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1. Is the system

$$\dot{x} = -\frac{2xt}{1+t^2}, \qquad x \in \mathbb{R}$$

stable? asymptotically stable? globally asymptotically stable? uniformly globally asymptotically stable? (All stability properties refer to the equilibrium at the origin.) Justify your answers with the help of the respective stability definitions.

Solution: Suppose $x(t_0) = x_0 \in \mathbb{R}$. Then

$$
\begin{aligned}
\dot{x} = -\frac{2xt}{1+t^2} \quad &\Rightarrow \quad \frac{\mathrm{d}x}{x} = -\frac{2t\mathrm{d}t}{1+t^2} \\
&\Rightarrow \quad \frac{\mathrm{d}x}{x} = -\frac{\mathrm{d}(1+t^2)}{1+t^2} \\
&\Rightarrow \quad \log(x(t)) - \log(x_0) = -(\log(1+t^2) - \log(1+t_0^2)) \\
&\Rightarrow \quad x(t) = \frac{1+t_0^2}{1+t^2}x_0.
\end{aligned}
$$

- Since $|x(t)| \le |x_0|$ for all $t \in [t_0, \infty)$, the system is clearly (uniformly) stable.

- Since the system is stable, and $\lim_{t\to\infty} x(t) = 0$ for all $x_0 \in \mathbb{R}$, the system is globally asymptotically stable, and thus asymptotically stable.

- Recall that the system is globally uniformly asymptotically stable (GUAS) if it is uniformly stable, and for all $\delta, \epsilon > 0$, there exists $T = T(\delta, \epsilon)$ such that

$$|x_0| < \delta \quad \Rightarrow \quad |x(t)| \le \epsilon \quad \forall\, t \in [t_0 + T, \infty).$$

On the other hand, consider the case $\epsilon = \delta/16$. For any $T > 0$, let $t_0 = T$ and $x_0 = \delta/2$. Then

$$|x(t_0 + T)| = \frac{1+t_0^2}{1+(t_0+T)^2}x_0 = \frac{1+T^2}{1+4T^2} \cdot \frac{\delta}{2} > \frac{1}{8}\delta > \epsilon.$$

Thus the system is not GUAS.

$\square$

2. Let $V : \mathbb{R}^n \to \mathbb{R}$ be a continuous and positive definite function, and consider the following three properties:

a) V is radially unbounded, i.e., $V(x) \to \infty$ whenever $|x| \to \infty$.

b) All *level sets* of V are bounded, i.e., $\{x : V(x) = c\}$ is bounded for every c in the range of V.

c) All *sub-level sets* of V are bounded, i.e., $\{x : V(x) \leq c\}$ is bounded for every $c \geq 0$.

For each of the six possible implications between these properties, either prove that it is true or give a counterexample. (20 Points)

Solution:

1. $a) \Rightarrow b)$:
   Assume there exists $c \geq 0$ such that $\{x : V(x) = c\}$ is unbounded, then for any $k \in \mathbb{Z}_{>0}$, there exists an $x_k \in \mathbb{R}^n$ such that $|x_k| > k$ and $V(x) = c$. Then $\lim_{k\to\infty} |x_k| = \infty$, while $\lim_{k\to\infty} V(x_k) = \lim_{k\to\infty} c = c$. Hence V is not radially unbounded.

2. $b) \nRightarrow a)$:
   Consider $V : \mathbb{R} \to \mathbb{R}$ defined as
   $$V(x) = \tan^{-1}(|x|).$$
   The range of V is $[0, \pi/2)$. Then $\{x : V(x) = c\} = \{\pm \tan(c)\}$ for every $c \in [0, \pi/2)$, which is always bounded. On the other hand, $\lim_{|x|\to\infty} V(x) = \pi/2$.

3. $b) \nRightarrow c)$:
   Same counterexample as in 2. The set $\{x : V(x) \leq \pi/2\} = \mathbb{R}$ is unbounded.

4. $c) \Rightarrow b)$:
   This is true since $\{x : V(x) = c\} \subset \{x : V(x) \leq c\}$.

5. $c) \Rightarrow a)$:
   Suppose V is not radially unbounded, that is, there exists a sequence $\{x_k\}_{k\in\mathbb{Z}_{>0}}$ such that $\lim_{k\to\infty} |x_k| = \infty$, and an $M \geq 0$ such that $V(x_k) \leq M$ for all k. Then $\{x : V(x) \leq M\}$ is unbounded, since for any $N > 0$, there exist $k^* \in \mathbb{Z}_{>0}$ such that $|x_{k^*}| > N$, but $V(x_{k^*}) \leq M$.

6. $a) \Rightarrow c)$:
   The proof is similar to that in 1. Assume there exists $c \geq 0$ such that $\{x : V(x) \leq c\}$ is unbounded, then for any $k \in \mathbb{Z}_{>0}$, there exists an $x_k \in \mathbb{R}^n$ such that $|x_k| > k$ and $V(x) \leq c$. Then $\lim_{k\to\infty} |x_k| = \infty$, while $\lim_{k\to\infty} V(x_k) \leq \sup_{k\in\mathbb{Z}_{>0}} V(x_k) \leq c$. Hence V is not radially unbounded.

$\square$

3. Suppose that a function $V : \mathbb{R}^n \to \mathbb{R}$ is positive definite and radially unbounded, so that (as shown in class) we have

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|)$$

for some class $\mathcal{K}_\infty$ functions α_1, α_2. Suppose also that the derivative of V along solutions of the system $\dot{x} = f(x)$ satisfies $\dot{V} \leq 0$. According to Lyapunov's stability theorem, the system is then stable, meaning that for every $\epsilon > 0$ there exists a $\delta > 0$ such that for initial conditions satisfying $|x(0)| \leq \delta$ we have $|x(t)| \leq \epsilon$ for all $t \geq 0$. Derive a specific expression for δ as a function of ϵ, in terms of α_1 and α_2

Solution: As $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, they are globally invertible. Let $\delta = \alpha_2^{-1}(\alpha_1(\epsilon))$. Since α_1 and α_2 are strictly increasing,

$$
\begin{aligned}
|x(0)| \leq \delta \quad &\Rightarrow \quad V(x(0)) \leq \alpha_2(|x(0)|) \leq \alpha_2(\delta) = \alpha_2(\alpha_2^{-1}(\alpha_1(\epsilon))) = \alpha_1(\epsilon) \\
&\Rightarrow \quad \alpha_1(|x(t)|) \leq V(x(t)) \leq V(x(0)) \leq \alpha_1(\epsilon) \\
&\Rightarrow \quad |x(t)| \leq \epsilon
\end{aligned}
$$

for all $t \geq 0$. $\qquad\square$

4. Give a rigorous proof of the following fact: if a sequence of points along a solution trajectory converges to a stable equilibrium, then the entire solution trajectory converges to this equilibrium.

Solution: Let x_e be the stable equilibrium. If a sequence of points $\{x(t_k)\}_{k \in \mathbb{N}} \to x_e$ then for all $\delta > 0, \exists\ K_\delta \in \mathbb{N}$ such that $|x(t_k) - x_e| \leq \delta$ whenever $k > K_\delta$. Since the equilibrium is known to be stable; given any $\varepsilon > 0$ (fixed) there exists $\delta_\varepsilon > 0$ such that $|x(t_0) - x_e| < \delta_\varepsilon \implies |x(t) - x_e| \leq \varepsilon$. For this δ_ε find the corresponding K in the previous statement, i.e. $N \triangleq K_{\delta_\varepsilon}$. Then $\{x(t_k)\}_{k > N} \to x_e$ and $|x(t) - x_e| < \delta_\varepsilon$ for all $t > t_N$. Since $\varepsilon > 0$ can be chosen arbitrarily small this shows the entire solution trajectory converges to x_e. $\qquad\square$

5. Suppose that there exists a positive definite C^1 function V whose derivative along solutions of the system $\dot{x} = f(x)$ satisfies $\dot{V}(x) > 0$ for all x in the set $\{x : 0 < |x| \leq \varepsilon\}$ for some $\varepsilon > 0$. Prove that then the system is not stable. (20 Points)

Note: there is a sharper version of this result, known as Chetaev's Instability Theorem (Theorem 4.3 in the textbook). However, you should prove the above result directly, without relying on the proof of Chetaev's theorem. (You can then compare your argument with the one given in the book.)

Solution: Suppose the system is stable. Then there exists a $\delta \in (0, \epsilon]$ such that $|x(0)| \le \delta$ implies $|x(t)| \le \epsilon$ for all $t \ge 0$. Consider an $x(0)$ such that $|x(0)| = \delta$, and let $V_0 := V(x(0))$. The fact that V is positive definite and $\mathcal{C}^1$ implies that there exist

$$b := \min_{|x|=\delta} V(x),$$

$$c := \max_{|x|\le\epsilon} V(x),$$

and an $r > 0$ such that $\max_{|x|\le r} V(x) \le b/2$. Since $|x(t)| \le \epsilon$ for all $t \ge 0$, we have that $\dot{V}(x(t)) \ge 0$, and thus $V(x(t)) \ge V(x(0)) \ge b$ for all $t \ge 0$. Hence $x(t) \in \{x : r \le |x| \le \epsilon\}$ for all $t \ge 0$. Since V is $\mathcal{C}^1$, and $\{x : r \le |x| \le \epsilon\}$ is compact, there exists

$$d := \min_{r\le|x\|\le\epsilon} \dot{V}(x) > 0.$$

Then for $t = 2(c - V_0)/d$, we have that

$$V(t) \ge V_0 + dt \ge 2c > c = \max_{|x|\le\epsilon} V(x).$$

This is a contradiction. Therefore, the system is unstable. $\square$

6. Prove that the system given by the second-order equation

$$\ddot{x} + \psi(x)\dot{x} + \varphi(x) = 0$$

is globally asymptotically stable for all continuous functions ψ and φ satisfying the following conditions:

1. $x\varphi(x) > 0$ for all $x \ne 0$.

2. $\psi(x) > 0$ for all $x \ne 0$.

3. The function $\Phi(x) := \int_0^x \varphi(z)dz$ is radially unbounded.

(20 Points)

Solution: The system can be represented by

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = -\psi(x_1)x_2 - \varphi(x_1). \end{cases}$$

First we show that the origin $(0, 0)$ is an equilibrium, which amounts to showing that $\varphi(0) = 0$. Suppose that $\varphi(0) \neq 0$. Without loss of generality, assume that $\varphi(0) > 0$. Since φ is continuous, then there exist $\delta > 0$ such that $\varphi(-\delta) > 0$, which implies that $-\delta\varphi(-\delta) < 0$. This is a contradiction. The case when $\varphi(0) < 0$ is similar. Hence $\varphi(0) = 0$, and $(0, 0)$ is an equilibrium.

Next we show that Φ is positive definite. It is clear that $\Phi(0) = 0$. For any $x > 0$, the the Mean Value Theorem implies that there exists a $y \in (0, x)$ such that $\Phi(x) = y\varphi(y) > 0$. The case when $x < 0$ is similar. Hence Φ is positive definite.

Consider the function $V : \mathbb{R}^2 \to \mathbb{R}$ defined as

$$V(x_1, x_2) := \frac{1}{2}x_2^2 + \Phi(x_1) = \frac{1}{2}x_2^2 + \int_0^{x_1} \varphi(z)\mathrm{d}z.$$

It is $\mathcal{C}^1$ as

$$\frac{\partial}{\partial x_1}V(x_1, x_2) = \varphi(x_1), \quad \frac{\partial}{\partial x_2}V(x_1, x_2) = x_2$$

are both continuous, positive definite, and radially unbounded as x_2^2 and $\Phi(x_1)$ are both positive definite and radially unbounded. We calculate

$$\dot{V}(x_1, x_2) = \frac{\partial}{\partial x_1}V(x_1, x_2)\dot{x}_1 + \frac{\partial}{\partial x_2}V(x_1, x_2)\dot{x}_2 = x_2\varphi(x_1) - \psi(x_1)x_2^2 - x_2\varphi(x_1) = -\psi(x_1)x_2^2.$$

As $\psi(x) > 0$ for all $x \neq 0$, $\dot{V}(x_1, x_2) \leq 0$ on $\mathbb{R}^2$ and

$$\{(x_1, x_2) : \dot{V}(x_1, x_2) = 0\} = \{(x_1, x_2) : \psi(x_1) = 0 \text{ or } x_2 = 0\} \subset \{(x_1, x_2) : x_1x_2 = 0\} =: S.$$

Let $M \subset S$ be the largest invariant set. Then any solution with $(x_1(0), x_2(0)) \in M$ satisfies $x_1(t)x_2(t) = 0$ for all $t \in [0, \infty)$, and thus

$$0 \equiv \frac{\mathrm{d}(x_1x_2)}{\mathrm{d}t} = \dot{x}_1x_2 + x_1\dot{x}_2 = x_2^2 - x_1\psi(x_1)x_2 - x_1\varphi(x_1) = x_2^2 - x_1\varphi(x_1).$$

The condition $x\varphi(x) > 0$ on $\mathbb{R}\backslash\{0\}$ implies that every $(x_1, x_2) \in M$ satisfies that $x_1 = 0$ if and only if $x_2 = 0$. This, together with the fact that $M \subset S$, implies that $M = \{(0, 0)\}$. Then LaSalle's invariance principle (and [Khalil, Corollary 4.2]) implies that the system is globally asymptotically stable. $\qquad\square$