1. (10 points) Let \( f : \mathbb{R}^n \to \mathbb{R} \) be a continuous function. Prove that its set of zeros \( \{ x : f(x) = 0 \} \) is always a closed set.

**Solution:** Let \( \{ x_k \} \) be a sequence in \( \{ x : f(x) = 0 \} \) (i.e., \( f(x_k) = 0 \) for all \( k \)) such that \( \lim_{k \to \infty} x_k = x \).

We need to show that \( f(x) = 0 \). However, \( f \) is continuous implies that

\[
f(x) = f\left( \lim_{k \to \infty} x_k \right) = \lim_{k \to \infty} f(x_k) = \lim_{k \to \infty} 0 = 0.
\]

\( \square \)

2. (20 points) Prove that the sequence defined recursively by

\[
x_{k+1} = \frac{x_k}{2} + \frac{2}{x_k}
\]

converges to 2 for every \( x_0 \in [\sqrt{2}, 2\sqrt{2}] \).

**Solution:** Define a function \( P : \mathbb{R} \to \mathbb{R} \) as

\[
P(x) := \frac{x}{2} + \frac{2}{x},
\]

a set \( S := [\sqrt{2}, 2\sqrt{2}] \), and consider applying the contracting mapping theorem. We need to show the following properties:

1. \( P(S) \subset S \), which follows from

\[
x \in S \quad \Rightarrow \quad \frac{\sqrt{2}}{2} \leq \frac{2}{x} \leq \sqrt{2} \quad \Rightarrow \quad \sqrt{2} \leq \frac{x}{2} + \frac{2}{x} \leq 2\sqrt{2} \quad \Rightarrow \quad P(x) \in S.
\]

2. \( P \) is a contraction. By the Mean Value Theorem [Khalil, p. 651], for all \( x, y \in S \),

\[
|P(x) - P(y)| \leq \max_{z \in S} |P'(z)||x - y| = \max_{z \in S} \left| \frac{1}{2} - \frac{2}{z^2} \right| |x - y| = \frac{1}{2} |x - y|.
\]

\(^1\)For this problem, it is not allowed to use the definitions or properties of continuous functions which were not discussed in class without proof, e.g., the pre-image of a closed set through a continuous function is closed.
3. $x^* = 2$ is a fixed point of $P$, which is quite obvious:
\[ 2 = \frac{2}{2} + \frac{2}{2}. \]
By the Contraction Mapping Theorem [Khalil, p. 655], we know that $x^* = 2$ is the unique fixed point of $P$ in $S$, and the sequence converges to 2 for every $x_0 \in S$. □

3. (20 points) Consider a function $f : \mathbb{R} \to \mathbb{R}$.
   a) If $f$ is locally Lipschitz, does this imply that $f$ is uniformly continuous?
   b) Conversely, if $f$ is uniformly continuous, does this imply that $f$ is locally Lipschitz?

Solution:

(a) Consider $f(x) = x^2$. Since $f'(x) = x$ it is clear the same Lipschitz constant cannot hold over all of $\mathbb{R}$; but indeed given a $[a, b] \subset \mathbb{R}$ we can find a local Lipschitz constant.
\[
|x^2 - y^2| = |(x + y)(x - y)| = |x + y||x - y| \\
\leq (|x| + |y|)|x - y| \leq 2 \max(|a|, |b|)|x - y|
\]
Now for $f : \mathbb{R} \to \mathbb{R}$ uniform continuity would require that for every $\varepsilon > 0$ there exists $\delta > 0$ (which only depends on $\varepsilon$) such that for all $x, y \in \mathbb{R}$ if $|x - y| < \delta$ then $|f(x) - f(y)| < \varepsilon$. We can show this is not the case. Let $\varepsilon = 2$ and let $\delta > 0$ be arbitrary. Let $n_\delta \in \mathbb{N}$ be such that $\frac{1}{n_\delta} < \delta$. Consider then setting
\[
x := n_\delta + 1/n_\delta, \quad y := n_\delta
\]
We have,
\[
f(x) - f(y) = (n_\delta + 1/n_\delta)^2 - n_\delta^2 = 2 + 1/n_\delta^2 > \varepsilon
\]
(b) No. See Khalil, paragraph before Lemma 3.1, where $f(x) = x^{1/3}$ is given as a counter example. From real analysis, we know the $n$-th root function defined on the real line is uniformly continuous. But $f(x)$ here is not locally Lipschitz as $f'(x) \to \infty$ as $x \to 0$.

Remark: One could also use $f = \sqrt{x}$ (but strictly speaking that only is defined $\mathbb{R}^+ \to \mathbb{R}^+$) and show work to get credit.

□

4. (20 points) Suppose that $f(t, x)$ is continuous in $t$ and locally Lipschitz in $x$ for each fixed $t$, and that $t$ takes values in a closed interval $[t_0, t_1]$. Does this imply that $f$ is locally Lipschitz in $x$ uniformly in $t \in [t_0, t_1]$? Prove or give a counterexample.
Solution: No, \( f \) is not necessarily locally Lipschitz in \( x \) uniformly in \( t \in [t_0, t_1] \). To find a counterexample, the idea is to find a function \( f(t, x) \) which is continuous in \( t \) and locally Lipschitz in \( x \) for each fixed \( t \), while \( \frac{\partial}{\partial x} f(t, x) \) is unbounded on \( [t_0, t_1] \times \mathbb{R} \). A counterexample is

\[
  f(t, x) = \begin{cases} \sqrt{t}\cos \frac{x}{t}, & t \in (0, 1], \\ 0, & t = 0 \end{cases}
\]

with \([t_0, t_1] = [0, 1]\). (The point is that \( f(t, x) \) is continuous in \( t \) and locally Lipschitz in \( x \) for each fixed \( t \) does not imply that \( \frac{\partial}{\partial x} f(t, x) \) is continuous, so the compactness of \([t_0, t_1]\) doesn’t save us.)

5. (30 points) Suppose that \( f(t, x) \) satisfies all hypotheses of the local existence and uniqueness theorem. Let \( W \) be a compact subset of \( \mathbb{R}^n \). Prove that there exists a \( \delta > 0 \) such that every solution with \( x(t_0) \in W \) can be extended to the interval \([t_0, t_0 + \delta] \).

(Note: \( \delta \) depends just on \( W \) but not on a particular initial condition in \( W \). This fact implies that if it is known that the solution \( x(t) \) remains in \( W \), then it is defined globally—without the need to assume global Lipschitzness.)

Hint: make appropriate modifications to the proof of the existence and uniqueness theorem given in class.

Solution: Consider the same Banach space \( X := C[t_0, t_0 + \delta] = C^0([t_0, t_0 + \delta], \mathbb{R}^n) \) as in the proof of [Khalil, Theorem 3.1]. For an arbitrary fixed \( r > 0 \), let

\[
  U := \{ x : \exists y \in W \text{ s.t. } |x - y| \leq r \}
\]

and

\[
  S := \{ x \in X : x(t) \in U, \forall t \in [t_0, t_0 + \delta] \}.
\]

For any \( x_0 \in W \), define \( P : X \to X \) as

\[
  (Px)(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds,
\]

and consider applying the contraction mapping theorem. We first show that \( P(S) \subseteq S \). Indeed,

\[
  |(Px)(t) - x_0| = \left| \int_{t_0}^{t} f(s, x(s))ds \right|
  \leq \int_{t_0}^{t} |(f(s, x(s)) - f(s, x_0)) + f(s, x_0)|ds
  \leq \int_{t_0}^{t} (L|x(s) - x_0| + H)ds
  \leq \delta (LR + H),
\]

where
\begin{itemize}
  \item $R := \max_{x,y \in U} |x - y|$;
  \item $L$ is the Lipschitz constant over $U$, which is independent of $x_0$; and
  \item $H := \max_{s \in [t_0, t_1], x_0 \in W} |f(s, x_0)|$.
\end{itemize}

Choosing a $\delta$ such that $\delta \leq \frac{r}{LR + H}$ ensures that $P(S) \subset S$. (Notice that this upper bound does not depend on $x_0$, only on $W$.) The rest of the proof is the same as that of [Khalil, Theorem 3.1]. \hfill \Box

---

2The existence of such an $H$ can be shown via a generalization of the Weierstrass theorem. However, all we need is that $f(t, x)$ is bounded over $[t_0, t_1] \times W$, that is, $\sup_{s \in [t_0, t_1], x_0 \in W} |f(s, x_0)|$ exists and is finite. But the boundedness is obvious from local Lipschitzness in $x$ uniform over $t$: consider an arbitrary fixed $y \in W$. Then $|f(t, x) - f(t, y)| \leq L|x - y|$ for all $x \in W$ and all $t \in [t_0, t_1]$, where $L$ is the Lipschitz constant over $W$. Clearly, $f(t, y)$ is bounded by piecewise continuity in $t$, and $|x - y|$ is bounded over $W$, so $f(t, x)$ is bounded.