilon\to 0}\lim_{h\to 0}\rho^\varepsilon\big(x,z^{\varepsilon,h}\big)=0$.

Proof: By a straightforward Application of the triangle equality and Theorems 3 and 5 we have that

$$\rho^{\varepsilon}\left(x, z^{\varepsilon,h}\right) \leq C(\varepsilon + h^{\omega}) \tag{32}$$

for some $C > 0$. The desired result follows by taking the appropriate limits. ∎

Note that the proof of Corollary 1 also demonstrates the rate of convergence. In cases where the hybrid Filippov solution may be non-unique, our discrete approximations still converge to a well-defined limit.

Corollary 2: Let $\mathcal{H}$ be a hybrid system. For data $x_0 \in \mathcal{M}$ and $u \in PC([0,T],U)$, let $x\colon [0,T] \to \mathcal{M}$ let $x^0\colon [0,T]\mathcal{M}$ be the limiting trajectory in Theorem 4. For each $\varepsilon > 0$ and $h > 0$ let $z^{\varepsilon,h}\colon [0,T] \to \mathcal{M}^{\varepsilon}$ be the numerical approximation of a relaxed execution corresponding to this data. Then $\lim_{\varepsilon\to 0}\lim_{h\to 0}\rho^{\varepsilon}\left(x^0, z^{\varepsilon,h}\right) = 0$.

Proof: The result follows from Theorems 4 and 5 and taking the appropriate limits. ∎

It should be noted that since the vector fields $\left\{f_j^{\varepsilon}\right\}_{j\in\mathcal{J}}$ are *stiff*, thus in practice in practice we expect the constant C in Theorem 5 to be large. However, there are two practical ways to overcome this issue. The first strategy is to simply use a high-order integrator to, such as a member of the Runge-Kutta family, to offset integrating vector fields with large Lipschitz constants. The other strategy is to reduce the step-size of the integrator when the discrete approximation is in a relaxed strip. Fortunately, these two effects compound.

Example 2: (Double Pendulum With a Mechanical Stop)Again, the double pendulum has two identical modes, $\mathcal{J}_{dp} = \{1,2\}$. For $j \in \{1,2\}$ the continuous dynamics are given by

$$D_j = \{x \in \mathbb{R}^x\colon x_1 \geq 0\} \text{ and } f_j(x) = f_L(x)^T,$$

where the state is ordered $x = \left(\theta_1, \dot{\theta}_1, \theta_2, \dot{\theta}_2\right)$, and an explicit representation of the *Lagrangian dynamics* prescribed by f_L may be found in [7]. Each mode has a single edge leaving it to the other mode:

$$G_{(j,1-j)} = \left\{x \in R^4\colon x_3 = 0,\ x_4 \leq 0\right\} \text{ and}$$
$$R_{(j,1-j)}(,) = (x_1, \dot{\theta}_1 + Mc\dot{\theta}_2, x_3, -cx_4)^T,$$

where $c \in (0,1]$ is the *coefficient of restitution*. We formally define the corresponding hybrid system in [19]. It was shown in [7] that when $c \in (0,1)$ classical constructions of hybrid execution for this system yield Zeno trajectories near points where $\theta_2 = \dot{\theta}_2 = 0$ and $\theta_1 \leq 0$. Physically, such trajectories correspond to the second arm being locked in place against the stop for a time by an imaginary force. The results of a numerical simulation of the system is depicted on the right of Figure 7. Time steps where the simulation is in a relaxed strip are colored black. The second arm is repeatedly locked into place (during intervals when the blackened time-steps accumulate) until the imaginary force dissipates, at which point the second arm swings freely again. In other words, our relaxation procedure automatically recovers the *completion* of *Lagrangian Hybrid Systems* [7], but does so using a smooth dynamical system.

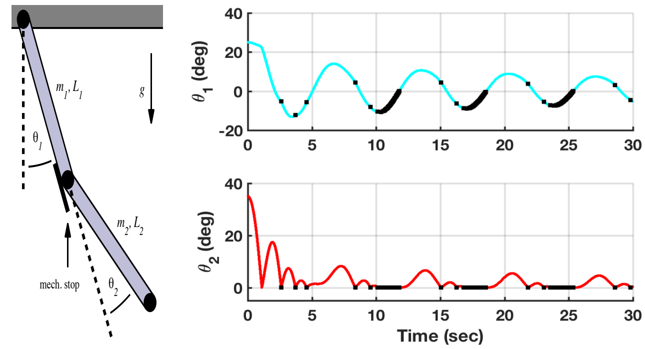

Fig. 7: Schematic of the double pendulum with a mechanical stop (left), and a numerical simulation for this system (right) using Euler integration on each domain with system parameters $c = 0.5$, and $m_1 = m_2 = l_1 = l_2 = g = 1$ with initial condition $(\theta_1(0), \dot{\theta}_1(0), \theta_2(0), \dot{\theta}_2(0)) = (25°, 0, 35°, 0)$, and simulation parameters $\varepsilon = 10^{-6}$ and a step-size of $h = 10^{-6}$.

IX. Conclusion and Future Work

In this paper we developed a novel solution concept for hybrid systems, which is a generalization of Filippov's method, and allows us to described each of the trajectories of the system using a single differential inclusion. We then demonstrated that these trajectories can be approximated using the flows of a parameterized family of *smooth control systems*, and discussed how these two solution concepts compare in the limit. Controller synthesis for smooth control systems is an established discipline [13], [14] providing a new angle to control hybrid systems using our relaxations. In particular, we are currently investigating algorithmic techniques for explicitly computing variations on relaxed trajectories, developing gradient-based techniques (see e.g. [20]) to solve optimal control problems over our relaxations, and constructing feedback controllers to stabilize mechanical systems undergoing impacts.

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APPENDIX

This Appendix contains proofs for Theorems 3 and 4. Each proof relies on a supportive lemma, which we develop before each of the main results.

A. Proof of Theorem 3

Lemma 3: Let $e = (j, j') \in \Gamma$ and assume that the hypothesis of Lemma 1 holds for f_e. Consider two initial conditions $x_0, x_0^\varepsilon \in \hat{D}_e^\varepsilon$ such that $\|x_0 - x_0^\varepsilon\| \leq \bar{C}\varepsilon$ for some $\bar{C} > 0$, and consider a single input $u \in PC([0,T], U)$. Let $\gamma\colon [0,T] \to \hat{D}_e$ be defined by $\frac{\mathrm{d}}{\mathrm{d}t}\gamma(t) \in \mathcal{F}[f_e](\gamma(t), u(t))$ and $\gamma(0) = x_0$, and let $\gamma^\varepsilon\colon [0,T] \to \hat{D}_e^\varepsilon$ be defined by $\frac{\mathrm{d}}{\mathrm{d}t}\gamma^\varepsilon(t) = f_e^\varepsilon(\gamma^\varepsilon(t), u(t))$ and $\gamma^\varepsilon(0) = x_0^\varepsilon$. Then for each ε small enough $\|\gamma - \gamma^\varepsilon\|_\infty \leq C\varepsilon$, for some $C > 0$.

Proof: First, we show that the result holds for each $\hat{u} \in PCD([0,T], U)$ (where PCD denotes the class of piecewise-continuously differentiable functions), and then extend the result to our desired class of inputs. We transform each of the vector fields into a corresponding autonomous vector field, so that we can inductively call the result from [18, Lemma 2]. Consider the autonomous vector field $\bar{f}_e(\gamma, z) = (f_e(\gamma, \hat{u}(z)), 1)^T$, and the solution to the differential equation $\frac{\mathrm{d}}{\mathrm{d}t}(\gamma, z) \in \mathcal{F}[\bar{f}_e](\gamma, z)$ where $(\gamma(0), z(0)) = (x_0, 0)$. Note that $z(t) = t, \forall t \in [0,T]$, and thus $\frac{\mathrm{d}}{\mathrm{d}t}\gamma(t) \in \mathcal{F}[f_e](\gamma(t), u(t))$, $\forall t \in [0,T]$, as desired. Let $\bar{f}_e^\varepsilon$ be the ε-relaxation of $\bar{f}_e$, namely, $\bar{f}_e^\varepsilon(\gamma^\varepsilon, z^\varepsilon) = (f_e^\varepsilon(\gamma^\varepsilon, \bar{u}(z^\varepsilon)), 1)^T$, and note that if we let $\frac{\mathrm{d}}{\mathrm{d}t}(\gamma^\varepsilon, z^\varepsilon) = \bar{f}_e^\varepsilon(\gamma^\varepsilon, z^\varepsilon)$ with initial condition $(\gamma^\varepsilon(0), z^\varepsilon(0)) = (x_0^\varepsilon, 0)$, then the solution of γ^ε is as desired. Next, note that $\hat{u}$ must be non differentiable on a finite number of points $0 = t_1 < t_2 < ... < t_p = T$, $p \in \mathbb{N}$. Thus, on each interval $(t_i, t_{i+1}), \forall i = 1, 2, \dots, p-1$, $\bar{f}_e$ is continuously differentiable in z. Thus, restricting both trajectories to the time interval $[t_1, t_2]$, we have $\left\|(\gamma, z)|_{[t_1,t_2]} - (\gamma^\varepsilon, z^\varepsilon)|_{[t_1,t_2]}\right\|_\infty \leq C_1\varepsilon$ for each ε small enough and some $C_1 > 0$, by an argument similar to [18, Lemma 2]. Then by a straightforward inductive argument we obtain $\|(\gamma, z) - (\gamma^\varepsilon, z^\varepsilon)\|_\infty \leq C_2\varepsilon$, for some $C_2 > 0$, and thus $\|\gamma - \gamma^\varepsilon\|_\infty \leq C\varepsilon$, for some $C > 0$. The result for our desired $u \in PC([0,T], U)$ follows from noting that $PCD([0,T], U)$ is dense in $PC([0,T], U)$ under the L^2 norm, meaning we may select $\hat{u}$ such that $\|u - \hat{u}\|_2 < \delta$ for arbitrarily small $\delta > 0$, and then note that γ^ε depends continuously on its input [20, Lemma 5.6.7]. ■

(Proof of Theorem 3): We will prove the result using the representations we constructed in Sections VI and VII, namely the vector fields $\{f_e\}_{e\in\Gamma}$ and $\{f_e^\varepsilon\}_{e\in\Gamma}$, and by inductive appeals to Lemma 3. Assume that for some $t' \in [0,T]$ that $x(t'), x^\varepsilon(t') \in D_e^\varepsilon$, and that $d_{\mathcal{M}^\varepsilon}(x(t), x^\varepsilon(t)) \leq C_1\varepsilon$, for some $C_1 > 0$. Here, as in [6], we view x as a piecewise continuous curve on $\mathcal{M}^\varepsilon$. Let $\gamma\colon [t', t''] \to \hat{D}_e$ (for some $t' > 0$) be the maximal Filippov solution of $\mathcal{F}[f_e](\gamma(t), u(t))$ with initial condition $\gamma(t') = \pi_e^{-1}(x(t'))$, and let $\gamma^\varepsilon\colon [t', t''_\varepsilon] \to \hat{D}_e^\varepsilon$ (for some $t''_\varepsilon > 0$) be the maximal integral curve of f_e^ε with initial condition $\gamma^\varepsilon(t') = (\pi_e^\varepsilon)^{-1}(x^\varepsilon(t'))$, and note that $\|\gamma(t') - \gamma^\varepsilon(t')\| \leq C_2\varepsilon$, for some $C_2 > 0$. Let $\bar{t} = \min\{t'', t''_\varepsilon\}$. Then by Lemma 3, $\left\|\gamma|_{[t',\bar{t}]} - \gamma^\varepsilon|_{[t',\bar{t}]}\right\|_\infty \leq C_3\varepsilon$ for some $C_3 > 0$. This implies that $d_{\mathcal{M}^\varepsilon}(x(t), x^\varepsilon(t)) \leq C_4\varepsilon$, for some $C_4 > 0$, and each $t \in [t', \bar{t}]$, since we have that $x|_{[t',\bar{t}]} = \pi_e \circ \gamma|_{[t',\bar{t}]}$ and $x^\varepsilon|_{[t',\bar{t}]} = \pi_e^\varepsilon \circ \gamma^\varepsilon|_{[t',\bar{t}]}$, and the result of [6, Theorem 21] guarantees we incur an additional error of order ε by including x on $\mathcal{M}$. Note that if ε is small enough then we can find $e' \in \Gamma$ such that $x(\bar{t}), x^\varepsilon(\bar{t}) \in D_{e'}^\varepsilon$. The results follows by noting that $x(0) = x^\varepsilon(0)$, an inductive application of the preceding argument.

B. Proof of Theorem 4

Lemma 4: Let $e \in \Gamma$, $x_0 \in \hat{D}_e$ and $u \in PC([0,T], U)$. For each $\varepsilon > 0$ let $\gamma^\varepsilon\colon [0,T] \to \hat{D}_e^\varepsilon$ be the solution to $\frac{\mathrm{d}}{\mathrm{d}t}\gamma^\varepsilon(t) = f_e^\varepsilon(\gamma^\varepsilon(t), u(t))$ with initial condition x_0^ε, and let the map $\varepsilon \to x_0^\varepsilon$ be Lipschitz continuous and such that $x_0^0 = x_0$. Then there exists $\gamma^0\colon [0,T] \to \hat{D}_e$ such that $\gamma^0(0) = x_0$ and $\lim_{\varepsilon\to 0}\left\|\gamma^\varepsilon - \gamma^0\right\|_\infty = 0$.

Proof: Note that f_e^ε is continuously differentiable in ε for each $\varepsilon > 0$, since it is constructed using a finite number of compositions and multiplications of functions which are each continuously differentiable in ε. Thus, $\frac{\partial f_e^\varepsilon}{\partial \varepsilon}$ must be Lipschitz continuous for each $\varepsilon \in [\underline{\varepsilon}, \bar{\varepsilon}]$, where $\bar{\varepsilon} > \underline{\varepsilon} > 0$, since continuous functions are Lipschitz on compact domains. By Lemma 5.6.8 of [20], $\gamma^\varepsilon(t)$ is a Lipschitz continuous function of ε, $\forall t \in [0,T]$ and $\varepsilon \in (\underline{\varepsilon}, \bar{\varepsilon})$. Thus we have $\lim_{\varepsilon \to \underline{\varepsilon}} \|\gamma - \gamma^{\varepsilon}\|_{\infty} = 0$ for some $\gamma^{\underline{\varepsilon}} \colon [0,T] \to \hat{D}_e^{\underline{\varepsilon}}$. The desired result follows by noting that $\underline{\varepsilon}$ is arbitrary. ■

(Proof of Theorem 4): The proof is entirely analogous to that of Theorem 3, except Lemma 4 is called inductively in place of Lemma 3.