ECE517, Fall 2021 PROBLEM SET 1 Solution

1. Consider the 2-D system

$$\dot{x}_1 = -\frac{6x_1}{(1+x_1^2)^2} + 2x_2,$$
$$\dot{x}_2 = -\frac{2(x_1+x_2)}{(1+x_1^2)^2}$$

and the candidate Lyapunov function

$$V(x_1,x_2) = \frac{x_1^2}{1+x_1^2} + x_2^2.$$

Compute the derivative of this V along solutions. Can you conclude that all solutions $x(t)$ are bounded? that all solutions $x(t)$ with initial conditions $x(0)$ sufficiently close to 0 are bounded? that all $x(t)$ converge to 0? that all $x(t)$ with $x(0)$ sufficiently close to 0 converge to 0?

For each question, explain which result you're using or give a reason why you can not.

Solution. The derivative of this V along solutions is

$$\begin{aligned}\dot{V} &= \frac{\partial V}{\partial x_1}\dot{x}_1 + \frac{\partial V}{\partial x_2}\dot{x}_2 \\ &= \frac{2x_1}{(1+x_1^2)^2}\left(-\frac{6x_1}{(1+x_1^2)^2} + 2x_2\right) + 2x_2\left(-\frac{2(x_1+x_2)}{(1+x_1^2)^2}\right) \\ &= -\frac{12x_1^2}{(1+x_1^2)^4} - \frac{4x_2^2}{(1+x_1^2)^2},\end{aligned}$$

which is negative definite. However, since

$$\lim_{x_1\to\infty} V(x_1,x_2) = 1 + x_2^2 < \infty,$$

the candidate Lyapunov function is not radially unbounded. Hence, by Theorem 1 from the Lecture notes

1) we *cannot* conclude that all solutions $x(t)$ are bounded;
2) we *can* conclude that all solutions $x(t)$ with initial condition $x(0)$ sufficiently close to 0 are bounded;
3) we *cannot* conclude that all $x(t)$ converge to 0; and
4) we *can* conclude that all solutions $x(t)$ with initial condition $x(0)$ sufficiently close to 0 converge to 0.

2. When proving Barbalat's lemma in class, we assumed for simplicity that $W(x) \geq 0$ (which is one of the hypotheses in Theorem 2). However, Barbalat's lemma itself is valid even if W is not sign-definite. Refine the proof of Barbalat's lemma from class so that it works for W possibly taking both positive and negative values.

Solution. We prove by contradiction. Suppose $W(x(t))$ doesn't converge to 0. Then there exist an $\epsilon > 0$ and an increasing sequence $(t_k)_{k\in\mathbb{N}}$ such that $t_k \to \infty$ as $k \to \infty$ and

$$|W(x(t_k))| \geq \epsilon \qquad \forall\, k \in \mathbb{N}.$$

As W is a continuous function of x and $x(t)$ is bounded, there exists a $\delta_x > 0$ such that

$$|x(t) - x(t_k)| \leq \delta_x \quad \Rightarrow \quad |W(x(t)) - W(x(t_k))| \leq \epsilon/2$$

for all $k \in \mathbb{N}$. Furthermore, as $\dot{x}(t)$ is also bounded, there exists a $\delta_t > 0$ such that

$$|t - t_k| \leq \delta_t \quad \Rightarrow \quad |x(t) - x(t_k)| \leq \delta_x$$

for all $k \in \mathbb{N}$. Combining the arguments above with the triangle inequality shows that for each $k \in \mathbb{N}$, either $W(x(t)) \geq \epsilon/2$ for all $t \in [t_k, t_k + \delta_t]$, or $W(x(t)) \leq -\epsilon/2$ for all $t \in [t_k, t_k + \delta_t]$. Therefore, the fact that $t_k \nearrow \infty$ as $k \to \infty$ implies that for each $T > 0$ there exists a $t_k > T$ such that

$$\left| \int_0^{t_k+\delta_t} W(x(s))\mathrm{d}s - \int_0^{t_k} W(x(s))\mathrm{d}s \right| = \left| \int_{t_k}^{t_k+\delta_t} W(x(s))\mathrm{d}s \right| = \int_{t_k}^{t_k+\delta_t} |W(x(s))|\mathrm{d}s \geq \epsilon\delta_t/2. \tag{1}$$

On the other hand, from Cauchy's convergence test (cf. Cauchy Sequence [Khalil, 2002, p. 654]) it follows that if

$$\int_0^{\infty} W(x(s))\mathrm{d}s = \lim_{t\to\infty} \int_0^{t} W(x(s))\mathrm{d}s$$

is well-defined and finite then for each $\delta_t > 0$ there exists a $T > 0$ such that

$$\left| \int_0^{t_k+\delta_t} W(x(s))\mathrm{d}s - \int_0^{t_k} W(x(s))\mathrm{d}s \right| < \epsilon\delta_t/2 \qquad \forall\, t_k > T,$$

which is a contradiction. Hence $W(x(t))$ converges to 0.

Notes:

(A) You cannot take the sum of (1) for all k and conclude that $\int_0^\infty W(x(s))\mathrm{d}s$ is infinite, since for different k the signs of $W(x(t))$ on $[t_k, t_k + \delta_t]$ may be different.

(B) It is *not* true that if $\int\limits_0^\infty W\left(x\left(s\right)\right) ds$ converges, so does $\int_0^\infty \left|W\left(x\left(s\right)\right)\right| ds$ for the Riemann integral. Consider $\int\limits_0^\infty \frac{\sin x}{x} dx$ for example (the only possible counterexamples are Riemann integrable, but not Lebesgue integrable).

3. Consider the system

$$\dot{x} = \theta + x + u \tag{2}$$

which is similar to Example 1 from the class notes except the unknown parameter θ enters additively and not multiplicatively. Suppose we want to make x converge to 0. Propose an adaptive control law that achieves this, and justify that it works. Base your design and analysis on ideas similar to the ones used to treat Example 1 in class: introduce an estimate $\hat{\theta}$ and a differential equation for it (tuning law); make the control law depend on $\hat{\theta}$; analyze the closed-loop system with the help of a Lyapunov function.

When discussing Example 1 in class, we commented that the estimate $\hat{\theta}$ does not necessarily converge to the true value θ (see also Problem 4 below). For the closed-loop system that you obtained from the system (2) with your controller, can you prove that $\hat{\theta}$ does in fact converge to θ?

Solution. Consider the candidate Lyapunov function

$$V(x, \hat{\theta}) = \frac{x^2}{2} + \frac{(\hat{\theta} - \theta)^2}{2}.$$

It's derivative along the solution of (2) is given by

$$\dot{V} = x\dot{x} + (\hat{\theta} - \theta)\dot{\hat{\theta}} = (x - \dot{\hat{\theta}})\theta + x^2 + xu + \hat{\theta}\dot{\hat{\theta}}.$$

To eliminate the unknown θ, we set $\dot{\hat{\theta}} = x$, and obtain

$$\dot{V} = (u + x + \hat{\theta})x.$$

Hence we set $u = -kx - \hat{\theta}$ with a $k > 1$, which gives

$$\dot{V} = -(k-1)x^2 < 0 \qquad \forall\, x \neq 0.$$

The closed-loop system becomes

$$\begin{aligned} \dot{x} &= \theta - \hat{\theta} + (1-k)x, \\ \dot{\hat{\theta}} &= x. \end{aligned} \tag{3}$$

It follows from Theorem 2 in the lecture notes that $W(x(t)) = (k-1)x(t)^2 \to 0$ as $t \to \infty$, that is, the state x converges to 0. Note that $x(t)$ remains bounded for all $t > 0$ because $V(x, \hat{\theta})$ is positive definite, radially unbounded, and $\dot{V}(x, \hat{\theta}) \leq 0$.

We use LaSalle's theorem (cf. [Khalil, 2002, Sec. 4.2]) to show that $\hat{\theta}$ does in fact converge to θ. Define $E := \{(x, \hat{\theta}) | \dot{V} = 0\} = \{(x, \hat{\theta}) | x = 0\}$. As

$$x \equiv 0 \quad \Rightarrow \quad \dot{x} \equiv 0 \quad \Rightarrow \quad \hat{\theta} = \theta,$$

the largest invariant $M \subset E$ is $M = \{(0, \theta)\}$. From LaSalle's theorem it follows that all solutions $(x, \hat{\theta})$ of (3) converge to M, that is, the estimate $\hat{\theta}$ converges to θ.

Alternatively, if we let $y := \hat{\theta} - \theta$ then (3) becomes

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 1-k & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix},$$

in which the matrix is Hurwitz as its eigenvalues

$$\lambda_{1,2} = \frac{1 - k \pm \sqrt{k^2 - 2k - 3}}{2}$$

have negative real parts (for $k > 1$).

As a side remark, let $e = x - 0$ be the error between the origin and the state x, and notice that $\hat{\theta}(t) = \int_0^t e(\tau)d\tau$. Then, one can rewrite the control law as $u(t) = -\left(ke + \int_0^t e(\tau)d\tau\right)$, which is a PI controller. It is known that the integral term in a PI controller can reject constant perturbations, such as θ in our case. This shows, that one can arrive at this same widely used controller by means of the adaptive control theory.

Note:

You cannot say that $x(t) \to 0$ as $t \to \infty$ implies $\dot{x}(t) \to 0$. A counterexample is $x(t) = \frac{\sin(t^2)}{t}$ for $t \neq 0$ and $x(0) = 0$. Note that such x is a $\mathcal{C}^1$ function.

4. a) Consider the adaptive control system introduced in class to treat Example 1:

$$\dot{x} = \theta x + u, \tag{4}$$

$$\dot{\hat{\theta}} = x^2, \tag{5}$$

$$u = -(\hat{\theta} + 1)x \tag{6}$$

where θ is an unknown real parameter. Using MATLAB (or any other software), simulate this system for some arbitrary choices of θ. Does the state x converge to 0? Does $\hat{\theta}$ remain bounded? Does it converge to θ?

b) Keep the same tuning/control law (5)–(6) but modify the plant dynamics (4) to

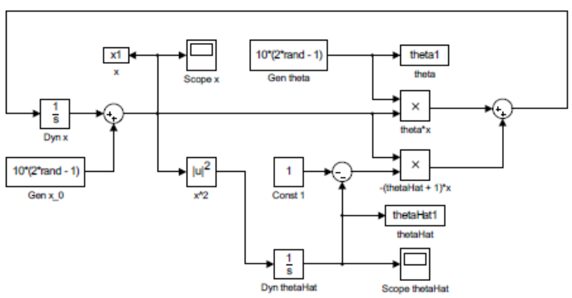
$$\dot{x} = \theta x + u + d$$

where d is a constant nonzero disturbance. Repeat the simulations and answer the same questions as in part a).

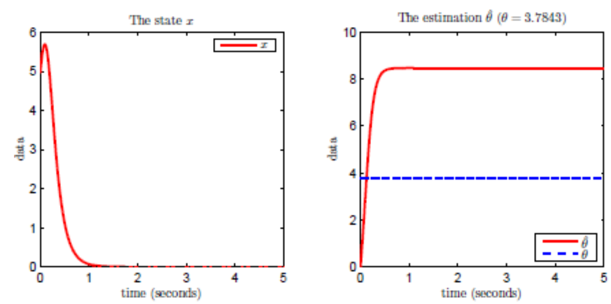
c) Keep the disturbance, and modify the tuning law (5) by turning the adaptation off (setting $\dot{\hat{\theta}} = 0$) whenever x is in some small interval around 0. This modification is known as *dead zone*. Repeat the simulations and answer the same questions as in part a); play with the size of the dead-zone interval and explain how it should be chosen.

d) Keep the disturbance, and modify the tuning law (5) by selecting some value $\hat{\theta}_{\max}$ and turning the adaptation off (setting $\dot{\hat{\theta}} = 0$) if $\hat{\theta}$ reaches $\hat{\theta}_{\max}$. This modification is known as *projection*. Repeat the simulations and answer the same questions as in part a); play with the value of $\hat{\theta}_{\max}$ and explain how it should be chosen.

Solution. a) The Simulink diagram and the simulation result can be found in Fig. 1. From the simulation result, we see that the state x converges to 0, and the estimate $\hat{\theta}$ remains bounded but does not converge to θ.



(a) Simulink diagram



(b) Simulation result

Fig. 1: Problem 4.a)

Now, note that by taking $V(x) = \frac{x^2}{2}$, we get by differentiations:

$$\dot{V} = x\dot{x} = \left(\theta - \hat{\theta} - 1\right)\dot{\hat{\theta}},$$

where the second equality comes from the closed-loop dynamics. This expressions can be integrated as:

$$\frac{x_t^2}{2} - \frac{{x_0}^2}{2} = \int_{\hat{\theta}_0}^{\hat{\theta}_t} (\theta - z - 1)\, dz = \left[(\theta - 1)\, z - \frac{z^2}{2}\right]_{\hat{\theta}_0}^{\hat{\theta}_t}$$

where (1) change of parameter $z = \hat{\theta}$ was introduced and (2) the argument t has been reduced to subscripts to avoid clutter. Let $\hat{\theta}_\infty$ denote the limiting value of $\hat{\theta}$ as $t \to \infty$. Since $x \to 0$ as time goes to infinity, we can find $\hat{\theta}_\infty$ as the positive solution of the quadratic equation:

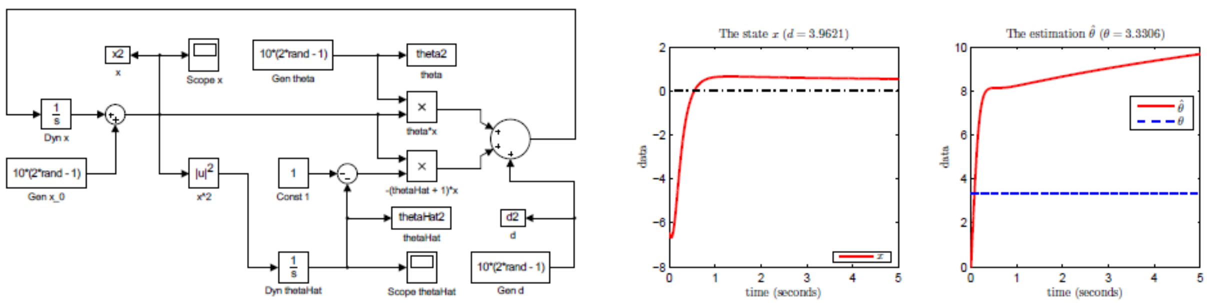
$$-{x_0}^2 = 2\,(\theta - 1)\left(\hat{\theta}_\infty - \hat{\theta}_0\right) + \left(\hat{\theta}_0^2 - \hat{\theta}_\infty^2\right).$$

Note that your solution will be a function of the initial conditions x_0, θ and $\hat{\theta}_0$.

b) The Simulink diagram and the simulation result can be found in Fig. 2. From the simulation result, we see that the state x does not converge to 0 (exponentially), and $\hat{\theta}$ goes unbounded and does not converge to θ. The closed-loop system is

$$\begin{aligned} \dot{x} &= (\theta - \hat{\theta} - 1)x + d, \\ \dot{\hat{\theta}} &= x^2. \end{aligned}$$

From the closed-loop dynamics we are able to conclude that $\hat{\theta}$ cannot be bounded. Indeed, if it is bounded then $\dot{x} > 0$ for all $|x| < d/|\hat{\theta} + 1 - \theta|$, meaning that the state x is bounded away from 0. But then from the dynamics of $\hat{\theta}$ we see that $\hat{\theta} \to \infty$ as $t \to \infty$, which is a contradiction. Hence $\hat{\theta}$ goes unbounded and does not converge to θ. However, we cannot definitively conclude that x converges to 0 from the simulation.



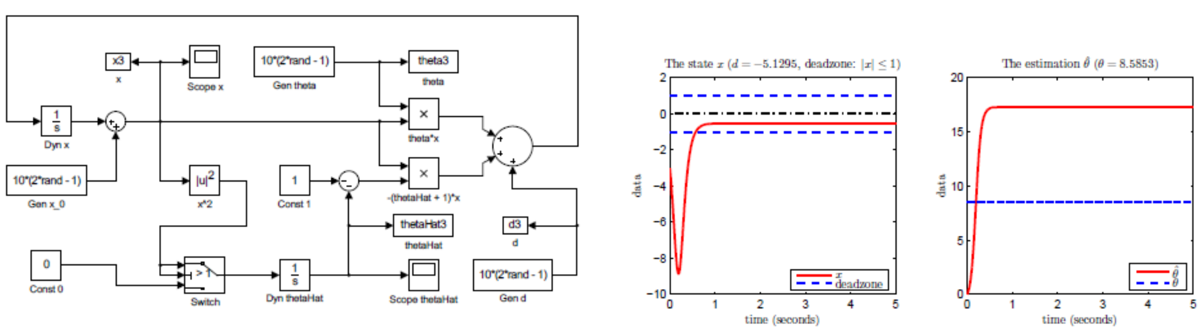
(a) Simulink diagram (b) Simulation result

Fig. 2: Problem 4.b)

c) The Simulink diagram and the simulation result can be found in Fig. 3. From the simulation result, we see that the state x does not converge to 0, and $\hat{\theta}$ remains bounded but does not converge to θ. In this case, instead of $\dot{\hat{\theta}} = x^2$, we use

$$\dot{\hat{\theta}} = \begin{cases} x^2, & \text{if } |x| > \delta, \\ 0 & \text{if } |x| \leq \delta \end{cases}$$

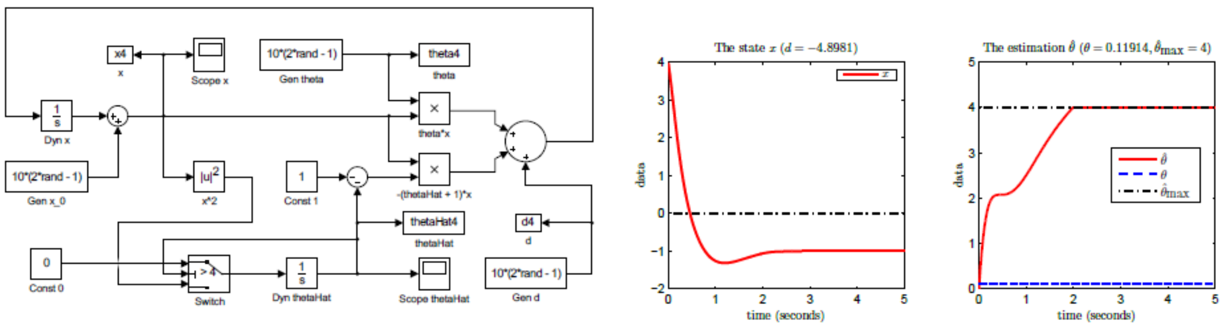
for some $\delta > 0$. Thus, knowledge of the disturbance d would help to choose the size of dead-zone in this case. Switching the adaptation off after $|x| \leq \delta$ then guarantees that $\hat{\theta}$ remains bounded but doesn't necessarily converge to θ.



(a) Simulink diagram (b) Simulation result

Fig. 3: Problem 4.c)

d) The Simulink diagram and the simulation result can be found in Fig. 4. From the simulation result, we see that the state x does not converge to 0, and $\hat{\theta}$ remains bounded and does not converge to θ. In this case, as $\dot{x} = (\theta - \hat{\theta} - 1)x + d$, the value of $\hat{\theta}_{\max}$ should be chosen so that $\theta - \hat{\theta}_{\max} - 1 < 0$. Thus, knowledge on the unknown parameter θ would help to choose the value of $\hat{\theta}_{\max}$ in this case.



(a) Simulink diagram

(b) Simulation result

Fig. 4: Problem 4.d)

It is not required to derive any closed-form formula, but the answer and explanation needs to be consistent with the plots and correct.

5. a) Modify the system in Example 1 from class to

$$\dot{x} = \theta(t)x + u$$

where $\theta(t)$ satisfies $|\theta(t) - \theta_{\text{av}}| \leq \delta$ for some known δ and unknown θ_{av}. Design an adaptive control law that makes $x(t)$ converge to 0.

b) Solve the same problem for the system $\dot{x} = \theta(t)x^2 + u$, under the same assumption on $\theta(t)$.

Solution. Solution: If we consider

$$V(x, \hat{\theta}) := \frac{x^2}{2} + \frac{(\hat{\theta} - \theta)^2}{2}$$

as in class, then $\dot{V}$ will have a term that depends on $\dot{\theta}$ and it's not clear how to deal with it. But we don't really need θ in V, because even for a constant θ we don't show convergence of θ. Instead, the trick is to consider the candidate Lyapunov function

$$V(x, \hat{\theta}) := \frac{x^2}{2} + \frac{(\hat{\theta} - \theta_{\text{av}})^2}{2}$$

For a), rewrite the system as

$$\dot{x} = \hat{\theta}x + u + (\theta - \hat{\theta})x$$

to obtain

$$\begin{aligned}\dot{V} &= \hat{\theta}x^2 + xu + (\theta - \hat{\theta})x^2 + (\hat{\theta} - \theta_{\text{av}})\dot{\hat{\theta}} \\ &= \hat{\theta}x^2 + xu + (\theta - \theta_{\text{av}})x^2 + (\theta_{\text{av}} - \hat{\theta})x^2 - (\theta_{\text{av}} - \hat{\theta})\dot{\hat{\theta}}\end{aligned}$$

Now the same tuning law $\dot{\hat{\theta}} = x^2$ as in class cancels the last two terms, and the feedback law

$$u = -(\hat{\theta} + 1 + \delta)x$$

gives

$$\dot{V} = -x^2 + (\theta - \theta_{\text{av}})x^2 - \delta x^2 \leq -x^2$$

by assumption on $\theta(t)$. From here the analysis proceeds in the same way as in class.

For b), the same V as above gives

$$\dot{V} = \hat{\theta}x^3 + xu + (\theta - \theta_{\text{av}})x^3 + (\theta_{\text{av}} - \hat{\theta})x^3 - (\theta_{\text{av}} - \hat{\theta})\dot{\hat{\theta}}$$

To cancel the last two terms, we now let $\dot{\hat{\theta}} = x^3$. The term $(\theta_{\text{av}} - \hat{\theta})x^3$ can be dominated if we choose the feedback law

$$u = -\hat{\theta}x^2 - (1 + \tfrac{\delta}{2})x - \tfrac{\delta}{2}x^3$$

Indeed, we then have

$$\dot{V} = -x^2 - \tfrac{\delta}{2}x^2 + (\theta - \theta_{\text{av}})x^3 - \tfrac{\delta}{2}x^4 \leq -x^2 - \tfrac{\delta}{2}x^2 + \delta|x|^3 - \tfrac{\delta}{2}x^4$$

from our assumption on $\theta(t)$, and by square completion it follows that

$$\dot{V} \leq -x^2$$

This method is called “congelation of variables”. Source: Chen and Astolfi, “Adaptive control for systems with time-varying parameters”, IEEE TAC, May 2021, pages 1986–2001.