**Reading:** Class Notes, Sections 6.3, 6.4, 1.6.

**Problems:**

1. Consider the LTI system

\[
\dot{x} = Ax \\
y = Cx
\]

and suppose that the eigenvalues of \(A\) have negative real parts. Consider the function \(V(x) = x^T M x\), where \(M\) denotes the observability Gramian for the infinite time horizon, i.e., \(M(0, \infty)\). Show that along solutions of the system we have

\[
\dot{V} = -|y|^2.
\]

2. a) For LTI systems, show that \((A, C)\) is observable if and only if \((-A, C)\) is observable.

b) Is the same statement true for LTV systems? Prove or give a counterexample.

3. Obtain a combined controllability/observability decomposition for the LTI system \(\dot{x} = Ax + Bu, y = Cx\) by following these steps:

   a) Ignoring the control for now, write down the Kalman observability decomposition.

   b) Now add the control, noting that the \(B\) matrix assumes no special structure in the coordinates that give the observability decomposition.

   c) For the observable part of the system, switch coordinates to get the Kalman controllability decomposition for it. Repeat separately for the unobservable part.

In the resulting system, make sure to specify all controllability and observability properties of various subsystems. Identify four types of modes: controllable and observable, uncontrollable but observable, controllable but unobservable, and uncontrollable and unobservable.

4. Consider the system \(\dot{x} = Ax + Bu, y = Cx\) and suppose that it is both controllable and observable. Now consider the feedback of the form \(u = Kx + v\), which leads to the system with new control \(v\):

\[
\dot{x} = (A + BK)x + Bv \\
y = Cx
\]

a) Is the new system controllable? Prove or give a counterexample.

b) Is the new system observable? Prove or give a counterexample.

5. Compute the transfer function of the system

\[
\begin{align*}
\dot{x}_1 &= x_2 - u \\
\dot{x}_2 &= -3x_1 - 2x_2 + u \\
y &= x_1 + x_2
\end{align*}
\]