1. Consider the 2-D system

$$\dot{x}_1 = -\frac{6x_1}{(1+x_1^2)^2} + 2x_2, \qquad \dot{x}_2 = -\frac{2(x_1+x_2)}{(1+x_1^2)^2}$$

and the candidate Lyapunov function $V(x_1, x_2) = \frac{x_1^2}{1+x_1^2} + x_2^2$. Compute the derivative of this V along solutions. Can you conclude that all solutions $x(t)$ are bounded? that all solutions $x(t)$ with initial conditions $x(0)$ sufficiently close to 0 are bounded? that all $x(t)$ converge to 0? that all $x(t)$ with $x(0)$ sufficiently close to 0 converge to 0? For each question, explain which result you're using or give a reason why you can not.

2. When proving Barbalat's lemma in class, we assumed for simplicity that $W(x) \geq 0$ (which is one of the hypotheses in Theorem 2). However, Barbalat's lemma itself is valid even if W is not sign-definite. Refine the proof of Barbalat's lemma from class so that it works for W possibly taking both positive and negative values.

3. Consider the system

$$\dot{x} = \theta + x + u \tag{1}$$

which is similar to Example 1 from the class notes except the unknown parameter θ enters additively and not multiplicatively. Suppose we want to make x converge to 0. Propose an adaptive control law that achieves this, and justify that it works. Base your design and analysis on ideas similar to the ones used to treat Example 1 in class: introduce an estimate $\hat{\theta}$ and a differential equation for it (tuning law); make the control law depend on $\hat{\theta}$; analyze the closed-loop system with the help of a Lyapunov function.

When discussing Example 1 in class, we commented that the estimate $\hat{\theta}$ does not necessarily converge to the true value θ (see also Problem 4 below). For the closed-loop system that you obtained from the system (1) with your controller, can you prove that $\hat{\theta}$ does in fact converge to θ?

4. a) Consider the adaptive control system introduced in class to treat Example 1:

$$\dot{x} = \theta x + u \tag{2}$$

$$\dot{\hat{\theta}} = x^2 \tag{3}$$

$$u = -(\hat{\theta} + 1)x \tag{4}$$

where θ is an unknown real parameter. Using MATLAB (or any other software), simulate this system for some arbitrary choices of θ. Does the state x converge to 0? Does $\hat{\theta}$ remain bounded? Does it converge to θ? Can you analytically derive the limiting value of $\hat{\theta}$?

b) Keep the same tuning/control law (3)–(4) but modify the plant dynamics (2) to

$$\dot{x} = \theta x + u + d$$

where d is a constant nonzero disturbance. Repeat the simulations and answer the same questions as in part a).

c) Keep the disturbance, and modify the tuning law (3) by turning the adaptation off (setting $\dot{\hat{\theta}} = 0$) whenever x is in some small interval around 0. This modification is known as *dead zone*. Repeat the simulations and answer the same questions as in part a); play with the size of the dead-zone interval and explain how it should be chosen.

d) Keep the disturbance, and modify the tuning law (3) by selecting some value $\hat{\theta}_{\max}$ and turning the adaptation off (setting $\dot{\hat{\theta}} = 0$) if $\hat{\theta}$ reaches $\hat{\theta}_{\max}$. This modification is known as *projection*. Repeat the simulations and answer the same questions as in part a); play with the value of $\hat{\theta}_{\max}$ and explain how it should be chosen.

5. a) Modify the system in Example 1 from class to

$$\dot{x} = \theta(t)x + u$$

where $\theta(t)$ is a time-varying parameter that satisfies $|\theta(t) - \theta_{\text{av}}| \leq \delta$ for some known δ and unknown θ_{av}. Design an adaptive control law that makes $x(t)$ converge to 0.

b) Solve the same problem as in part a) for the system $\dot{x} = \theta(t)x^2 + u$, under the same assumption on $\theta(t)$ as in part a).