1. (20 Points) Consider the 2-D system

\[
\begin{align*}
\dot{x}_1 &= -\frac{6x_1}{(1 + x_1^2)^2} + 2x_2, \\
\dot{x}_2 &= -\frac{2(x_1 + x_2)}{(1 + x_1^2)^2}
\end{align*}
\]

and the candidate Lyapunov function

\[V(x_1, x_2) = \frac{x_1^2}{1 + x_1^2} + x_2^2.
\]

Compute the derivative of this \(V\) along solutions. Can you conclude that all solutions \(x(t)\) are bounded? that all solutions \(x(t)\) with initial conditions \(x(0)\) sufficiently close to 0 are bounded? that all \(x(t)\) with \(x(0)\) sufficiently close to 0 converge to 0? For each question, explain which result you're using or give a reason why you can not.

**Solution.** The derivative of this \(V\) along solutions is

\[
\dot{V} = \frac{\partial V}{\partial x_1} \dot{x}_1 + \frac{\partial V}{\partial x_2} \dot{x}_2
\]

\[
= \frac{2x_1}{(1 + x_1^2)^2} \left( -\frac{6x_1}{(1 + x_1^2)^2} + 2x_2 \right) + 2x_2 \left( -\frac{2(x_1 + x_2)}{(1 + x_1^2)^2} \right)
\]

\[
= -\frac{12x_2^2}{(1 + x_1^2)^4} - \frac{4x_2^2}{(1 + x_1^2)^2},
\]

which is negative definite. However, since

\[
\lim_{x_1 \to \infty} V(x_1, x_2) = 1 + x_2^2 < \infty,
\]

the candidate Lyapunov function is not radially unbounded. Hence

1) we cannot conclude that all solutions \(x(t)\) are bounded;
2) we can conclude that all solutions \(x(t)\) with initial condition \(x(0)\) sufficiently close to 0 are bounded;
3) we cannot conclude that all \(x(t)\) converge to 0; and
4) we can conclude that all solutions \(x(t)\) with initial condition \(x(0)\) sufficiently close to 0 converge to 0.

2. (30 Points) When proving Barbalats lemma in class, we assumed for simplicity that \(W(x) \geq 0\) (which is one of the hypotheses in Theorem 2). However, Barbalats lemma itself is valid even if \(W\) is not sign-definite. Refine the proof of Barbalats lemma from class so that it works for \(W\) possibly taking both positive and negative values.

**Solution.** We prove by contradiction. Suppose \(W(x(t))\) doesn’t converge to 0. Then there exist an \(\epsilon > 0\) and an increasing sequence \(\{t_k\}_{k \in \mathbb{N}}\) such that \(t_k \uparrow \infty\) as \(k \to \infty\) and

\[
|W(x(t_k))| \geq \epsilon \quad \forall \ k \in \mathbb{N}.
\]

As \(W\) is a continuous function of \(x\), there exists a \(\delta_x > 0\) such that

\[
|x(t) - x(t_k)| \leq \delta_x \quad \Rightarrow \quad |W(x(t)) - W(x(t_k))| \leq \epsilon/2
\]

for all \(k \in \mathbb{N}\). Furthermore, as \(x\) is a continuous function of \(t\), there exists a \(\delta_t > 0\) such that

\[
|t - t_k| \leq \delta_t \quad \Rightarrow \quad |x(t) - x(t_k)| \leq \delta_x
\]
for all $k \in \mathbb{N}$. Combining the above arguments with the triangle inequality shows that for all $k \in \mathbb{N}$, either $W(x(t)) \geq \epsilon/2$ for all $t \in [t_k, t_k + \delta_t]$, or $W(x(t)) \leq -\epsilon/2$ for all $t \in [t_k, t_k + \delta_t]$. Therefore, the fact that $t_k \uparrow \infty$ as $k \to \infty$ implies that for each $T > 0$ there exists a $t_k > T$ such that

$$\left| \int_{t_k}^{t_k+\delta_t} W(x(s))ds - \int_{t_k}^{t_k+\delta_t} W(x(s))ds \right| = \left| \int_{t_k}^{t_k+\delta_t} W(x(s))ds \right| = \int_{t_k}^{t_k+\delta_t} |W(x(s))|ds \geq \epsilon\delta_t/2.$$  

Hence the Cauchy’s convergence test (cf. [Khalil, 2002, p. 654]) shows that

$$\int_0^\infty W(x(s))ds = \lim_{t \to \infty} \int_0^t W(x(s))ds$$

cannot be well-defined and finite, which is a contradiction. Hence $W(x(t))$ converges to 0.

3. (20 Points) Consider the system

$$\dot{x} = \theta + x + u \quad \text{(1)}$$

which is similar to Example 1 from the class notes except the unknown parameter $\theta$ enters additively and not multiplicatively. Suppose we want to make $x$ converge to 0. Propose an adaptive control law that achieves this, and justify that it works. Base your design and analysis on ideas similar to the ones used to treat Example 1 in class: introduce an estimate $\hat{\theta}$ and a differential equation for it (tuning law); make the control law depend on $\hat{\theta}$; analyze the closed-loop system with the help of a Lyapunov function.

When discussing Example 1 in class, we commented that the estimate $\hat{\theta}$ does not necessarily converge to the true value $\theta$ (see also Problem 4 below). For the closed-loop system that you obtained from the system (1) with your controller, can you prove that $\hat{\theta}$ does in fact converge to $\theta$?

**Solution.** Consider the candidate Lyapunov function

$$V(x, \hat{\theta}) = \frac{x^2}{2} + \frac{(\hat{\theta} - \theta)^2}{2}.$$  

It’s derivative along the solution of (1) is given by

$$\dot{V} = x\dot{x} + (\hat{\theta} - \theta)\dot{\theta} = (x - \hat{\theta})\theta + x^2 + xu + \hat{\theta}\dot{\theta}.$$  

To eliminate the unknown $\theta$, we set $\dot{\theta} = x$, and obtain

$$\dot{V} = (u + x + \hat{\theta})x.$$  

Hence we set $u = -kx - \hat{\theta}$ with a $k > 1$, which gives

$$\dot{V} = -(k - 1)x^2 < 0 \quad \forall x \neq 0.$$  

The closed-loop system becomes

$$\dot{x} = \theta - \hat{\theta} + (1 - k)x,$$

$$\dot{\hat{\theta}} = x.$$  

(2)

It follows from Theorem 2 in the lecture notes that $W(x(t)) = (k - 1)x(t)^2 \to 0$ as $t \to \infty$, that is, the state $x$ converges to 0.

We use LaSalle’s invariance principle [Khalil, 2002, Section 4.2] to show that $\hat{\theta}$ does in fact converge to $\theta$. Define $E := \{(x, \hat{\theta})|\dot{V} = 0\} = \{(x, \hat{\theta})|x = 0\}$. As

$$x \equiv 0 \implies \dot{x} \equiv 0 \implies \hat{\theta} = \theta,$$

the largest invariant $M \subset E$ is $M = \{(0, \theta)\}$. The invariance principle gives that all solutions $(x, \hat{\theta})$ of (2) converge to $M$, that is, the estimate $\hat{\theta}$ converges to $\theta$. 
4. (30 Points)
   a) Consider the adaptive control system introduced in class to treat Example 1:

   \[ \dot{x} = \theta x + u, \]
   \[ \dot{\theta} = x^2, \]
   \[ u = -\left(\hat{\theta} + 1\right)x \]

   where \( \theta \) is an unknown real parameter. Using MATLAB (or any other software), simulate this system for some arbitrary choices of \( \theta \). Does the state \( x \) converge to 0? Does \( \hat{\theta} \) remain bounded? Does it converge to \( \theta \)?

   b) Keep the same tuning/control law (4)–(5) but modify the plant dynamics (3) to

   \[ \dot{x} = \theta x + u + d \]

   where \( d \) is a constant nonzero disturbance. Repeat the simulations and answer the same questions as in part a).

   c) Keep the disturbance, and modify the tuning law (4) by turning the adaptation off (setting \( \dot{\hat{\theta}} = 0 \)) whenever \( x \) is in some small interval around 0. This modification is known as dead zone. Repeat the simulations and answer the same questions as in part a); play with the size of the dead-zone interval and explain how it should be chosen.

   d) Keep the disturbance, and modify the tuning law (4) by selecting some value \( \hat{\theta}_{\text{max}} \) and turning the adaptation off (setting \( \dot{\hat{\theta}} = 0 \)) if \( \hat{\theta} \) reaches \( \hat{\theta}_{\text{max}} \). This modification is known as projection. Repeat the simulations and answer the same questions as in part a); play with the value of \( \hat{\theta}_{\text{max}} \) and explain how it should be chosen.

Solution. a) The Simulink diagram and the simulation result can be found in Fig. 1. From the simulation result, we see that the state \( x \) converges to 0, and the estimate \( \hat{\theta} \) remains bounded but does not converge to \( \theta \).
b) The Simulink diagram and the simulation result can be found in Fig. 2. From the simulation result, we see that the state $x$ does not converge to 0 (exponentially), and $\dot{\theta}$ goes unbounded and does not converge to $\theta$. The closed-loop system is

$$\dot{x} = (\theta - \dot{\theta} - 1)x + d,$$

$$\dot{\theta} = x^2. $$

From the closed-loop dynamics we are able to conclude that $\dot{\theta}$ cannot be bounded. Indeed, if it is bounded then $\dot{x} > 0$ for all $|x| < d/|\dot{\theta} + 1 - \theta|$, meaning that the state $x$ is bounded away from 0. But then from the dynamics of $\dot{\theta}$ we see that $\dot{\theta} \to \infty$ as $t \to \infty$, which is a contradiction. Hence $\dot{\theta}$ goes unbounded and does not converge to $\theta$. However, we cannot definitively conclude that $x$ converges to 0 from the simulation.

![Simulink diagram](image1)

**Fig. 2: Problem 4.b)**

(c) The Simulink diagram and the simulation result can be found in Fig. 3. From the simulation result, we see that the state $x$ does not converge to 0, and $\dot{\theta}$ remains bounded but does not converge to $\theta$. In this case, instead of $\dot{\theta} = x^2$, we use

$$\dot{\theta} = \begin{cases} x^2, & \text{if } |x| > \delta, \\ 0, & \text{if } |x| \leq \delta \end{cases}$$

for some $\delta > 0$. Thus, knowledge of the disturbance $d$ would help to choose the size of dead-zone in this case. Switching the adaptation off after $|x| \leq \delta$ then guarantees that $\dot{\theta}$ remains bounded but doesn’t necessarily converge to $\theta$.

![Simulink diagram](image2)

**Fig. 3: Problem 4.c)**
d) The Simulink diagram and the simulation result can be found in Fig. 4. From the simulation result, we see that the state $x$ does not converge to 0, and $\hat{\theta}$ remains bounded and does not converge to $\theta$. In this case, as $\dot{x} = (\theta - \hat{\theta} - 1)x + d$, the value of $\hat{\theta}_{\text{max}}$ should be chosen so that $\theta - \hat{\theta}_{\text{max}} - 1 < 0$. Thus, knowledge on the unknown parameter $\theta$ would help to choose the value of $\hat{\theta}_{\text{max}}$ in this case.

Fig. 4: Problem 4.d)