1. Which of the following scalar systems are ISS? (Justify your answers.)

   a)  $\dot{x} = -x^3 + xd$

   b)  $\dot{x} = -x^3 + x^2d_1 - xd_2 + d_1d_2$

   c)  $\dot{x} = -\text{sat}(x) + d$, where $\text{sat}(x) = \begin{cases} x & \text{if } -1 \leq x \leq 1 \\ -1 & \text{if } x < -1 \\ 1 & \text{if } x > 1 \end{cases}$

**Solution:**

   a) Consider the function $V : \mathbb{R} \rightarrow \mathbb{R}_{\geq 0}$ defined as

   $$ V(x) = \frac{1}{2} x^2. $$

   Then

   $$ \dot{V}(x) = -x^4 + x^2d \leq -x^4 + \frac{1}{2}(x^4 + d^2) = -\frac{1}{2}x^4 + \frac{1}{2}d^2. $$

   Hence the system is ISS.

   b) Consider the function $V : \mathbb{R} \rightarrow \mathbb{R}_{\geq 0}$ defined as

   $$ V(x) = \frac{1}{2} x^2. $$

   Then

   $$ \dot{V}(x) = -x^4 + x^3d_1 - x^2d_2 + xd_1d_2 = -(x^2 + d_2)(x^2 - xd_1), $$

   and thus

   $$ |x| \geq \max\{2|d_1|, \sqrt{2}|d_2|\} \quad \Rightarrow \quad -\frac{1}{2}x^2 \leq xd_1 \leq \frac{1}{2}x^2, \quad -\frac{1}{2}x^2 \leq d_2 \leq \frac{1}{2}x^2 $$

   $$ \Rightarrow \quad \dot{V}(x) \leq -\frac{1}{4}x^4. $$

   Let $d := (d_1, d_2)^T$. Then

   $$ |x| \geq \max\{2|d|, \sqrt{2}|d|\} \quad \Rightarrow \quad \dot{V}(x) \leq -\frac{1}{4}x^4. $$

   Hence the system is ISS.
c) Consider \( d \equiv 2 \). Then \( \dot{x} \geq -1 + d \equiv 1 \), and thus

\[
x(0) > 0 \implies \lim_{t \to \infty} x(t) = \infty.
\]

Hence the system is not ISS.

\( \square \)

2. Prove that if the system \( \dot{x} = f(x, d) \) is ISS and \( d(t) \to 0 \) as \( t \to \infty \), then \( x(t) \) converges to zero.

**Solution:** Since the system \( \dot{x} = f(x, d) \) is ISS, there exist \( \beta \in \mathcal{KL} \) and \( \gamma \in \mathcal{K}_\infty \) such that

\[
|x(t)| \leq \beta(|x(t_0)|, t - t_0) + \gamma(|d|_{[t_0, t]}) \quad \forall t \geq t_0 \geq 0.
\]

In particular, \( \gamma \in \mathcal{K}_\infty \) implies that it is invertible. On the other hand, \( \lim_{t \to \infty} d(t) = 0 \) implies that, for an arbitrary \( \epsilon > 0 \), there exists a \( T_1 \geq 0 \) such that

\[
|d(t)| \leq \gamma^{-1}(\epsilon/2) \quad \forall t \geq T_1.
\]

Moreover, \( \beta \in \mathcal{KL} \) implies that there also exists a \( T_2 \geq T_1 \) such that

\[
\beta(x(T_1), t - T_1) \leq \epsilon/2 \quad \forall t \geq T_2.
\]

Hence

\[
|x(t)| \leq \beta(x(T_1), t - T_1) + \gamma(|d|_{[T_1, t]}) \leq \epsilon \quad \forall t \geq T_2,
\]

that is, \( \lim_{t \to \infty} x(t) = 0. \) \( \square \)

3. Finish the calculation started in class which shows that the cascade of a GAS system and an ISS system is GAS.

**Solution:** By the hypotheses, there exist \( \beta_1, \beta_2 \in \mathcal{KL} \) and \( \gamma \in \mathcal{K}_\infty \) such that for all \( t \geq t_0 \geq 0 \), we have

\[
\|x(t)\| \leq \beta_1(\|x(t_0)\|, t - t_0)
\]

and

\[
\|z(t)\| \leq \beta_2(\|z(t_0)\|, t - t_0) + \gamma(\|x\|_{[t_0, t]}).
\]

Consider an arbitrary \( \tau \geq 0 \). Letting \( t_0 = 0 \) and \( t = \tau \) in (1) gives that

\[
\|x(\tau)\| \leq \beta_1(\|x(0)\|, \tau).
\]

On the other hand, letting \( t_0 = \tau/2 \) and \( t = \tau \) in (2) gives that

\[
\|z(\tau)\| \leq \beta_2(\|z(\tau/2)\|, \tau/2) + \gamma(\|x\|_{[\tau/2, \tau]});
\]

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while letting $t_0 = 0$ and $t = \tau/2$ in (2) gives that
\[
\|z(\tau/2)\| \leq \beta_2(\|z(0)\|, \tau/2) + \gamma(\|x\|_{[0, \tau/2]}).
\]

Since $\beta_1$ is monotonically decreasing, (1) implies that
\[
\|x\|_{[0, \tau/2]} = \sup_{s \in [0, \tau/2]} \|x(s)\| \leq \sup_{s \in [0, \tau/2]} \beta_1(\|x(0)\|, s) = \beta_1(\|x(0)\|, 0),
\]
and
\[
\|x\|_{[\tau/2, \tau]} = \sup_{s \in [\tau/2, \tau]} \|x(s)\| \leq \sup_{s \in [\tau/2, \tau]} \beta_1(\|x(0)\|, s) = \beta_1(\|x(0)\|, \tau/2).
\]

Define $w := (x^T, z^T)^T$. Combining the inequalities above gives that
\[
\|w(\tau)\| \leq \|x(\tau)\| + \|z(\tau)\|
\]
\[
\leq \beta_1(\|x(0)\|, \tau) + \beta_2(\|z(0)\|, \tau/2) + \gamma(\|x\|_{[\tau/2, \tau]})
\]
\[
\leq \beta_1(\|x(0)\|, \tau) + \beta_2(\beta_2(\|z(0)\|, \tau/2) + \gamma(\beta_1(\|x(0)\|, 0)), \tau/2) + \gamma(\beta_1(\|x(0)\|, \tau/2))
\]
\[
\leq \beta(\|w(0)\|, \tau),
\]
where
\[
\beta(r, t) := \beta_1(r, t) + \beta_2(\beta_2(r, 0) + \gamma(\beta_1(r, 0)), t/2) + \gamma(\beta_1(r, t/2)).
\]

Since $\beta_1, \beta_2 \in KL$ and $\gamma \in K_\infty$, it is clear that $\beta(\cdot, t) \in K_\infty$ for each fixed $t \geq 0$, and $\lim_{s \to \infty} \beta(x, s) = 0$ for each fixed $x$, that is, $\beta \in KL$. Hence the cascade system is GAS.

4. Consider the scalar control system $\dot{x} = \phi(x) + u$ where $\phi$ satisfies $|\phi(x)| \leq \psi(|x|)$ for some function $\psi$. Let the control law be given by the following state feedback:
\[
u(|x|)x
\]
where the function $\nu$ is positive, non-decreasing, and satisfies
\[(1 - c)r \nu((1 - c)r) \geq \frac{\ell}{2} r + \psi(r)
\]
for some constants $c \in (0, 1)$ and $\ell > 0$. Now suppose that the state measurements used for feedback are corrupted by an additive disturbance $d$ (this can be quantization error, for example), so that the actual control applied to the system is $u = k(x + d)$ with $k$ as above. Show that the closed-loop system is ISS with respect to $d$.

**Solution:** Consider the function $V : \mathbb{R} \to \mathbb{R}_{\geq 0}$ defined as
\[V(x) = \frac{1}{2} x^2.
\]
Suppose
\[V(x) \geq \frac{d^2}{2c^2},
\]

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or equivalently
\[ |d| \leq c|x|. \]

Then
\[ \dot{V}(x) = V(x)(\phi(x) + u) = x\phi(x) - \nu(|x + d|)(x + d)x. \]

First, \(|\phi(x)| \leq \psi(|x|)\) implies that
\[ x\phi(x) \leq |x||\phi(x)| \leq |x|\psi(|x|). \]

On the other hand, \(\nu\) being increasing and \(|d| \leq c|x|\) implies that
\[ \nu(|x + d|) \geq \nu((1 - c)|x|); \]
and
\[(x + d)x = x^2 + xd \geq (1 - c)x^2.\]

Hence
\[ \dot{V}(x) \leq |x|\psi(|x|) - (1 - c)|x|\nu((1 - c)|x|)|x| \]
\[ \leq |x|\psi(|x|) - \frac{l}{2}x^2 - |x|\psi(|x|) \]
\[ = -\frac{l}{2}x^2. \]

Therefore,
\[ V(x) \geq \frac{d^2}{2c^2} \Rightarrow \dot{V}(x) \leq -\frac{l}{2}x^2, \]
which implies that the closed-loop system is ISS with respect to \(d\). \(\square\)

5. Consider a feedback interconnection of two systems, \(\dot{x}_1 = f_1(x_1, x_2)\) and \(\dot{x}_2 = f_2(x_2, x_1)\). Suppose that there exist positive definite, radially unbounded, continuously differentiable \((C^1)\) functions \(V_1(x_1)\) and \(V_2(x_2)\) satisfying the following “gain-margin” ISS-Lyapunov function conditions:
\[ V_1(x_1) \geq \chi_1(V_2(x_2)) \Rightarrow \frac{\partial V_1}{\partial x_1}f_1(x_1, x_2) \leq -\alpha_1(V_1(x_1)), \]
\[ V_2(x_2) \geq \chi_2(V_1(x_1)) \Rightarrow \frac{\partial V_2}{\partial x_2}f_2(x_2, x_1) \leq -\alpha_2(V_2(x_2)) \]
where \(\chi_1, \chi_2\) are class \(K_\infty\) functions and \(\alpha_1, \alpha_2\) are positive definite functions. (These conditions differ from the ones discussed in class in that the left-hand sides involve the Lyapunov functions and not the norms of the states/inputs, but the ISS properties they capture are the same.) Assume that the following small-gain condition holds:
\[ \chi_1(\chi_2(r)) < r \ \ \ \forall r > 0. \]
It is not hard to show that then there exists a class \(K_\infty\) function \(\rho\) that is \(C^1\) on \((0, \infty)\) and satisfies \(\chi_1(r) < \rho(r) < \chi_2^{-1}(r)\) and \(\rho'(r) > 0\) for all \(r > 0\); you can assume the existence of such a function \(\rho\).
Consider the function
\[ V(x_1, x_2) := \max\{V_1(x_1), \rho(V_2(x_2))\}. \]

Prove that \( V \) is a Lyapunov function for the overall \((x_1, x_2)\)-system—and consequently the system is GAS—by studying the derivative of \( V \) along solutions on the two sets
\[ A := \{(x_1, x_2) : V_1(x_1) < \rho(V_2(x_2))\} \]
and
\[ B := \{(x_1, x_2) : V_1(x_1) > \rho(V_2(x_2))\} \]
You can ignore the set where \( V_1(x_1) = \rho(V_2(x_2)) \).

**Solution:** We write \( x \) for the composite state \((x_1, x_2)\). Consider first \( x \in A \). In this case \( V(x) = \rho(V_2(x_2)) \) and we have that
\[ V_1(x_1) < \rho(V_2(x_2)) \implies V_2(x_2) > \chi_2(V_1(x_1)) \]
using the small-gain condition and the construction of \( \rho \). Hence, by the second ISS-Lyapunov function condition above, we have
\[ \dot{V}(x) = \dot{V}_1(x) \leq -\alpha_1(V_1(x_1)) = -\alpha_1(V(x)) \]
Next, consider \( x \in B \). Since \( V_1(x_1) > \rho(V_2(x_2)) \), we have using the small-gain condition and the construction of \( \rho \) that \( V_1(x_1) > \chi_1(V_2(x_2)) \) and \( V(x) = V_1(x_1) \). Hence, by the first ISS-Lyapunov function condition above, we have
\[ \dot{V}(x) = \dot{V}_1(x) \leq -\alpha_1(V_1(x_1)) = -\alpha_1(V(x)) \]
Combining the two cases, we have \( \dot{V}(x) \leq -\alpha(V(x)) \) where
\[ \alpha(r) := \min\{\rho' \circ \rho^{-1}(r) \cdot \alpha_2 \circ \rho^{-1}(r), \alpha_1(r)\} \]
is a positive definite function, which proves the claim. □