1. Consider the class of scalar plants

\[
\dot{y} = ay + bu, \quad a \in \mathbb{R}, \ b > 0
\]  

(1)

In Section 3.1.1 of class notes, it is shown that the controller (19) is a universal regulator for this class of plants, with the help of the Lyapunov function \(V(y) = \frac{y^2}{2}\). For the same controller, find a Lyapunov function more similar to the Lyapunov function (3) used in Example 1 (i.e., one that depends also on \(a, b, k\)) such that convergence of \(y\) to 0 in closed loop can be proved by a direct application of Theorem 2.

2. Consider again the class of scalar plants (1). Show that there doesn’t exist a linear universal regulator for this class of plants, i.e., a universal regulator of the form (22) from class notes with \(f\) and \(h\) linear functions. Here the dimension of \(z\) can be arbitrary. (Thus you cannot use the non-existence result for rational controllers proved in class, because it is restricted to scalar \(z\).)

3. Design a universal regulator for the class of scalar plants

\[
\dot{y} = a\varphi(y) + bu, \quad a \in \mathbb{R}, \ b > 0
\]

where \(\varphi(\cdot)\) is a fixed known function. Justify rigorously that it works.

4. Consider a linear system

\[
\dot{x} = Ax + Bu
\]

and assume that \(A\) is Hurwitz, so that we have \(\|e^{At}\| \leq ce^{-\lambda_0 t}\) for some \(c, \lambda_0 > 0\). Prove the following:

a) If \(u \in L_2\) or \(u\) is bounded, then \(x\) is bounded. (Hint: use the variation-of-constants formula and the Cauchy-Schwartz and Hölder’s inequalities.)

b) If \(u \in L_2\) or \(u \to 0\), then \(x \to 0\). (Hint: use part a.).

Note that both in part a) and in part b) you need to consider two separate cases.