1. Investigate whether or not the following system is asymptotically stable around the origin:

$$\dot{x}_1 = x_2 - x_1^3$$
$$\dot{x}_2 = 2x_1 - 3x_1x_2 - 4x_2$$

2. The following statement is known to be false: If $V(x)$ is a C^1 positive definite function whose derivative along solutions of $\dot{x} = f(t,x)$ satisfies $\dot{V}(t,x) \le -W(x) \le 0$ for all x, then every bounded solution converges to the largest invariant set inside the set $\{x : W(x) = 0\}$. If one tries to prove this statement using the argument given in class to prove LaSalle's theorem for time-invariant systems, there is exactly one place where this argument breaks down and cannot be easily fixed. Explain where this happens and why.

3. Consider the LTI system

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

Suppose that there is a positive semidefinite matrix P that solves the Riccati equation

$$PA + A^TP + PBB^TP + C^TC = 0.$$

Prove that the system has an $\mathcal{L}_2$ gain $\gamma \le 1$.

4. Consider a linear time-varying system

$$\dot{x} = A(t)x + B(t)u, \qquad y = C(t)x \tag{1}$$

Assume that the system $\dot{x} = A(t)x$ (without input) is uniformly exponentially stable and that the time-varying matrices $B(t)$ and $C(t)$ are uniformly bounded over time. Then the system (1) has finite L_2-to-L_2, L_∞-to-L_∞, and L_2-to-L_∞ induced gains (from u to y). Prove this by computing explicit upper bounds for at least two of these three gains.

5. Investigate stability of the system $\ddot{x} + \dot{x} = u$ under nonlinear feedbacks of the form $u = -\varphi(x)$ by applying (if possible) small-gain theorem, passivity criterion, circle criterion, and Popov's criterion. Explain what you get from each of these results. For each criterion that is applicable, also try to directly find a Lyapunov function in the suitable form guaranteed to exist by that criterion.