Problems:

1. Consider the system $\dot{x} = Ax + Bu + d_1$, $y = Cx + d_2$.
   Suppose the objective is to make $y$ asymptotically approach a reference signal $r$, in spite of the disturbances. In this problem, take $r$ to be the “ramp” signal: $r(t) = t$, $t \geq 0$. Solve the problem by following these steps, along the lines of the construction given in class for tracking constants (steps):
   a) Reduce the problem to asymptotic stabilization of an auxiliary system whose state contains the tracking error $e := y - r$. You can assume here that the disturbances belong to the same class as $r$, but make this assumption precise and explain how it helps your analysis.
   b) Assuming controllability of this auxiliary system, state what type of control law can be used to stabilize it (you don’t need to investigate conditions for controllability in terms of the original data $A$, $B$, $C$ like we did in class). Your final answer for the controller should be a state-space dynamical system (don’t leave it in the form containing integrators).
   c) Comment in what sense your results reflect the internal model principle.

2. Repeat Problem 1 but for a sinusoidal reference signal: $r(t) = \sin t$, $t \geq 0$.

3. Let $X$ and $Y$ be linear vector spaces over $\mathbb{R}$ equipped with inner products $\langle \cdot, \cdot \rangle_X$ and $\langle \cdot, \cdot \rangle_Y$, respectively. Let $L : X \to Y$ be a linear operator. We define the adjoint of $L$ to be a linear operator $L^* : Y \to X$ with the property that
   \[ \langle y, Lx \rangle_Y = \langle L^*y, x \rangle_X \quad \forall x \in X, y \in Y \]
   Assume that the map $LL^* : Y \to Y$ is invertible. Then the equation $Lx = y_0$ has a solution
   \[ x_0 = L^*(LL^*)^{-1}y_0 \]
   for each $y_0 \in Y$. Prove that if $x_1$ is any other solution of $Lx = y_0$, then $\langle x_1, x_1 \rangle \geq \langle x_0, x_0 \rangle$.
   (Hint: Let $y_1 := (LL^*)^{-1}y_0$. Using the definition of adjoint, show that $\langle y_1, Lx_0 \rangle = \langle x_0, x_0 \rangle$ and also that $\langle x_0, x_1 \rangle = \langle y_1, Lx_0 \rangle$. Complete the proof by using the fact that $\langle x_1 - x_0, x_1 - x_0 \rangle \geq 0$.)

4. Use the result of Problem 3 to give an alternative proof of the fact that the controller given in class:
   \[ u(t) = B^T(t)\Phi^T(t_0, t)\eta, \quad \eta = (W(t_0, t_1))^{-1} \cdot (\Phi(t_0, t_1)x_1 - x_0) \]
   has the minimal $L_2$ norm among all controllers that transfer the state of the LTV system $\dot{x} = A(t)x + B(t)u$ from $x_0$ at time $t_0$ to $x_1$ at time $t_1$. 