

If you use any facts in your solutions that were not discussed in class, you must prove them. It is expected, however, that your solutions will be based on class material.

1. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a continuous function. Prove that its set of zeros $\{x : f(x) = 0\}$ is always a closed set.

2. Prove that the sequence defined recursively by

$$x_{k+1} = \frac{x_k}{2} + \frac{2}{x_k}$$

converges to 2 for every $x_0 \in [\sqrt{2}, 2\sqrt{2}]$.

3. Consider a function $f : \mathbb{R} \rightarrow \mathbb{R}$.

a) If f is locally Lipschitz, does this imply that f is uniformly continuous?

b) Conversely, if f is uniformly continuous, does this imply that f is locally Lipschitz?

4. Suppose that $f(t, x)$ is continuous in t and locally Lipschitz in x for each fixed t , and that t takes values in a closed interval $[t_0, t_1]$. Does this imply that f is locally Lipschitz in x uniformly in $t \in [t_0, t_1]$? Prove or give a counterexample.

5. Suppose that $f(t, x)$ satisfies all hypotheses of the local existence and uniqueness theorem. Let W be a compact subset of \mathbb{R}^n . Prove that there exists a $\delta > 0$ such that every solution with $x(t_0) \in W$ can be extended to the interval $[t_0, t_0 + \delta]$.

(Note: δ depends just on W but not on a particular initial condition in W . This fact implies that if it is known that the solution $x(t)$ remains in some compact set W , then it is defined globally—without the need to assume global Lipschitzness.)

Hint: make appropriate modifications to the proof of the existence and uniqueness theorem given in class.