ECE517, Fall 2021 PROBLEM SET 4 Solution

1. Solve Exercise 4.9, parts (b)–(e) on page 247 in [Ioannou-Sun].

Solution. From part (a) we get

$$m\ddot{y} + \beta\dot{y} + ky = u.$$

Taking Laplace transform of both sides gives

$$\mathcal{L}\{y\}(s) = \frac{1/m}{s^2 + (\beta/m)s + k/m}\mathcal{L}\{u\}(s).$$

Filtering both sides with a second order stable filter

$$\frac{1}{\Lambda(s)} := \frac{1}{(s+\lambda_1)(s+\lambda_2)}, \qquad \lambda_1, \lambda_2 \in (-\infty, 0)$$

gives

$$z = \theta^\top \phi$$

where

$$z(s) := \frac{s^2}{\Lambda(s)}\mathcal{L}\{y\}(s),$$

and

$$\theta := \frac{1}{m}\begin{pmatrix} 1 \\ \beta \\ k \end{pmatrix}, \qquad \phi(s) := \frac{1}{\Lambda(s)}\begin{pmatrix} \mathcal{L}\{u\}(s) \\ -s\mathcal{L}\{y\}(s) \\ -\mathcal{L}\{y\}(s) \end{pmatrix}.$$

(b) The estimate of z is defined by

$$\hat{z} := \hat{\theta}^\top \phi,$$

where $\hat{\theta}$ is the estimate of θ. Normalizing the estimates gives

$$\bar{\phi} := \frac{\phi}{n}, \quad \bar{z} := \theta^\top \bar{\phi}, \quad \hat{\bar{z}} := \hat{\theta}^\top \bar{\phi},$$

where n is the normalizing signal defined by $n := \sqrt{1 + \phi^\top \phi}$ (so that $\bar{\phi} = \phi/n \in L_\infty$). Let $\tilde{\theta} := \hat{\theta} - \theta$ and $\bar{e} := \hat{\bar{z}} - \bar{z}$ be the estimation errors. Then

$$\bar{e} = \tilde{\theta}^\top \bar{\phi}.$$

Define the cost function

$$J(\hat{\theta}) := \frac{\bar{e}^2}{2} = \frac{(\tilde{\theta}^\top \bar{\phi})^2}{2}$$

and calculate its gradient

$$\nabla J(\hat{\theta}) = (\tilde{\theta}^\top \phi)\bar{\phi} = \bar{e}\bar{\phi}.$$

Hence the gradient adaptive law is given as

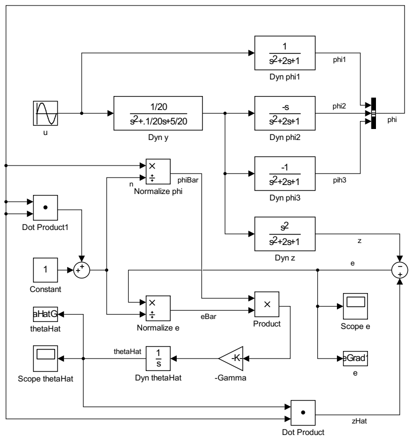
$$\dot{\hat{\theta}} = -\Gamma \bar{e}\bar{\phi},$$

where $\Gamma > 0$ is a scaling matrix (adaptive gain).

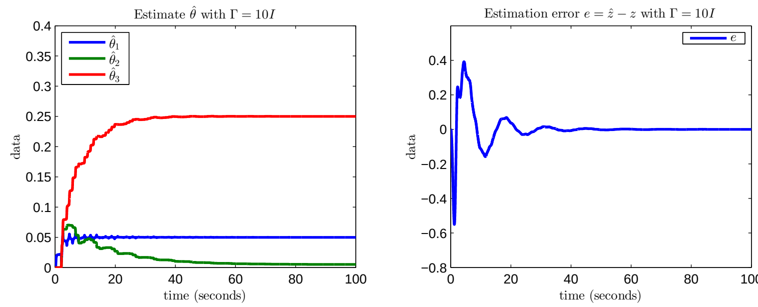
(c) The continuous-time recursive least-squares algorithm is given as

$$\dot{\hat{\theta}} = -P\bar{e}\bar{\phi}, \qquad \dot{P} = -P\bar{\phi}\bar{\phi}^\top P.$$

(d) For the gradient method, we set $\Gamma = 10I$. A single sinusoidal source is used with frequency π and amplitude 100. The Simulink diagram and the simulation result can be found in Fig. 1. We see from the simulation result that the estimate $\hat{\theta}$ goes to $\theta = (1, \beta, k)/m = (.05, .005, .25)$.

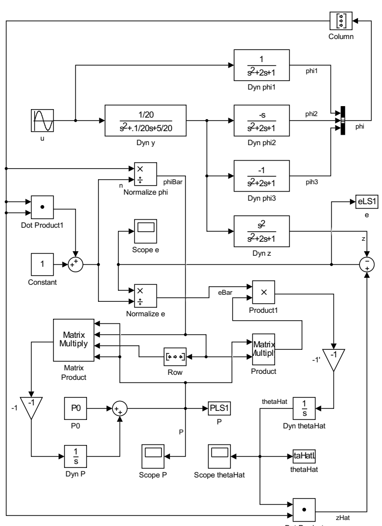


(a) Simulink diagram

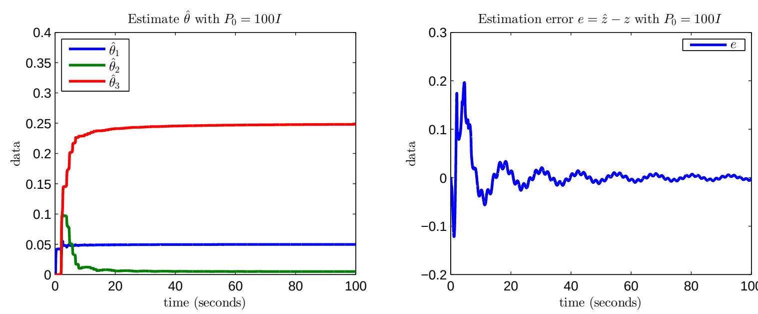


(b) Simulation result

Fig. 1: Problem 1(d). Gradient method



(a) Simulink diagram



(b) Simulation result

Fig. 2: Problem 1(d). Least-square method

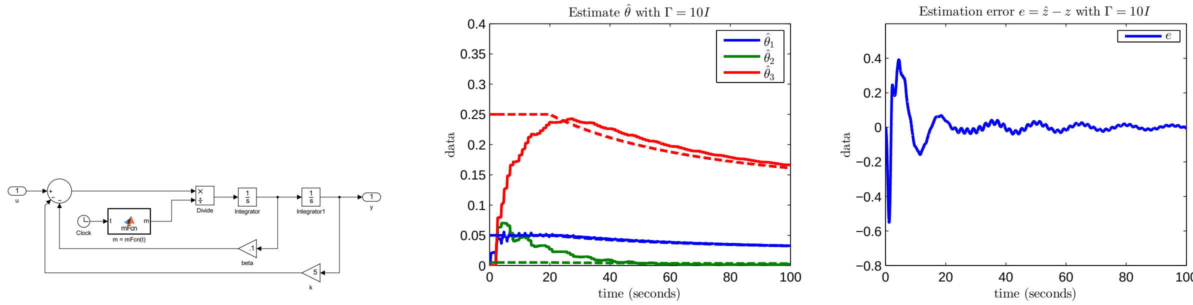
For the least-square method, we set $P(0) = 100I$. The same sinusoidal source with frequency π and amplitude 100 is used. The Simulink diagram and the simulation result can be found in Fig. 2.

We see from the simulation result that the estimate $\hat{\theta}$ goes to $\theta = (1, \beta, k)/m = (.05, .005, .25)$.

(e) Since m is time-varying, we need to substitute the dynamics of y in both Simulink diagrams with the subsystem in Fig. 3a.

For the gradient method, we set $\Gamma = 10I$. The same sinusoidal source with frequency π and amplitude 100 is used. The simulation result can be found in Fig. 3b.

We see from the simulation result that the estimate $\hat{\theta}$ goes to θ (dashed-line), which is slowly time-varying in this case.



(a) Dynamics of y (b) Simulation result

Fig. 3: Problem 1(e). Gradient method

For the least-square method, we set $P(0) = 100I$. The same sinusoidal source with frequency π and amplitude 100 is used. The simulation result can be found in Fig. 4.

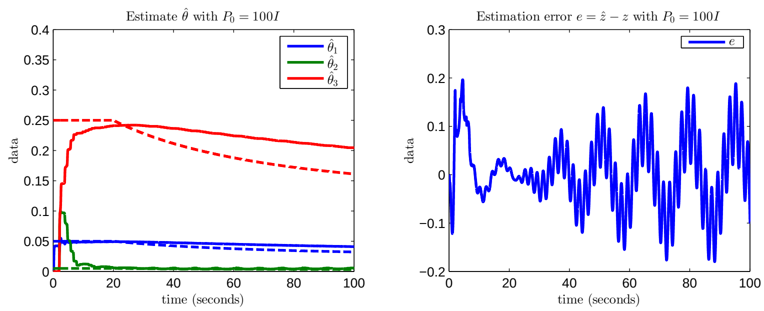


Fig. 4: Problem 1(e). Least-square method simulation result

We see from the simulation result that the estimate $\hat{\theta}$ goes to θ (dashed-line), which is slowly time-varying in this case. However, the estimation error e increases significantly when the system becomes time-varying, and goes to 0 very slowly, as shown in the longer simulation in Fig. 5.

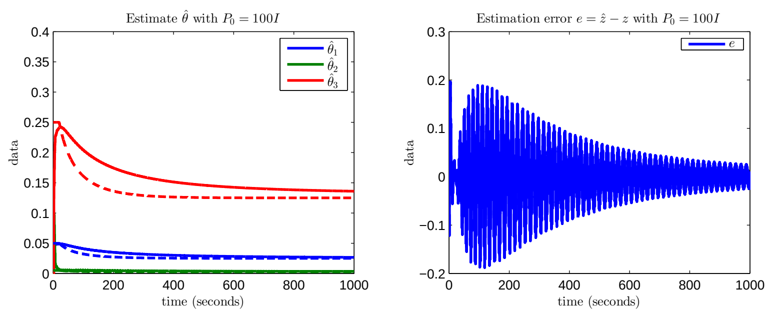


Fig. 5: Problem 1(e). Least-square method simulation result (1000 sec)

In both (d) and (e), the performance improves if an input with more frequency components is used.

2. Implement the indirect MRAC control design example given in class via computer simulation. Plot the behavior of the plant state as well as control gains over time. Does loss of stabilizability ($\hat{b} \to 0$) actually happen? Investigate this in the following cases:

a) $b > 0$ and $\hat{b}$ is initialized with the correct sign ($\hat{b}(0) > 0$).

b) $b > 0$ but $\hat{b}$ is initialized with the wrong sign ($\hat{b}(0) < 0$).

In each case, if you experience loss of stabilizability, modify the update law for $\hat{b}$ to get rid of the problem. Submit both sets of plots (with and without modification) and compare them.

Solution. The indirect MRAC model is given by

Plant :

$$\dot{y} = ay + bu;$$

Reference model :

$$\dot{y}_m = -a_m y_m + b_m r;$$

Controller :

$$u = -\hat{k}y + \hat{l}r, \qquad \hat{k} = \frac{\hat{a} + a_m}{\hat{b}}, \qquad \hat{l} = \frac{b_m}{\hat{b}};$$

Adaptation :

$$e = y_m - y, \qquad \dot{\hat{a}} = -\gamma e y, \qquad \dot{\hat{b}} = -\gamma e u.$$

In the simulation, we set $(a, b) = (5, 3)$ and $(a_m, b_m) = (3, 1)$.

a) When $\hat{b}$ is initialized with the correct sign, if the gain γ is relatively small (e.g. $\hat{b}(0) = 1, \gamma = 0.1$) then there is no loss of stabilizability. The simulation result can be found in Fig. 6.

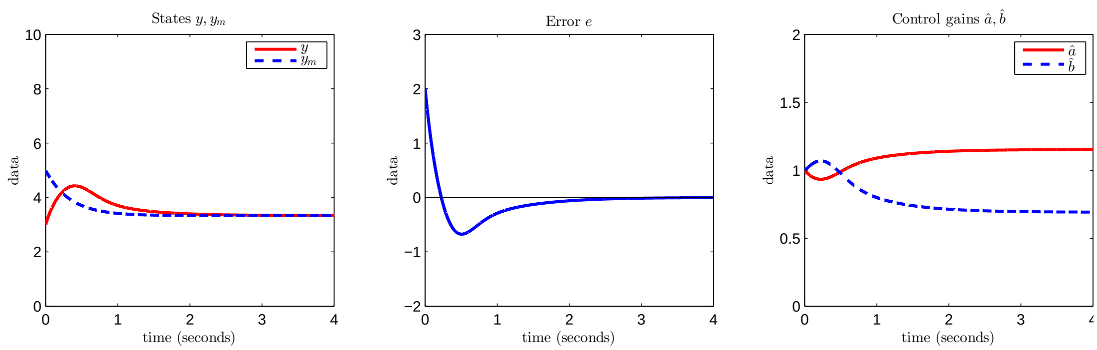


Fig. 6: Problem 2(a). Initialized with correct sign, no loss of stabilizability

On the other hand, if the gain γ is relatively large (e.g. $\hat{b}(0) = 5, \gamma = 10$) then there is loss of stabilizability. The simulation result can be found in Fig. 7.

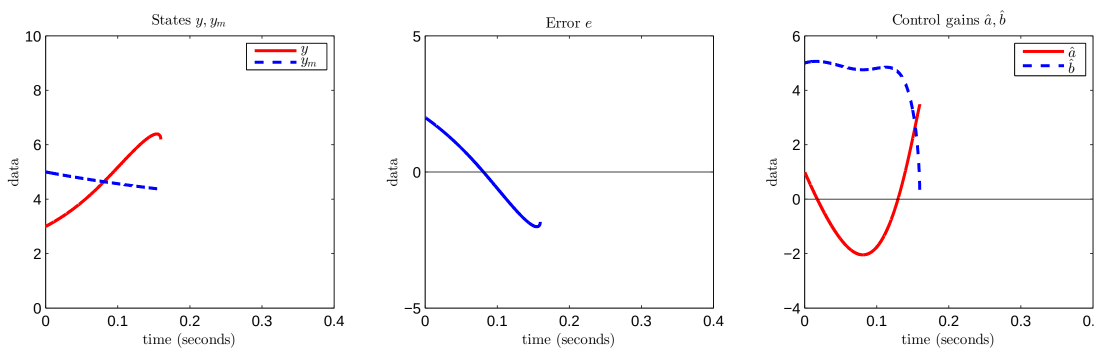


Fig. 7: Problem 2(a). Initialized with correct sign, loss of stabilizability

In this case, we prevent the loss of stabilizability using projection with $b_0 = 0.2$. The simulation result can be found in Fig. 8.

b) When $\hat{b}$ is initialized with the incorrect sign, there is always loss of stabilizability. The simulation result can be found in Fig. 9. In this case, we cannot prevent the loss of stabilizability using projection.

3. Consider the scalar system $\dot{x} = f(x) + g(x)u$ where $f(x) = -x\sin^2(x^2)$, $g(x) = \cos(x^2)$.

a) Construct a feedback law $u = k(x)$ which makes the closed-loop system GAS. (Justify this.)

b) Now suppose that the state measurements available to the feedback law are affected by an additive disturbance, resulting in the system $\dot{x} = f(x) + g(x)k(x + d)$ where f, g are the same as before and k is the feedback you found in part a). Is this system ISS with respect to d? Prove or disprove.

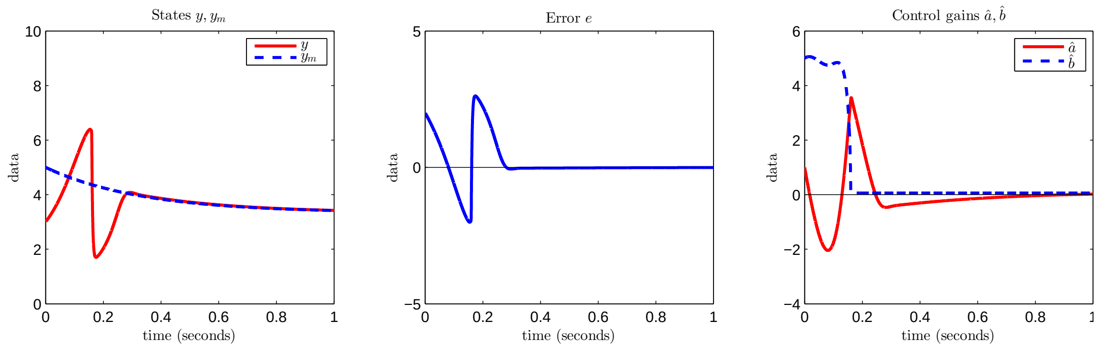


Fig. 8: Problem 2(a). Initialized with correct sign, projection

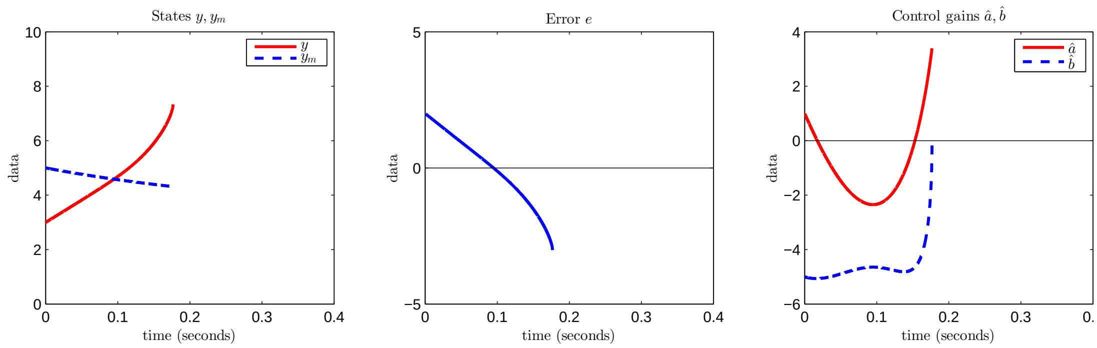


Fig. 9: Problem 2(b). Initialized with incorrect sign, loss of stabilizability

Solution. The system is given by

$$\dot{x} = -x\sin^2(x^2) + \cos(x^2)u.$$

a) Setting $u = k(x) := -x\cos(x^2)$ gives the closed-loop system

$$\dot{x} = -x(\sin^2(x^2) + \cos^2(x^2)) = -x.$$

Consider $V : \mathbb{R} \to [0, \infty)$ defined by $V(x) := x^2/2$. Then $\dot{V}(x) = x\dot{x} = -x^2 \leq 0$ and $\dot{V}(x) < 0$ for all $x \neq 0$. Hence Theorem 2 implies that the closed-loop system is GAS.

b) The closed-loop system is given by

$$\dot{x} = -x\sin^2(x^2) + \cos(x^2)k(x+d) = -x\sin^2(x^2) - (x+d)\cos(x^2)\cos((x+d)^2) =: h(x,d).$$

Notice that h is a continuous function in both arguments. Consider the sequences $(a_n)_{n=1}^{\infty}, (d_n)_{n=1}^{\infty}$ defined by

$$a_n := \sqrt{n\pi},$$
$$d_n := \sqrt{(n+1)\pi} - \sqrt{n\pi}.$$

It is straight forward to see that all $d_n > 0$, and $a_n \to \infty$, $d_n \to 0$ as $n \to \infty$. Moreover,

$$h(0, d_n) = -d_n\cos(d_n^2) < 0$$

for all $\sqrt{n+1} - \sqrt{n} < 1/\sqrt{2}$, that is, all $n \geq 1$; and

$$h(a_n, d_n) = -\sqrt{(n+1)\pi}\cos(n\pi)\cos((n+1)\pi) = \sqrt{(n+1)\pi} > 0$$

for all $n \geq 1$. By the continuity of h, we get that for each $n \geq 1$, there exist a $b_n \in (0, a_n)$ such that $h(b_n, d_n) = 0$ and $h(x, d_n) > 0$ for all $x \in (b_n, a_n]$. Hence if the initial value $x_n(0) \in (b_n, a_n]$, the state satisfies $x_n(t) \geq x_n(0) > b_n > 0$ for all $t \geq 0$. Since $a_n \to \infty$, $d_n \to 0$ as $n \to \infty$, this means that, when $d_n \to 0$, the state does not necessarily go to 0 as $t \to \infty$. On the other hand, the system is ISS with respect to d only if there are $\beta \in \mathcal{KL}, \gamma \in \mathcal{K}_\infty$ such that

$$|x(t)| \leq \beta(|x(0)|, t) + \gamma(|d|_t) \qquad \forall\, t \geq 0,$$

where $|d|_t$ is the essential supremum of $|d|$ on $[0,t]$. This implies that when $d_n \to 0$, we get $x \to 0$ as $t \to \infty$. Therefore, the system is not ISS with respect to d.

4. Consider again the system from the previous homework:

$$\dot{x} = \theta x + \xi_1 \qquad \dot{\xi}_1 = \xi_2 \qquad \dot{\xi}_2 = u$$

Use the adaptive ISS backstepping procedure given in Section 7.4.1 in the notes to design a feedback law that makes the closed-loop system ISS with respect to $(\tilde{\theta}, \dot{\tilde{\theta}})^T$ (for an arbitrary tuning law). You need to do the initialization step as well as the two recursion steps.

Solution. *Step 0.* Consider the system

$$\dot{x} = \theta x + u.$$

By setting

$$u := k_0(x, \hat{\theta}) := -\hat{\theta}x - x - x^3$$

we get the closed-loop system

$$\dot{x} = -\tilde{\theta}x - x - x^3,$$

where $\tilde{\theta} = \hat{\theta} - \theta$. Let $V_0 : \mathbb{R} \to [0,\infty)$ be defined by

$$V_0(x) := \frac{1}{2}x^2.$$

Its derivative along the state trajectory is

$$\begin{aligned} \dot{V}_0(x) &= x(\theta x + k_0) && (1) \\ &= -\tilde{\theta}x^2 - x^4 - x^2 \le -x^2 + \tilde{\theta}^2 - \left(x^2 + \frac{1}{2}\tilde{\theta}\right)^2 \\ &\le -x^2 + \tilde{\theta}^2. && (2) \end{aligned}$$

Step 1. Consider the system

$$\dot{x} = \theta x + \xi_1, \qquad \dot{\xi}_1 = u.$$

Let $V_1 : \mathbb{R}^3 \to [0,\infty)$ be defined by

$$V_1(x, \hat{\theta}, \xi_1) := V_0(x) + \frac{1}{2}(\xi_1 - k_0(x, \hat{\theta}))^2.$$

Its derivative along the state trajectory is

$$\begin{aligned} \dot{V}_1(x, \hat{\theta}, \xi_1) &= x(\theta x + \xi_1) + (\xi_1 - k_0)\left(u - \frac{\partial k_0}{\partial x}(\theta x + \xi_1) - \frac{\partial k_0}{\partial \hat{\theta}}\dot{\hat{\theta}}\right) \\ &= x(\theta x + \boldsymbol{k_0}) + (\xi_1 - k_0)\left(u + \boldsymbol{x} - \frac{\partial k_0}{\partial x}(\theta x + \xi_1) - \frac{\partial k_0}{\partial \hat{\theta}}\dot{\hat{\theta}}\right) && (3) \end{aligned}$$

By (1) and (2)

$$\begin{aligned} &\dot{V}_1(x, \hat{\theta}, \xi_1) \le \boldsymbol{-x^2 + \tilde{\theta}^2} + (\xi_1 - k_0)\left(u + x - \frac{\partial k_0}{\partial x}(\theta x + \xi_1) - \frac{\partial k_0}{\partial \hat{\theta}}\dot{\hat{\theta}}\right) \\ &= -x^2 + \tilde{\theta}^2 + (\xi_1 - k_0)\left(u + x - \frac{\partial k_0}{\partial x}(\boldsymbol{\hat{\theta}}x + \xi_1)\right) + \frac{\partial k_0}{\partial x}(\xi_1 - k_0)\boldsymbol{\tilde{\theta}}x - \frac{\partial k_0}{\partial \hat{\theta}}(\xi_1 - k_0)\dot{\tilde{\theta}} \\ &\le -x^2 + \tilde{\theta}^2 \boldsymbol{- (\xi_1 - k_0)^2} + (\xi_1 - k_0)\left(u + x + \boldsymbol{(\xi_1 - k_0)} - \frac{\partial k_0}{\partial x}(\hat{\theta}x + \xi_1)\right) \\ &\quad + \boldsymbol{\left(\frac{\partial k_0}{\partial x}\right)^2 (\xi_1 - k_0)^2 x^2 + \frac{1}{4}\tilde{\theta}^2 + \left(\frac{\partial k_0}{\partial \hat{\theta}}\right)^2 (\xi_1 - k_0)^2 + \frac{1}{4}\dot{\tilde{\theta}}^2} \\ &= -x^2 - (\xi_1 - k_0)^2 + \frac{5}{4}\tilde{\theta}^2 + \frac{1}{4}\dot{\tilde{\theta}}^2 \\ &\quad + (\xi_1 - k_0)\left(u + x + (\xi_1 - k_0) - \frac{\partial k_0}{\partial x}(\hat{\theta}x + \xi_1) + \left(\frac{\partial k_0}{\partial x}\right)^2 (\xi_1 - k_0)x^2 + \left(\frac{\partial k_0}{\partial \hat{\theta}}\right)^2 (\xi_1 - k_0)\right). \end{aligned}$$

By setting

$$u := k_1(x,\hat{\theta},\xi_1) := -x - (\xi_1 - k_0) + \frac{\partial k_0}{\partial x}(\hat{\theta}x + \xi_1) - \left(\frac{\partial k_0}{\partial x}\right)^2 (\xi_1 - k_0)x^2 - \left(\frac{\partial k_0}{\partial \hat{\theta}}\right)^2 (\xi_1 - k_0)$$

we get

$$\dot{V}_1(x,\hat{\theta},\xi_1) \leq -x^2 - (\xi_1 - k_0)^2 + \frac{5}{4}\tilde{\theta}^2 + \frac{1}{4}\dot{\tilde{\theta}}^2. \tag{4}$$

Step 2. Consider the system

$$\begin{aligned} \dot{x} &= \theta x + \xi_1, \\ \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= u. \end{aligned}$$

Let $V_2 : \mathbb{R}^4 \to [0,\infty)$ be defined by

$$V_2(x,\hat{\theta},\xi_1,\xi_2) := V_1(x,\hat{\theta},\xi_1) + \frac{1}{2}(\xi_2 - k_1(x,\hat{\theta},\xi_1))^2.$$

Its derivative along the state trajectory is

$$\begin{aligned} &\dot{V}_2(x,\hat{\theta},\xi_1,\xi_2) \\ &= x(\theta x + \xi_1) + (\xi_1 - k_0)\left(\xi_2 - \frac{\partial k_0}{\partial x}(\theta x + \xi_1) - \frac{\partial k_0}{\partial \hat{\theta}}\dot{\hat{\theta}}\right) + (\xi_2 - k_1)\left(u - \frac{\partial k_1}{\partial x}(\theta x + \xi_1) - \frac{\partial k_1}{\partial \hat{\theta}}\dot{\hat{\theta}} - \frac{\partial k_1}{\partial \xi_1}\xi_2\right) \\ &= x(\theta x + \boldsymbol{k_0}) + (\xi_1 - k_0)\left(\boldsymbol{k_1 + x} - \frac{\partial k_0}{\partial x}(\theta x + \xi_1) - \frac{\partial k_0}{\partial \hat{\theta}}\dot{\hat{\theta}}\right) \\ &\quad + (\xi_2 - k_1)\left(u + \boldsymbol{(\xi_1 - k_0)} - \frac{\partial k_1}{\partial x}(\theta x + \xi_1) - \frac{\partial k_1}{\partial \hat{\theta}}\dot{\hat{\theta}} - \frac{\partial k_1}{\partial \xi_1}\xi_2\right) \end{aligned}$$

Then by (3) and (4),

$$\begin{aligned} \dot{V}_2(x,\hat{\theta},\xi_1,\xi_2) &\leq \boldsymbol{-x^2 - (\xi_1 - k_0)^2 + \frac{5}{4}\tilde{\theta}^2 + \frac{1}{4}\dot{\tilde{\theta}}^2} + (\xi_2 - k_1)\left(u + (\xi_1 - k_0) - \frac{\partial k_1}{\partial x}(\theta x + \xi_1) - \frac{\partial k_1}{\partial \hat{\theta}}\dot{\hat{\theta}} - \frac{\partial k_1}{\partial \xi_1}\xi_2\right) \\ &= -x^2 - (\xi_1 - k_0)^2 + \frac{5}{4}\tilde{\theta}^2 + \frac{1}{4}\dot{\tilde{\theta}}^2 + (\xi_2 - k_1)\left(u + (\xi_1 - k_0) - \frac{\partial k_1}{\partial x}(\boldsymbol{\hat{\theta}}x + \xi_1) - \frac{\partial k_1}{\partial \xi_1}\xi_2\right) \\ &\quad + \frac{\partial k_1}{\partial x}(\xi_2 - k_1)\boldsymbol{\tilde{\theta}}x - \frac{\partial k_1}{\partial \hat{\theta}}(\xi_2 - k_1)\dot{\hat{\theta}} \end{aligned}$$

Therefore,

$$\begin{aligned} \dot{V}_2(x,\hat{\theta},\xi_1,\xi_2) &\leq -x^2 - (\xi_1 - k_0)^2 + \frac{5}{4}\tilde{\theta}^2 + \frac{1}{4}\dot{\tilde{\theta}}^2 \boldsymbol{- (\xi_2 - k_1)^2} \\ &\quad + (\xi_2 - k_1)\left(u + (\xi_1 - k_0) + \boldsymbol{(\xi_2 - k_1)} - \frac{\partial k_1}{\partial x}(\hat{\theta}x + \xi_1) - \frac{\partial k_1}{\partial \xi_1}\xi_2\right) \\ &\quad + \boldsymbol{\left(\frac{\partial k_1}{\partial x}\right)^2 (\xi_2 - k_1)^2 x^2 + \frac{1}{4}\tilde{\theta}^2 + \left(\frac{\partial k_1}{\partial \hat{\theta}}\right)^2 (\xi_2 - k_1)^2 + \frac{1}{4}\dot{\tilde{\theta}}^2} \\ &= -x^2 - (\xi_1 - k_0)^2 - (\xi_2 - k_1)^2 + \frac{3}{2}\tilde{\theta}^2 + \frac{1}{2}\dot{\tilde{\theta}}^2 + (\xi_2 - k_1)\Big(u + (\xi_1 - k_0) \\ &\quad + (\xi_2 - k_1) - \frac{\partial k_1}{\partial x}(\hat{\theta}x + \xi_1) - \frac{\partial k_1}{\partial \xi_1}\xi_2 + \left(\frac{\partial k_1}{\partial x}\right)^2 (\xi_2 - k_1)x^2 + \left(\frac{\partial k_1}{\partial \hat{\theta}}\right)^2 (\xi_2 - k_1)\Big). \end{aligned}$$

By setting

$$\begin{aligned} u &:= k_2(x,\hat{\theta},\xi_1,\xi_2) \\ &:= -(\xi_1 - k_0) - (\xi_2 - k_1) + \frac{\partial k_1}{\partial x}(\hat{\theta}x + \xi_1) + \frac{\partial k_1}{\partial \xi_1}\xi_2 - \left(\frac{\partial k_1}{\partial x}\right)^2 (\xi_2 - k_1)x^2 - \left(\frac{\partial k_1}{\partial \hat{\theta}}\right)^2 (\xi_2 - k_1) \end{aligned}$$

we get

$$\dot{V}_2(x,\hat{\theta},\xi_1,\xi_2) \le -x^2 - (\xi_1 - k_0)^2 - (\xi_2 - k_1)^2 + \frac{3}{2}\tilde{\theta}^2 + \frac{1}{2}\dot{\tilde{\theta}}^2$$
$$\le -\alpha\left(\left| \begin{matrix} x \\ \xi_1 \\ \xi_2 \end{matrix} \right| \right) + \gamma\left(\left| \begin{matrix} \tilde{\theta} \\ \dot{\tilde{\theta}} \end{matrix} \right| \right)$$

for suitable $\alpha, \gamma \in \mathcal{K}_\infty$. Hence the closed-loop system is ISS with respect to $(\tilde{\theta}, \dot{\tilde{\theta}})$.