ECE517, Fall 2021 PROBLEM SET 3 Solution

1. The purpose of this exercise is to extend the backstepping procedure beyond pure integrator backstepping. Consider the system

$$\dot{x} = f(x) + g(x)\xi$$
$$\dot{\xi} = f_1(x,\xi) + g_1(x,\xi)u$$

where x, ξ, u are all scalar variables and $g_1(x,\xi) \neq 0\ \forall x, \xi$. Assume that a Lyapunov function $V_0(x)$ and a feedback law $k_0(x)$ (with $k_0(0) = 0$) are known such that

$$\frac{\partial V_0}{\partial x}\big(f(x) + g(x)k_0(x)\big) \leq -W(x) < 0 \qquad \forall\, x \neq 0$$

Construct a control Lyapunov function $V_1(x,\xi)$ and a stabilizing feedback law $k_1(x,\xi)$ for the 2-D system.

Solution. Consider the candidate Lyapunov function

$$V_1(x,\xi) := V_0(x) + \frac{(\xi - k_0(x))^2}{2}.$$

Its derivative along the state trajectory is

$$\begin{aligned}
\dot{V}_1(x,\xi) &= \frac{\partial V_0}{\partial x}(f(x) + g(x)\xi) + (\xi - k_0(x))\left(f_1(x,\xi) + g_1(x,\xi)u - \frac{\partial k_0}{\partial x}(f(x) + g(x)\xi)\right) \\
&= \frac{\partial V_0}{\partial x}(f(x) + g(x)\boldsymbol{k_0(x)}) + (\xi - k_0(x))\left(\boldsymbol{\frac{\partial V_0}{\partial x} g(x)} + f_1(x,\xi) + g_1(x,\xi)u - \frac{\partial k_0}{\partial x}(f(x) + g(x)\xi)\right) \\
&\leq -W(x) + (\xi - k_0(x))\left(\frac{\partial V_0}{\partial x} g(x) + f_1(x,\xi) + g_1(x,\xi)u - \frac{\partial k_0}{\partial x}(f(x) + g(x)\xi)\right).
\end{aligned}$$

Note the $g_1(x,\xi) \neq 0$. Letting

$$u := k_1(x,\xi) := \frac{1}{g_1(x,\xi)}\left(-\frac{\partial V_0}{\partial x}(x)g(x) - f_1(x,\xi) + \frac{\partial k_0}{\partial x}(f(x) + g(x)\xi) - (\xi - k_0(x))\right)$$

gives that

$$\dot{V}_1(x,\xi) = -W(x) - (\xi - k_0(x))^2 \leq 0,$$

and $\dot{V}_1(x,\xi) < 0$ for all $(x,\xi) \neq (0, k_0(0)) = (0,0)$. Hence $V_1(x,\xi)$ is a control Lyapunov function, and $k_1(x,\xi)$ is a stabilizing feedback law.

2. Consider the system

$$\dot{x} = \theta x + \xi_1$$
$$\dot{\xi}_1 = \xi_2$$
$$\dot{\xi}_2 = u$$

where x, ξ_1, ξ_2, u are all scalar variables and θ is an unknown parameter. Continue the adaptive backstepping procedure given in Section 5.2 in the lecture notes and design an adaptive control law that asymptotically stabilizes this system.

Solution. Consider the candidate Lyapunov function

$$V_2(x,\xi_1,\xi_2,\hat{\theta}) := V_1(x,\hat{\theta},\xi_1) + \frac{1}{2}(\xi_2 - k_1)^2$$

where

$$V_1(x,\hat{\theta},\xi_1) = \frac{1}{2}(x^2 + (\hat{\theta} - \theta)^2) + \frac{1}{2}(\xi_1 - k_0)^2,$$

and

$$k_0(x,\hat{\theta}) := -(\hat{\theta}+1)x, \tag{1}$$

$$k_1(x,\hat{\theta},\xi_1) := -x + \frac{\partial k_0}{\partial x}(\hat{\theta}x+\xi_1) + \frac{\partial k_0}{\partial \hat{\theta}}(\tau_0+\tau_1) - (\xi_1 - k_0), \tag{2}$$

$$\tau_0(x) := x^2, \tag{3}$$

$$\tau_1(x,\hat{\theta},\xi_1) := -\frac{\partial k_0}{\partial x}(\xi_1-k_0)x \tag{4}$$

as defined in the lecture notes. Calculating the derivative of V_1 along solutions gives that

$$\begin{aligned}
\dot{V}_1(x,\hat{\theta},\xi_1) &= x(\theta x+\xi_1) + (\hat{\theta}-\theta)\dot{\hat{\theta}} + (\xi_1-k_0)\left(\xi_2 - \frac{\partial k_0}{\partial x}(\theta x+\xi_1) - \frac{\partial k_0}{\partial \hat{\theta}}\dot{\hat{\theta}}\right)\\
&= x(\theta x+\boldsymbol{k_0}) + (\hat{\theta}-\theta)\dot{\hat{\theta}} + (\xi_1-k_0)\left(\xi_2 + \boldsymbol{x} - \frac{\partial k_0}{\partial x}(\theta x+\xi_1) - \frac{\partial k_0}{\partial \hat{\theta}}\dot{\hat{\theta}}\right)\\
&\overset{(1)}{=} -x^2 + (\hat{\theta}-\theta)(\dot{\hat{\theta}}-x^2) + (\xi_1-k_0)\left(\xi_2 + x - \frac{\partial k_0}{\partial x}(\theta x+\xi_1) - \frac{\partial k_0}{\partial \hat{\theta}}\dot{\hat{\theta}}\right)\\
&= -x^2 + (\hat{\theta}-\theta)\left(\dot{\hat{\theta}}-x^2 + \boldsymbol{\frac{\partial k_0}{\partial x}(\xi_1-k_0)x}\right) + (\xi_1-k_0)\left(\xi_2 + x - \frac{\partial k_0}{\partial x}(\boldsymbol{\hat{\theta}}x+\xi_1) - \frac{\partial k_0}{\partial \hat{\theta}}\dot{\hat{\theta}}\right)\\
&= -x^2 \boldsymbol{-(\xi_1-k_0)^2} + (\hat{\theta}-\theta)\left(\dot{\hat{\theta}}-x^2 + \frac{\partial k_0}{\partial x}(\xi_1-k_0)x\right)\\
&\quad + (\xi_1-k_0)\left(\xi_2 + x - \frac{\partial k_0}{\partial x}(\hat{\theta}x+\xi_1) - \frac{\partial k_0}{\partial \hat{\theta}}\dot{\hat{\theta}} + \boldsymbol{(\xi_1-k_0)}\right)\\
&\overset{(2)(3)(4)}{=} -x^2 - (\xi_1-k_0)^2 + (\hat{\theta}-\theta)(\dot{\hat{\theta}}-\tau_0-\tau_1) + (\xi_1-k_0)\left(\xi_2 - k_1 - \frac{\partial k_0}{\partial \hat{\theta}}(\dot{\hat{\theta}}-\tau_0-\tau_1)\right).
\end{aligned}$$

Then calculating the derivative of V_2 along solutions gives that

$$\begin{aligned}
\dot{V}_2(x,\hat{\theta},\xi_1,\xi_2) &= \dot{V}_1(x,\hat{\theta},\xi_1) + (\xi_2-k_1)\left(u - \frac{\partial k_1}{\partial x}(\theta x+\xi_1) - \frac{\partial k_1}{\partial \hat{\theta}}\dot{\hat{\theta}} - \frac{\partial k_1}{\partial \xi_1}\xi_2\right)\\
&= -x^2 - (\xi_1-k_0)^2 + (\hat{\theta}-\theta)(\dot{\hat{\theta}}-\tau_0-\tau_1) + (\xi_1-k_0)\left(\xi_2 - k_1 - \frac{\partial k_0}{\partial \hat{\theta}}(\dot{\hat{\theta}}-\tau_0-\tau_1)\right)\\
&\quad + (\xi_2-k_1)\left(u - \frac{\partial k_1}{\partial x}(\theta x+\xi_1) - \frac{\partial k_1}{\partial \hat{\theta}}\dot{\hat{\theta}} - \frac{\partial k_1}{\partial \xi_1}\xi_2\right)\\
&= -x^2 - (\xi_1-k_0)^2 + (\hat{\theta}-\theta)\left(\dot{\hat{\theta}}-\tau_0-\tau_1 + \boldsymbol{\frac{\partial k_1}{\partial x}(\xi_2-k_1)x}\right)\\
&\quad + (\xi_1-k_0)\left(\xi_2 - k_1 - \frac{\partial k_0}{\partial \hat{\theta}}(\dot{\hat{\theta}}-\tau_0-\tau_1)\right)\\
&\quad + (\xi_2-k_1)\left(u - \frac{\partial k_1}{\partial x}(\boldsymbol{\hat{\theta}}x+\xi_1) - \frac{\partial k_1}{\partial \hat{\theta}}\dot{\hat{\theta}} - \frac{\partial k_1}{\partial \xi_1}\xi_2\right)\\
&= -x^2 - (\xi_1-k_0)^2 + (\hat{\theta}-\theta)\left(\dot{\hat{\theta}}-\tau_0-\tau_1 + \frac{\partial k_1}{\partial x}(\xi_2-k_1)x\right)\\
&\quad + (\xi_1-k_0)\left(-\frac{\partial k_0}{\partial \hat{\theta}}\left(\dot{\hat{\theta}}-\tau_0-\tau_1 + \boldsymbol{\frac{\partial k_1}{\partial x}(\xi_2-k_1)x}\right)\right)\\
&\quad + (\xi_2-k_1)\left(u - \frac{\partial k_1}{\partial x}(\hat{\theta}x+\xi_1) - \frac{\partial k_1}{\partial \hat{\theta}}\dot{\hat{\theta}} - \frac{\partial k_1}{\partial \xi_1}\xi_2 + \boldsymbol{(\xi_1-k_0)\left(1+\frac{\partial k_0}{\partial \hat{\theta}}\frac{\partial k_1}{\partial x}x\right)}\right)
\end{aligned}$$

Hence setting

$$\begin{aligned}
\dot{\hat{\theta}} &:= \tau_0 + \tau_1 + \tau_2, \\
\tau_2 &:= -\frac{\partial k_1}{\partial x}(\xi_2 - k_1)x, \\
u &:= \frac{\partial k_1}{\partial x}(\hat{\theta}x + \xi_1) + \frac{\partial k_1}{\partial \hat{\theta}}\dot{\hat{\theta}} + \frac{\partial k_1}{\partial \xi_1}\xi_2 - (\xi_1 - k_0)\left(1 + \frac{\partial k_0}{\partial \hat{\theta}}\frac{\partial k_1}{\partial x}x\right) - (\xi_2 - k_1)
\end{aligned}$$

gives

$$\dot{V}_2(x, \hat{\theta}, \xi_1, \xi_2) = -x^2 - (\xi_1 - k_0)^2 - (\xi_2 - k_1)^2 < 0 \qquad \forall\ (x, \xi_1, \xi_2) \neq (0,0,0).$$

Then Theorem 2 implies that $x, \hat{\theta}, \xi_1, \xi_2$ are bounded and $x, \xi_1, \xi_2 \to 0$.

3. Prove Lemma 4 from the lecture notes. Be sure to give explicit expressions for λ, c in terms of α_0, T_0 and vice versa.

Solution. Recall that

$$\tilde{\theta}(t) = e^{-\gamma \int_{t_0}^{t} u(s)^2 \mathrm{d}s}\tilde{\theta}(t_0)$$

For all $s \in \mathbb{R}$, let $\lfloor s \rfloor$ denote the largest integer less or equal to s, i.e., $\lfloor s \rfloor := \max\{z \in \mathbb{Z} | z \leq s\}$.

First, suppose there exist $\alpha_0, T_0 > 0$ such that u is PE. Then

$$\begin{aligned}
|\tilde{\theta}(t)| &= e^{-\gamma \int_{t_0}^{t} u(s)^2 \mathrm{d}s}|\tilde{\theta}(t_0)| \\
&\leq e^{-\gamma \int_{t_0}^{t_0 + \left\lfloor \frac{t-t_0}{T_0} \right\rfloor T_0} u(s)^2 \mathrm{d}s}|\tilde{\theta}(t_0)| \\
&\overset{\text{(PE)}}{\leq} e^{-\gamma \left\lfloor \frac{t-t_0}{T_0} \right\rfloor \alpha_0 T_0}|\tilde{\theta}(t_0)| \\
&\leq e^{-\gamma \left(\frac{t-t_0}{T_0} - 1\right)\alpha_0 T_0}|\tilde{\theta}(t_0)| \\
&= e^{\gamma \alpha_0 T_0} e^{-\gamma \alpha_0 (t - t_0)}|\tilde{\theta}(t_0)|
\end{aligned}$$

for all $t \geq t_0 \geq 0$. Hence $\tilde{\theta}$ is UEC with $c := e^{\gamma \alpha_0 T_0} > 0$ and $\lambda := \gamma \alpha_0 > 0$.

Second, suppose there exist $c, \lambda > 0$ such that $\tilde{\theta}$ is UEC. Then

$$e^{-\gamma \int_t^\tau u(s)^2 \mathrm{d}s}|\tilde{\theta}(t)| \leq c e^{-\lambda(\tau - t)}|\tilde{\theta}(t)| \qquad \forall\ \tau \geq t \geq 0.$$

Taking logarithms on both sides gives

$$\int_t^\tau u(s)^2 \mathrm{d}s \geq \frac{\lambda}{\gamma}(\tau - t) - \frac{\log c}{\gamma} \qquad \forall\ \tau \geq t \geq 0.$$

By selecting $\alpha_0 < \lambda/\gamma$ and $T_0 \geq (\log c)/(\lambda - \gamma \alpha_0)$ and letting $\tau = t + T_0$, we get

$$\int_t^{t+T_0} u(s)^2 \mathrm{d}s \geq \frac{\lambda}{\gamma}T_0 - \frac{\log c}{\gamma} \geq \frac{\lambda}{\gamma}T_0 - \frac{\lambda - \gamma\alpha_0}{\gamma}T_0 = \alpha_0 T_0 \qquad \forall\ t \geq 0,$$

i.e., u is PE.

4. As defined in class, a vector-valued signal ϕ is persistently exciting (PE) if for some $\alpha_0, T_0 > 0$ we have $\int_t^{t+T_0} \phi(s)\phi^T(s) ds \geq \alpha_0 T_0 I$ for all t, where the inequality is to be understood in the matrix sense. Prove that the vector signal

$$\phi(t) := \begin{pmatrix} A\sin(\omega t + \alpha) \\ \sin \omega t \end{pmatrix}$$

is PE under suitable constraints on the numbers A, ω, α. Be sure to specify these constraints.

Solution. First we notice that if $\omega = 0$ then ϕ cannot be PE. Hence Integrating the matrix gives

$$\begin{aligned}
&\int_t^{t+T_0} \phi(s)\phi(s)^\top \mathrm{d}s \\
&= \int_t^{t+T_0} \begin{bmatrix} A^2 \sin^2(\omega s + \alpha) & A\sin(\omega s + \alpha)\sin \omega s \\ A\sin(\omega s + \alpha)\sin \omega s & \sin^2 \omega s \end{bmatrix} \mathrm{d}s \\
&= \int_t^{t+T_0} \frac{1}{2} \begin{bmatrix} A^2 - A^2\cos(2(\omega s + \alpha)) & A\cos\alpha - A\cos(2\omega s + \alpha) \\ A\cos\alpha - A\cos(2\omega s + \alpha) & 1 - \cos(2\omega s) \end{bmatrix} \mathrm{d}s \\
&= \frac{T_0}{2} \begin{bmatrix} A^2 & A\cos\alpha \\ A\cos\alpha & 1 \end{bmatrix} + \frac{1}{4\omega} \begin{bmatrix} A^2 \sin(2(\omega s + \alpha)) & A\sin(2\omega s + \alpha) \\ A\sin(2\omega s + \alpha) & \sin(2\omega s) \end{bmatrix} \Bigg|_{s=t}^{t+T_0} \\
&= \frac{T_0}{2} \begin{bmatrix} A^2 & A\cos\alpha \\ A\cos\alpha & 1 \end{bmatrix} + \frac{\sin(\omega T_0)}{2\omega} \begin{bmatrix} A^2 \cos(2\omega t + 2\alpha + \omega T_0) & A\cos(2\omega t + \alpha + \omega T_0) \\ A\cos(2\omega t + \alpha + \omega T_0) & \cos(2\omega t + \omega T_0) \end{bmatrix}
\end{aligned}$$

Letting $T_0 = \pi/|\omega|$ gives

$$H(t) := \int_t^{t+T_0} \phi(s)\phi(s)^\top \mathrm{d}s = \frac{\pi}{2|\omega|} \begin{bmatrix} A^2 & A\cos\alpha \\ A\cos\alpha & 1 \end{bmatrix} \qquad \forall\ t.$$

Note that we can always select a sufficiently small $\alpha_0 > 0$ so that $H(t) \geq \alpha_0 T_0 I$ for all t, provided that $H(t) > 0$ for all t, or equivalently, that both of the principal minors of H are always positive, that is,

$$m_1 = \frac{\pi}{2|\omega|} A^2 > 0,$$

and

$$m_2 = \frac{\pi}{2|\omega|} \begin{vmatrix} A^2 & A\cos\alpha \\ A\cos\alpha & 1 \end{vmatrix} = \frac{\pi}{2|\omega|} A^2(1 - \cos^2\alpha) > 0.$$

Therefore, ϕ is PE if $\omega \neq 0$, $A \neq 0$ and $\alpha \neq n\pi, n \in \mathbb{Z}$.

5. Exercise 4.2 on page 246 in [Ioannou-Sun].[1]

Consider the second order stable system

$$\dot{x} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & 0 \end{bmatrix} x + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u$$

where x, u are available for measurement, $u \in \mathcal{L}_\infty$ and $a_{11}, a_{12}, a_{21}, b_1, b_2$ are unknown parameters. Design an on-line estimator to estimate the unknown parameters. Simulate your scheme using $u = 10\sin 2t$ and

$$A = \begin{bmatrix} -0.25 & 3 \\ -5 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ 2.2 \end{bmatrix}$$

Repeat the simulation when $u = 10\sin 2t + 7\cos 3.6t$.

Solution. Since the plant is stable, consider the estimator

$$\dot{\hat{x}} = A_m(\hat{x} - x) + \hat{A}x + \hat{B}u,$$

where A_m is Hurwtiz, and $\hat{x}, \hat{A}, \hat{B}$ are the estimations of x, A, B respectively. The update laws are given by

$$\begin{aligned}
\dot{\hat{A}} &= -P(\hat{x} - x)x^\top, \\
\dot{\hat{B}} &= -P(\hat{x} - x)u,
\end{aligned}$$

where P is the solution to the Lyapunov function $A_m^\top P + PA_m = -I$. In the simulation, we select $A_m = -5I$, which gives $P = 0.1I$.

[1]Use the link on the class homepage to access the electronic version of [Ioannou-Sun].

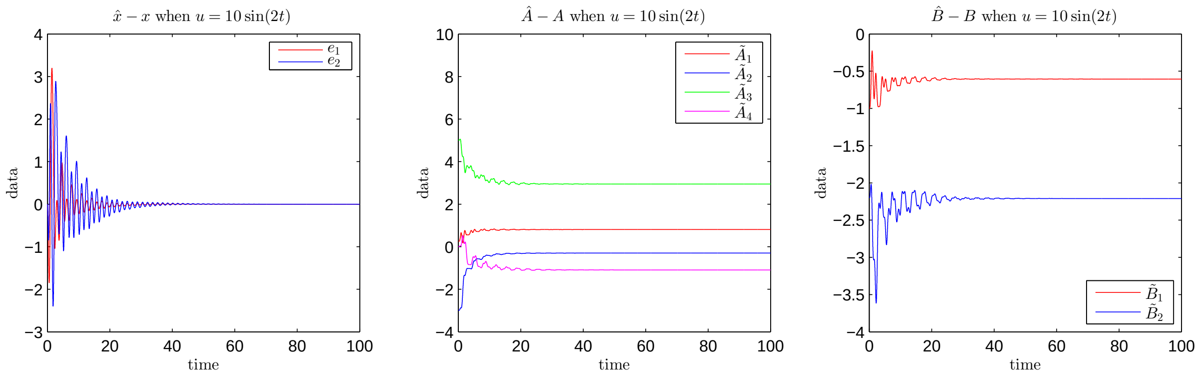

Fig. 1: Simulation result using $u = 10\sin(2t)$

1) The simulation using $u = 10\sin(2t)$ can be found in Fig. 1. We observe that $e \to 0$, but $\hat{A} \nrightarrow A$ and $\hat{B} \nrightarrow B$. This is due to the input u not being sufficiently rich (in term of frequency components).
2) The simulation result using $u = 10\sin(2t) + 7\cos(3.6t)$ can be found in Fig. 2. We observe that $e \to 0$, $\hat{A} \to A$ and $\hat{B} \to B$ as the input has one more frequency component.

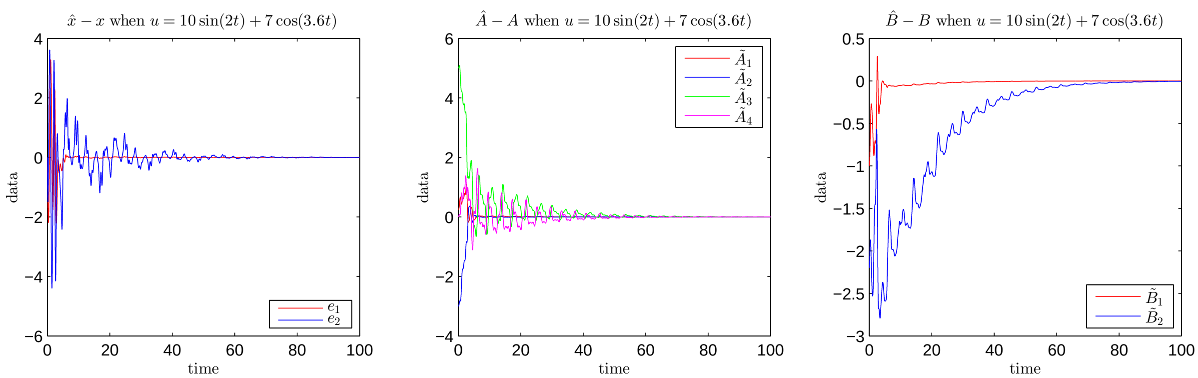

Fig. 2: Simulation result using $u = 10\sin(2t) + 7\cos(3.6t)$