Reading: For minimum-norm optimal control, see Brockett’s book, Section 22. For time-optimal control, see the handout [http://liberzon.csl.illinois.edu/teaching/time-optimal.pdf]. For LQR and introduction to optimal control theory, see Class Notes, Sections 10.1–10.5, 11.1, 11.4.

Problems (Problems 2–6 rely on material to be discussed in class on Nov 29 and Dec 1):

1. Use the result of Problem 4 from Problem Set 4 to give an alternative proof of the fact that the controller given in class:

\[ u(t) = B^T(t)\Phi^T(t_0, t)\eta, \quad \eta = (W(t_0, t_1))^{-1} \cdot (\Phi(t_0, t_1)x_1 - x_0) \]

(here you can assume that \( W^{-1} \) exists) has the minimal \( L_2 \) norm among all controllers that transfer the state of the LTV system \( \dot{x} = A(t)x + B(t)u \) from \( x_0 \) at time \( t_0 \) to \( x_1 \) at time \( t_1 \).

2. Consider the optimal control problem

\[ \ddot{x} = u, \quad J(u) = \int_{t_0}^{t_1} (x^4(t) + u^2(t)) \, dt + x^3(t_1) \]

a) Write down a partial differential equation for the optimal cost \( V \), and a boundary condition for it.

b) Simplify the PDE by computing the minimum in it. Using this minimum calculation, write down an expression for the optimal control law in state feedback form. (This expression can contain partial derivatives of the optimal cost, evaluated along the optimal trajectory.)

3. Consider the minimum-time parking problem discussed in class: bring a car modeled by the system \( \ddot{x} = u \), \( u \in [-1, 1] \) to rest at the origin in shortest possible time. Answer the same questions a) and b) as in the previous problem.

4. Consider the optimal control problem given by the system \( \dot{x} = xu \) with \( x \in \mathbb{R} \) and \( u \in [-1, 1] \), no running cost (\( L = 0 \)), and terminal cost \( M(x) = x \). In other words, the cost functional is \( J(u) = x(t_1) \). The final time \( t_1 \) is fixed and finite.

a) Find the optimal cost function (also called the value function) \( V(t, x) \) by inspection (without using the HJB equation).

b) Write down the HJB equation for this problem, and simplify it by computing the minimum in it.

c) Does the optimal cost function from part a) satisfy the HJB equation from part b) everywhere?

5. Consider the LQR problem

\[ \begin{align*}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= u \\
J(u) &= \int_{t_0}^{t_1} (x_2^2 + u^2) \, dt
\end{align*} \tag{1} \]

Write down the Riccati differential equation (with its boundary condition) and the expressions for the optimal cost \( V \) and the optimal state feedback control \( u^* \) (these expressions will depend on the solution \( P \) to the Riccati equation, but you don’t need to compute this solution).

6. Consider the infinite-horizon LQR problem

\[ \begin{align*}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= u \\
J(u) &= \int_{t_0}^{\infty} (x_2^2 + u^2) \, dt
\end{align*} \tag{2} \]

Find the optimal cost \( V \) and the optimal control \( u^* \) in state feedback form. Show that the closed-loop system is stable but not asymptotically stable. Which condition of the theorem that guarantees closed-loop asymptotic stability is violated?