ECE517, Fall 2021 PROBLEM SET 2 Solution

1. Consider the class of scalar plants

$$\dot{y} = ay + bu, \qquad a \in \mathbb{R},\ b > 0 \tag{1}$$

In Section 3.1.1 of class notes, it is shown that the controller (19) is a universal regulator for this class of plants with the help of the Lyapunov function $V(y) = y^2/2$. For the same controller, find a Lyapunov function more similar to the Lyapunov function (3) used in Example 1 (i.e., one that depends also on a, b, k) such that convergence of y to 0 in closed loop can be proved by a direct application of Theorem 2.

Solution. The controller is given by eq. (19)

$$\begin{aligned} \dot{k} &= y^2, \\ u &= -ky. \end{aligned}$$

Then the closed-loop system is in the form

$$\begin{aligned} \dot{y} &= (a - bk)y, \\ \dot{k} &= y^2. \end{aligned}$$

Consider the Lyapunov function

$$V(y, k) := \frac{y^2}{2b} + \frac{(k - a/b - 1)^2}{2}.$$

Its derivative along the solution to the closed-loop system satisfies

$$\begin{aligned} \dot{V}(y, k) &= \frac{y}{b} \cdot \dot{y} + \left(k - \frac{a}{b} - 1\right) \dot{k} \\ &= \frac{a}{b} y^2 - ky^2 + ky^2 + \frac{a}{b} y^2 - y^2 \\ &= -y^2 \\ &= -W(y, k) \le 0, \end{aligned}$$

where $W(y, k) := y^2$ is positive semidefinite (nonnegative definite). *As V is radially unbounded, all solutions (y, k) remain bounded.* Hence Theorem 2 implies that $W(y(t), k(t)) \to 0$ as $t \to 0$, that is, all y converge to 0.

2. Consider again the class of scalar plants (1). Show that there doesn't exist a *linear* universal regulator for this class of plants, i.e., a universal regulator of the form (22) from class notes with f and h linear functions. Here the dimension of z can be arbitrary. (Thus you cannot use the non-existence result for rational controllers proved in class, because it is restricted to scalar z.)

Solution. A linear regulator is of the form

$$\begin{aligned} \dot{z} &= Az + By, \\ u &= Hz + ky, \end{aligned}$$

where $z \in \mathbb{R}^n$ and $u \in \mathbb{R}$ (and A, B, H, k are of suitable dimensions). Then the closed-loop system is

$$\begin{pmatrix} \dot{z} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} A & B \\ bH & a + bk \end{pmatrix} \begin{pmatrix} z \\ y \end{pmatrix}.$$

Let

$$F = \begin{pmatrix} A & B \\ bH & a + bk \end{pmatrix}.$$

We will show that, for any A, B, H, k, there exists $a \in \mathbb{R}$ and $b > 0$ such that F is not Hurwitz. Indeed, recall that the trace of a matrix is the sum of its eigenvalues. Hence a necessary condition for F to be Hurwitz is that

$\operatorname{tr}(F) = \operatorname{tr}(A) + a + bk < 0$. However, for any $A \in \mathbb{R}^{n\times n}$ and $k \in \mathbb{R}$, there are sufficiently large $a \in \mathbb{R}$ and sufficiently small $b > 0$ such that $\operatorname{tr}(F) = a + bk + \operatorname{tr}(A) > 0$ (e.g., $b = 1$ and $a = |\operatorname{tr}(A)| + |k| + 1$). Hence there doesn't exist a linear universal regulator for this class of plants.

3. Design a universal regulator for the class of scalar plants

$$\dot{y} = a\varphi(y) + bu, \qquad a \in \mathbb{R},\ b > 0$$

where $\varphi(\cdot)$ is a fixed known function. Justify rigorously that it works.

Solution. Consider the candidate Lyapunov function

$$V(y,k) := \frac{y^2}{2b} + \frac{(k-a/b)^2}{2}.$$

Its derivative along the state trajectory is

$$\begin{aligned}\dot{V}(y,k) &= \frac{y}{b}\dot{y} + \left(k - \frac{a}{b}\right)\dot{k}\\ &= \frac{a}{b}(y\phi(y) - \dot{k}) + k\dot{k} + yu.\end{aligned}$$

Hence we select the regulator

$$\begin{aligned}\dot{k} &= y\varphi(y),\\ u &= -y - k\varphi(y),\end{aligned}$$

which gives $\dot{V}(y,k) = -y^2 = -W(y,k) \le 0$, where $W(y,k) := y^2$ is positive semidefinite (nonnegative definite). Since V is radially unbounded, all solutions (y,k) remain bounded. Hence Theorem 2 implies that $W(y(t),k(t)) \to 0$ as $t \to 0$, that is, all y converge to 0.

4. Consider a linear system

$$\dot{x} = Ax + Bu$$

and assume that A is Hurwitz, so that we have $\|e^{At}\| \le ce^{-\lambda_0 t}$ for some $c, \lambda_0 > 0$. Prove the following:

a) If $u \in L_2$ or u is bounded, then x is bounded. (Hint: use the variation-of-constants formula and the Cauchy-Schwartz and Hölder's inequalities.)

b) If $u \in L_2$ or $u \to 0$, then $x \to 0$. (Hint: use part a).)

Solution.

a) By the variation-of-constants formula, we have

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-s)}Bu(s)\mathrm{d}s$$

for all $t \in [0,\infty)$. Hence

$$\begin{aligned}\|x(t)\| &= \left\| e^{At}x(0) + \int_0^t e^{A(t-s)}Bu(s)\mathrm{d}s\| \right\|\\ &\le \|e^{At}x(0)\| + \left\|\int_0^t e^{A(t-s)}Bu(s)\mathrm{d}s\right\|\\ &\le \|e^{At}\|\|x(0)\| + \int_0^t \|e^{A(t-s)}\|\|B\|\|u(s)\|\mathrm{d}s\\ &\le ce^{-\lambda_0 t}\|x(0)\| + \int_0^t ce^{-\lambda_0(t-s)}\|B\|\|u(s)\|\mathrm{d}s.\end{aligned} \tag{2}$$

If $u \in L_2$, then there exists $M_1 > 0$ such that

$$\int_0^\infty \|u(s)\|^2\mathrm{d}s \le M_1.$$

Hence applying the Cauchy-Schwartz inequality to (2) gives that

$$\begin{aligned}\|x(t)\| &\le ce^{-\lambda_0 t}\|x(0)\| + \int_0^t ce^{-\lambda_0(t-s)}\|B\|\|u(s)\|\mathrm{d}s \\ &\le ce^{-\lambda_0 t}\|x(0)\| + c\|B\|\sqrt{\left(\int_0^t e^{-2\lambda_0(t-s)}\mathrm{d}s\right)\left(\int_0^t \|u(s)\|^2\mathrm{d}s\right)} \\ &\le ce^{-\lambda_0 t}\|x(0)\| + c\|B\|\sqrt{\frac{(1-e^{-2\lambda_0 t})M_1}{2\lambda_0}} \\ &\le c\|x(0)\| + c\|B\|\sqrt{\frac{M_1}{2\lambda_0}},\end{aligned}$$

that is, x is bounded.
On the other hand, if u is bounded, then there exists $M_2 > 0$ such that

$$\|u(s)\| \le M_2 \qquad \forall\ s \in [0, \infty).$$

Hence (2) implies that

$$\begin{aligned}\|x(t)\| &\le ce^{-\lambda_0 t}\|x(0)\| + \int_0^t ce^{-\lambda_0(t-s)}\|B\|\|u(s)\|\mathrm{d}s \\ &\le ce^{-\lambda_0 t}\|x(0)\| + c\|B\|M_2\int_0^t e^{-\lambda_0(t-s)}\mathrm{d}s \\ &= ce^{-\lambda_0 t}\|x(0)\| + \frac{(1-e^{-\lambda_0 t})c\|B\|M_2}{\lambda_0} \\ &\le c\|x(0)\| + \frac{c\|B\|M_2}{\lambda_0},\end{aligned}$$

that is, x is bounded.

b) Consider an arbitrary $\epsilon > 0$. If $u \in L_2$, the Cauchy's convergence test shows that there exists $T_1 > 0$ such that

$$\int_{T_1}^{\tau}\|u(s)\|^2\mathrm{d}s \le \frac{\lambda_0\epsilon^2}{2c^2\|B\|^2} \qquad \forall\ \tau \ge T_1.$$

As the system of interest is time-invariant, the variation-of-constants formula implies that for all $t \ge T_1$, we have

$$x(t) = e^{A(t-T_1)}x(T_1) + \int_{T_1}^t e^{A(t-s)}Bu(s)\mathrm{d}s,$$

and thus

$$\begin{aligned}\|x(t)\| &= \left\|e^{A(t-T_1)}x(T_1) + \int_{T_1}^t e^{A(t-s)}Bu(s)\mathrm{d}s\|\right\| \\ &\le ce^{-\lambda_0(t-T_1)}\|x(T_1)\| + c\|B\|\int_{T_1}^t e^{-\lambda_0(t-s)}\|u(s)\|\mathrm{d}s \\ &\le ce^{-\lambda_0(t-T_1)}\|x(T_1)\| + c\|B\|\sqrt{\left(\int_{T_1}^t e^{-2\lambda_0(t-s)}\mathrm{d}s\right)\left(\int_{T_1}^t \|u(s)\|^2\mathrm{d}s\right)} \\ &= ce^{-\lambda_0(t-T_1)}\|x(T_1)\| + c\|B\|\sqrt{\frac{1-e^{-2(t-T_1)}}{2\lambda_0}\frac{\lambda_0\epsilon^2}{2c^2\|B\|^2}} \\ &\le ce^{-\lambda_0(t-T_1)}\|x(T_1)\| + c\|B\|\sqrt{\frac{1}{2\lambda_0}\frac{\lambda_0\epsilon^2}{2c^2\|B\|^2}} \\ &= ce^{-\lambda_0(t-T_1)}\|x(T_1)\| + \frac{\epsilon}{2}.\end{aligned}$$

Moreover, from a) we see that

$$\|x(T_1)\| \leq c\|x(0)\| + c\|B\|\sqrt{\frac{M_1}{2\lambda_0}} =: E_1.$$

Hence for all

$$t \geq T_1 + \frac{|\ln(2cE_1/\epsilon)|}{\lambda_0},$$

we have $\|x(t)\| \leq \epsilon$. As $\epsilon > 0$ is arbitrary, we get $x \to 0$.
On the other hand, if $u \to 0$. Then there exists $T_2 > 0$ such that

$$u(s) \leq \frac{\lambda_0 \epsilon}{2c\|B\|} \qquad \forall\ s \geq T_2.$$

Again, by the time-invariance property of the system and the variation-of-constants formula, for all $t \geq T_2$ we get

$$x(t) = e^{A(t-T_2)}x(T_2) + \int_{T_2}^{t} e^{A(t-s)}Bu(s)\mathrm{d}s,$$

and thus

$$\begin{aligned}
\|x(t)\| &= \left\| e^{A(t-T_2)}x(T_2) + \int_{T_2}^{t} e^{A(t-s)}Bu(s)\mathrm{d}s\| \right\| \\
&\leq ce^{-\lambda_0(t-T_2)}\|x(T_2)\| + c\|B\| \int_{T_2}^{t} e^{-\lambda_0(t-s)}\|u(s)\|\mathrm{d}s \\
&\leq ce^{-\lambda_0(t-T_2)}\|x(T_2)\| + \frac{\lambda_0\epsilon}{2} \int_{T_2}^{t} e^{-\lambda_0(t-s)}\mathrm{d}s \\
&= ce^{-\lambda_0(t-T_2)}\|x(T_2)\| + \frac{\lambda_0\epsilon}{2} \frac{1 - e^{-\lambda_0(t-T_2)}}{\lambda_0} \\
&\leq ce^{-\lambda_0(t-T_2)}\|x(T_2)\| + \frac{\epsilon}{2}.
\end{aligned}$$

Moreover, from a) we see that

$$\|x(T_2)\| \leq c\|x(0)\| + \frac{c\|B\|M_2}{\lambda_0} =: E_2.$$

Hence for all

$$t \geq T_2 + \frac{|\ln(2cE_2/\epsilon)|}{\lambda_0},$$

we have $\|x(t)\| \leq \epsilon$. As $\epsilon > 0$ is arbitrary, we get $x \to 0$.
We cannot use Barbalat's lemma to conclude that $u \in L_2$ implies $u \to 0$, since u is not necessarily continuous.

5. Simulate the control systems described in Examples 13.16 and 13.17 in Khalil's Nonlinear Systems book (3rd edition, pp. 532–534) and confirm the unstable behavior of closed-loop solutions.

Solution.

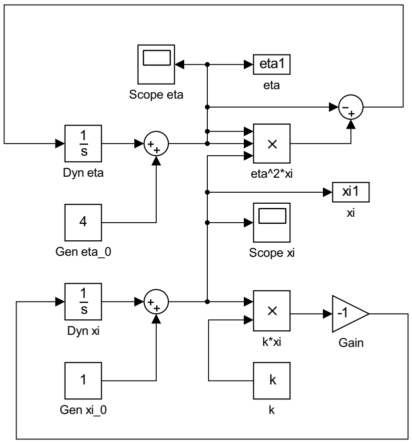
1) In Examples 13.16, we consider the second-order system

$$\begin{aligned}
\dot{\eta} &= -\eta + \eta^2\xi, \\
\dot{\xi} &= v,
\end{aligned}$$

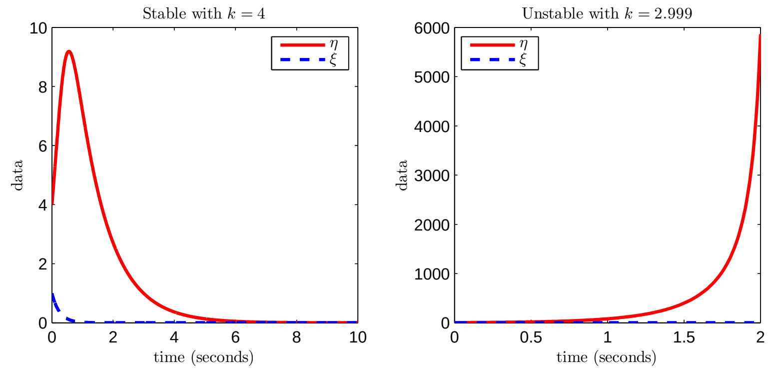
and the linear feedback control

$$v = -k\xi, \qquad k > 0.$$

The origin is (locally) exponentially stable, and the region of attraction is $\{\eta\xi < 1+k\}$. The Simulink diagram and the simulation result can be found in Fig. 1.



(a) Simulink diagram



(b) Simulation result ($\eta(0) = 4$, $\xi(0) = 1$)

Fig. 1: Problem 5.1)

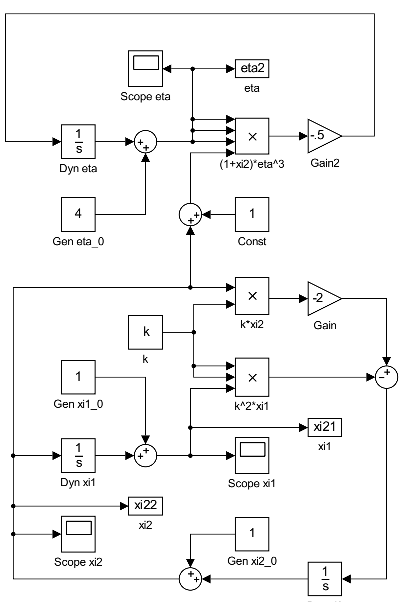
2) In Examples 13.17, we consider the third-order system

$$\begin{aligned} \dot{\eta} &= -\frac{1}{2}(1+\xi_2)\eta^3, \\ \dot{\xi}_1 &= \xi_2, \\ \dot{\xi}_2 &= v, \end{aligned}$$

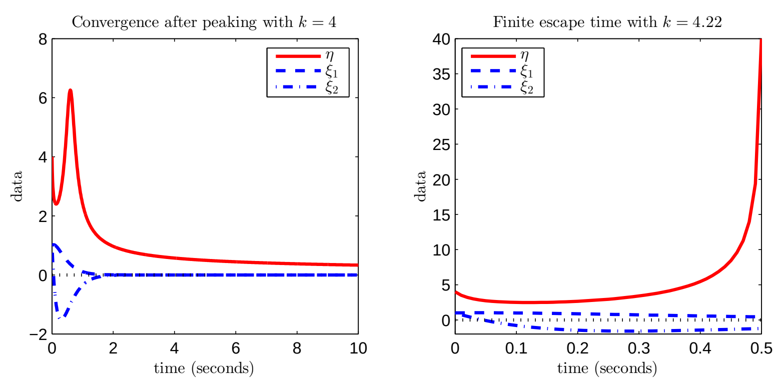
and the linear feedback control

$$v = -k^2\xi_1 - 2k\xi_2, \qquad k > 0.$$

If $\eta_0^2 > 1$, the system will have a finite escape time if k is chosen large enough. The Simulink diagram and the simulation result can be found in Fig. 2.



(a) Simulink diagram



(b) Simulation result ($\eta(0) = 4$, $\xi_1(0) = \xi_2(0) = 1$)

Fig. 2: Problem 5.2)