1. (25 Points) Implement the indirect MRAC control design example given in class via computer simulation. Plot the behavior of the plant state as well as control gains over time. Does loss of stabilizability ($\hat{b} \to 0$) actually happen? Investigate this in the following cases:

a) $b > 0$ and $\hat{b}$ is initialized with the correct sign ($\hat{b}(0) > 0$).

b) $b > 0$ but $\hat{b}$ is initialized with the wrong sign ($\hat{b}(0) < 0$).

In each case, if you experience loss of stabilizability, modify the update law for $\hat{b}$ to get rid of the problem. Submit both sets of plots (with and without modification) and compare them.

Solution. The indirect MRAC model is given by

Plant:

$$\dot{y} = ay + bu;$$

Reference model:

$$\dot{y}_m = -a_my_m + b_mr;$$

Controller:

$$u = -\hat{k}y + \hat{l}r,$$

$$\dot{\hat{k}} = \frac{\hat{a} + a_m}{b},$$

$$\dot{\hat{l}} = \frac{b_m}{b};$$

Adaptation:

$$e = y_m - y,$$

$$\dot{\hat{a}} = -\gamma ey,$$

$$\dot{\hat{b}} = -\gamma eu.$$

In the simulation, we set $(a, b) = (5, 3)$ and $(a_m, b_m) = (3, 1)$.

a) When $\hat{b}$ is initialized with the correct sign, if the gain $\gamma$ is relatively large (e.g. $\hat{b}(0) = 1, \gamma = 0.1$) then there is no loss of stabilizability. The simulation result can be found in Fig. 1. (Note that you need to assign the values of $bHat0 = 1$, gamma = .1 in MatLab before simulating the Simulink file on course website.)

On the other hand, if the gain $\gamma$ is relatively small (e.g. $\hat{b}(0) = 5, \gamma = 10$) then there is loss of stabilizability. The simulation result can be found in Fig. 2. (Note that you need to assign the values of $bHat0 = 5$, gamma = 10 in MatLab before simulating the Simulink file on course website.)

Fig. 1: Problem 1 a). Initialized with correct sign, no loss of stabilizability
In this case, we prevent the loss of stabilizability using projection with $b_0 = 0.2$. The simulation result can be found in Fig. 3. (Note that you need to assign the values of $bHat0 = 5$, gamma = 10 in MatLab before simulating the Simulink file on course website.)

b) When $\hat{b}$ is initialized with the incorrect sign, there is always loss of stabilizability. The simulation result can be found in Fig. 4. (Note that you need to assign the values of $bHat0 = -5$, gamma = 10 in MatLab before simulating the Simulink file on course website.) In this case, we cannot prevent the loss of stabilizability using projection.

2. (25 Points) Consider the scalar system $\dot{x} = f(x) + g(x)u$ where $f(x) = -x \sin^2(x^2)$, $g(x) = \cos(x^2)$.
   
a) Construct a feedback law $u = k(x)$ which makes the closed-loop system GAS. (Justify this.)
   
b) Now suppose that the state measurements available to the feedback law are affected by an additive disturbance, resulting in the system $\dot{x} = f(x) + g(x)k(x + d)$ where $f$, $g$ are the same as before and $k$ is the feedback you found in part a). Is this system ISS with respect to $d$? Prove or disprove.
Solution. The system is given by
\[ \dot{x} = -x \sin^2(x^2) + \cos(x^2)u. \]
a) Setting \( u = k(x) := -x \cos(x^2) \) gives the closed-loop system
\[ \dot{x} = -x \sin^2(x^2) + \cos^2(x^2)) = -x. \]
Consider \( V : \mathbb{R} \to [0, \infty) \) defined by \( V(x) := x^2/2 \). Then \( \dot{V}(x) = x \dot{x} = -x^2 \leq 0 \) and \( \dot{V}(x) < 0 \) for all \( x \neq 0 \). Hence Theorem 2 implies that the closed-loop system is GAS.
b) The closed-loop system is given by
\[ \dot{x} = -x \sin^2(x^2) + \cos^2(x^2)k(x + d) = -x \sin^2(x^2) - (x + d) \cos(x^2) \cos((x + d)^2) =: h(x, d). \]
Notice that \( h \) is a continuous function in both arguments. Consider the sequences \( (a_n)_{n=1}^{\infty}, (d_n)_{n=1}^{\infty} \) defined by
\[
\begin{align*}
a_n & := \sqrt{n\pi}, \\
d_n & := \sqrt{(n+1)\pi} - \sqrt{n\pi}.
\end{align*}
\]
It is straightforward to see that all \( d_n > 0 \), and \( a_n \to \infty, d_n \to 0 \) as \( n \to \infty \). Moreover,
\[ h(0, d_n) = -d_n \cos(d_n^2) < 0 \]
for all \( \sqrt{n+1} - \sqrt{n} < 1/\sqrt{2} \), that is, all \( n \geq 1 \); and
\[ h(a_n, d_n) = -\sqrt{(n+1)\pi} \cos(n\pi) \cos((n+1)\pi) = \sqrt{(n+1)\pi} > 0 \]
for all \( n \geq 1 \). By the continuity of \( h \), we get that for each \( n \geq 1 \), there exist a \( b_n \in (0, a_n) \) such that \( h(b_n, d_n) = 0 \) and \( h(x, d_n) > 0 \) for all \( x \in (b_n, a_n] \). Hence if the initial value \( x_n(0) \in (b_n, a_n] \), the state satisfies \( x_n(t) \geq x_n(0) > b_n > 0 \) for all \( t \geq 0 \). Since \( a_n \to \infty, d_n \to 0 \) as \( n \to \infty \), this means that, when \( d_n \to 0 \), the state does not necessarily go to 0 as \( t \to \infty \). On the other hand, the system is ISS with respect to \( d \) only if there are \( \beta \in \mathcal{KL}, \gamma \in \mathcal{K}_\infty \) such that
\[
|x(t)| \leq \beta(|x(0)|, t) + \gamma(|d|_t) \quad \forall t \geq 0,
\]
where \( |d|_t \) is the essential supremum of \( |d| \) on \([0, t] \). This implies that when \( d_n \to 0 \), we get \( x \to 0 \) as \( t \to \infty \). Therefore, the system is not ISS with respect to \( d \).

3. (30 Points) Consider again the system from the previous homework:
\[
\begin{align*}
\dot{x} &= \theta x + \xi_1 \\
\dot{\xi}_1 &= \xi_2 \\
\dot{\xi}_2 &= u
\end{align*}
\]
Use the adaptive ISS backstepping procedure given in Section 7.4.1 in the notes to design a feedback law that makes the closed-loop system ISS with respect to \((\tilde{\theta}, \tilde{\xi})^T\) (for an arbitrary tuning law). You need to do the initialization step as well as the two recursion steps.

Solution. Step 0. Consider the system
\[ \dot{x} = \theta x + u. \]
By setting
\[ u := k_0(x, \hat{\theta}) := -\hat{\theta} x - x - x^3 \]
we get the closed-loop system
\[ \dot{x} = -\hat{\theta} x - x - x^3, \]
where \( \hat{\theta} = \tilde{\theta} - \theta \). Let \( V_0 : \mathbb{R} \to [0, \infty) \) be defined by
\[ V_0(x) := \frac{1}{2} x^2. \]
Its derivative along the state trajectory is
\[
\dot{V}_0(x) = x(\theta x + k_0)
\]
\[
= -\dot{\theta}x^2 - x^4 - x^2
\]
\[
\leq -x^2 + \frac{\dot{\theta}}{2} - \left(x^2 + \frac{1}{2}\right)^2
\]
\[
\leq -x^2 + \frac{\dot{\theta}}{2}.
\]  

**Step 1.** Consider the system
\[
\dot{x} = \theta x + \xi_1,
\]
\[
\dot{\xi}_1 = u.
\]
Let \( V_1 : \mathbb{R}^3 \to [0, \infty) \) be defined by
\[
V_1(x, \hat{\theta}, \xi_1) := V_0(x) + \frac{1}{2}(\xi_1 - k_0(x, \hat{\theta}))^2.
\]
Its derivative along the state trajectory is
\[
\dot{V}_1(x, \hat{\theta}, \xi_1)
\]
\[
= x(\theta x + \xi_1) + (\xi_1 - k_0) \left( u - \frac{\partial k_0}{\partial x}(\theta x + \xi_1) - \frac{\partial k_0}{\partial \hat{\theta}} \right)
\]
\[
= x(\theta x + k_0) + (\xi_1 - k_0) \left( u + x - \frac{\partial k_0}{\partial x}(\theta x + \xi_1) - \frac{\partial k_0}{\partial \hat{\theta}} \right)
\]
\[
(1) (2)
\]
\[
\leq -x^2 + \frac{\dot{\theta}}{2} + (\xi_1 - k_0) \left( u + x - \frac{\partial k_0}{\partial x}(\theta x + \xi_1) - \frac{\partial k_0}{\partial \hat{\theta}} \right)
\]
\[
+ \left( \frac{\partial k_0}{\partial x} \right)^2 (\xi_1 - k_0)^2 x^2 + \frac{1}{4} \frac{\partial k_0}{\partial \hat{\theta}}^2 + \left( \frac{\partial k_0}{\partial x} \right)^2 (\xi_1 - k_0)^2 + \frac{1}{4} \frac{\partial k_0}{\partial \hat{\theta}}^2
\]
\[
= -x^2 - (\xi_1 - k_0)^2 + \frac{5}{4} \frac{\partial k_0}{\partial \hat{\theta}}^2 + \frac{1}{4} \frac{\partial k_0}{\partial \hat{\theta}}^2
\]
\[
+ (\xi_1 - k_0) \left( u + x + (\xi_1 - k_0) - \frac{\partial k_0}{\partial x}(\theta x + \xi_1) + \left( \frac{\partial k_0}{\partial x} \right)^2 (\xi_1 - k_0)x^2 + \left( \frac{\partial k_0}{\partial \hat{\theta}} \right)^2 (\xi_1 - k_0) \right).
\]

By setting
\[
u := k_1(x, \hat{\theta}, \xi_1) := -x - (\xi_1 - k_0) + \frac{\partial k_0}{\partial x}(\theta x + \xi_1) - \left( \frac{\partial k_0}{\partial x} \right)^2 (\xi_1 - k_0)x^2 - \left( \frac{\partial k_0}{\partial \hat{\theta}} \right)^2 (\xi_1 - k_0)
\]  
we get
\[
\dot{V}_1(x, \hat{\theta}, \xi_1) \leq -x^2 - (\xi_1 - k_0)^2 + \frac{5}{4} \frac{\partial k_0}{\partial \hat{\theta}}^2 + \frac{1}{4} \frac{\partial k_0}{\partial \hat{\theta}}^2.
\]  

**Step 2.** Consider the system
\[
\dot{x} = \theta x + \xi_1,
\]
\[
\dot{\xi}_1 = \xi_2
\]
\[
\dot{\xi}_2 = u.
\]
Let \( V_2 : \mathbb{R}^4 \to [0, \infty) \) be defined by
\[
V_2(x, \hat{\theta}, \xi_1, \xi_2) := V_1(x, \hat{\theta}, \xi_1) + \frac{1}{2}(\xi_2 - k_1(x, \hat{\theta}, \xi_1))^2.
\]
Its derivative along the state trajectory is
\[
\dot{V}_2(x, \hat{\theta}, \xi_1, \xi_2) = x(\theta x + \xi_1) + (\xi_1 - k_0) \left( \xi_2 - \theta x + \xi_1 - \frac{\partial k_0}{\partial \theta} \right) + (\theta x + \xi_1) \left( u - \frac{\partial k_1}{\partial x} (\theta x + \xi_1) - \frac{\partial k_1}{\partial \theta} \right) + (\theta x - \xi_1 - k_0) \left( \xi_2 - k_1 \right)
\]
\[
= x(\theta x + k_0) + (\xi_1 - k_0) \left( k_1 + x - \frac{\partial k_0}{\partial x} (\theta x + \xi_1) - \frac{\partial k_0}{\partial \theta} \right)
\]
\[
+ (\xi_2 - k_1) \left( u + (\xi_1 - k_0) - \frac{\partial k_1}{\partial x} (\theta x + \xi_1) - \frac{\partial k_1}{\partial \theta} \right) \frac{\partial k_1}{\partial \xi_1} \xi_2
\]
\[
\leq -x^2 - (\xi_1 - k_0)^2 + \frac{5}{4} \theta^2 + \frac{1}{4} \hat{\theta}^2 + (\xi_2 - k_1) \left( u + (\xi_1 - k_0) - \frac{\partial k_1}{\partial x} (\theta x + \xi_1) - \frac{\partial k_1}{\partial \theta} \right) \frac{\partial k_1}{\partial \xi_1} \xi_2
\]
\[
+ \frac{\partial k_1}{\partial x} (\xi_2 - k_1) \hat{\theta} - \frac{\partial k_1}{\partial \theta} (\xi_2 - k_1) \hat{\theta}
\]
\[
\leq -x^2 - (\xi_1 - k_0)^2 + \frac{5}{4} \theta^2 + \frac{1}{4} \hat{\theta}^2 - (\xi_2 - k_1)^2
\]
\[
+ (\xi_2 - k_1) \left( u + (\xi_1 - k_0) + (\xi_2 - k_1) - \frac{\partial k_1}{\partial x} (\theta x + \xi_1) - \frac{\partial k_1}{\partial \theta} \right) \frac{\partial k_1}{\partial \xi_1} \xi_2
\]
\[
+ \left( \frac{\partial k_1}{\partial x} \right)^2 (\xi_2 - k_1)^2 x^2 + \frac{1}{4} \theta^2 + \left( \frac{\partial k_1}{\partial \theta} \right)^2 (\xi_2 - k_1)^2 + \frac{1}{4} \hat{\theta}^2
\]
\[
= -x^2 - (\xi_1 - k_0)^2 - (\xi_2 - k_1)^2 + \frac{3}{2} \theta^2 + \frac{1}{2} \hat{\theta}^2 + (\xi_2 - k_1) \left( u + (\xi_1 - k_0) + (\xi_2 - k_1) - \frac{\partial k_1}{\partial x} (\theta x + \xi_1) - \frac{\partial k_1}{\partial \theta} \right) \frac{\partial k_1}{\partial \xi_1} \xi_2
\]
\[
+ (\xi_2 - k_1) \frac{\partial k_1}{\partial x} (\theta x + \xi_1) + \frac{\partial k_1}{\partial \theta} \xi_2 + \left( \frac{\partial k_1}{\partial x} \right)^2 (\xi_2 - k_1) x^2 + \left( \frac{\partial k_1}{\partial \theta} \right)^2 (\xi_2 - k_1)
\]
By setting
\[
u := k_2(x, \hat{\theta}, \xi_1, \xi_2)
\]
\[
:= - (\xi_1 - k_0) - (\xi_2 - k_1) + \frac{\partial k_1}{\partial x} (\theta x + \xi_1) + \frac{\partial k_1}{\partial \theta} \xi_2 - \left( \frac{\partial k_1}{\partial x} \right)^2 (\xi_2 - k_1) x^2 - \left( \frac{\partial k_1}{\partial \theta} \right)^2 (\xi_2 - k_1)
\]
we get
\[
\dot{V}_2(x, \hat{\theta}, \xi_1, \xi_2) \leq -x^2 - (\xi_1 - k_0)^2 - (\xi_2 - k_1)^2 + \frac{3}{2} \theta^2 + \frac{1}{2} \hat{\theta}^2
\]
\[
\leq -\alpha \left( \left| \begin{array}{c} x \\ \xi_1 \\ \xi_2 \end{array} \right| + \gamma \left| \begin{array}{c} \hat{\theta} \\ \hat{\theta} \end{array} \right| \right)
\]
for suitable \(\alpha, \gamma \in K_\infty\). Hence the closed-loop system is ISS with respect to \((\hat{\theta}, \hat{\theta})\).

4. (20 Points) Consider the following adaptive control system.

\text{Plant:} \dot{y} = ay + bu, \text{ where } a \text{ and } b \neq 0 \text{ are unknown parameters.}

\text{Control law:} u = -ky. \text{ Update law for } k: \dot{k} = b(a - bk + 1).

(Interpretation: drive } k \text{ to the equilibrium value } k = \frac{a + 1}{b}, \text{ but stop if } \dot{k} \rightarrow 0 \text{ to keep } k \text{ bounded.})

\text{Estimator:} \dot{\hat{y}} = -(\hat{y} - y) + \hat{a}y + \hat{b}u.

\text{Update laws for } \hat{a}, \hat{b} \text{ (as in class): } \dot{\hat{a}} = -\gamma \hat{e}y, \dot{\hat{b}} = -\gamma eu, \text{ where } \gamma > 0 \text{ and } e = \hat{y} - y.

Show that there exist values of \(a, b\) and initial values of \(y, \dot{y}, k, \hat{a}, \hat{b}\) for which we get a trajectory along which \(e \equiv 0\) but \(y \not\to \infty\). Interpret this situation in terms of lack of detectability.
Solution. The closed-loop system is given by

\[
\begin{align*}
\dot{y} &= (a - bk)y, \\
\dot{\hat{y}} &= -(\hat{y} - y) + (\hat{a} - \hat{b}k)y, \\
\dot{\hat{a}} &= -\gamma(\hat{y} - y)y, \\
\dot{\hat{b}} &= \gamma(\hat{y} - y)ky, \\
\dot{k} &= \hat{b}(\hat{a} - \hat{b}k + 1).
\end{align*}
\]

(5) (6) (7) (8) (9)

Combining (5) and (6) gives that

\[
\dot{e} = \dot{\hat{y}} - \dot{y} = -(\hat{y} - y) + \hat{a}y - \hat{b}ky - ay + bky = -e + ((\hat{a} - \hat{b}k) - (a - bk))y.
\]

In order to get \(e \equiv 0\), we need \(e(0) = 0\) and \(\dot{e} \equiv 0\), that is, \(\hat{y}(0) = y(0)\) and

\[
\hat{a} - \hat{b}k \equiv a - bk \tag{10}
\]

(as we need \(y \nearrow \infty\)). Moreover, substituting \(\hat{y} \equiv y\) into (7) and (8) gives that \(\hat{a} \equiv 0\) and \(\hat{b} \equiv 0\). Then (10) implies that \((\hat{b} - b)\hat{k} \equiv 0\), which, combined with (9) and (10), gives that \((\hat{b} - b)(a - bk + 1) \equiv 0\). In order to get \(y \nearrow \infty\), we need the differential equation (5) to be unstable, that is, \(a - bk > 0\). Hence we should select \(b \equiv b\) or \(\hat{b} \equiv 0\). For simplicity, we select \(\hat{b} \equiv 0\). Then (9) implies that \(\hat{k} \equiv 0\), and (10) implies that \(\hat{a} \equiv a - bk(0) > 0\). In summary, given the values of \(a, b\) and \(y(0)\), by selecting \(k(0)\) such that \(a - bk(0) > 0\), and

\[
\begin{align*}
\hat{y}(0) &= y(0), \\
\hat{a}(0) &= a - bk(0), \\
\hat{b}(0) &= 0,
\end{align*}
\]

we get \(\hat{y} = (a - bk(0))y\), which implies that \(e \equiv 0\) and \(y \nearrow \infty\).

Consider the \((y, e)\)-system with output \(e\). It is in the form

\[
\begin{pmatrix}
\dot{y} \\
\dot{e}
\end{pmatrix} = \begin{pmatrix}
a - bk & 0 \\
0 & -1
\end{pmatrix} \begin{pmatrix} y \\ e \end{pmatrix},
\]

\[
e = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} y \\ e \end{pmatrix}.
\]

It is clear that \(y\) is unobservable. If the coefficient \(a - bk\) remains constant and positive, then the dynamics of \(y\) is unstable, and the system under consideration becomes undetectable.