Systems & Control Letters 61 (2012) 973–979

Contents lists available at SciVerse ScienceDirect

Systems & Control Letters

journal homepage: www.elsevier.com/locate/sysconle

Stability analysis of nonlinear systems using higher order derivatives of Lyapunov function candidates

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ARTICLE INFO

Article history:
Received 23 December 2008
Received in revised form
14 July 2012
Accepted 16 July 2012
Available online 5 September 2012

Keywords:
Nonlinear systems
Stability analysis
Lyapunov function
Higher order derivatives

ABSTRACT

The Lyapunov method for stability analysis of an equilibrium state of a nonlinear dynamic system requires a Lyapunov function $v(t, x)$ having the following properties: (1) v is a positive definite function, and (2) $\dot{v}$ is at least a negative semi-definite function. Finding such a function is a challenging task. The first theorem presented in this paper simplifies the second property for a Lyapunov function candidate, i.e. this property is replaced by negative definiteness of some weighted average of the higher order time derivatives of v. This generalizes the well-known Lyapunov theorem. The second theorem uses such weighted average of the higher order time derivatives of a Lyapunov function candidate to obtain a suitable Lyapunov function for nonlinear systems' stability analysis. Even if we have a suitable Lyapunov function then this theorem can be used to prove a bigger region of attraction. The approach is illustrated by some examples.

1. Introduction

Consider the following n-dimensional dynamic system with a zero equilibrium state

$$\dot{x} = f(t, x) \quad t \geq 0,\ x \in \mathbb{R}^n \tag{1}$$

where the function $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is Lipschitz continuous and so that solutions to (1) depend uniquely on initial values. The Lyapunov direct method to prove asymptotic stability of a zero equilibrium state of such a system requires a Lyapunov function, i.e. a function $v(t, x)$ with the following properties: (1) $v(t, x)$ is a locally positive definite function, and (2) the time derivative along the system's trajectories $\dot{v}$ is a negative definite function.

A Lyapunov function candidate v is a function with the first property. Even if an equilibrium state is asymptotically stable, it can easily happen, that for a given Lyapunov function candidate v the derivative along trajectories is not negative definite. See [1,2] for details of this theory.

There have been attempts to rectify the above shortcoming. The first by LaSalle for autonomous systems $\dot{x} = f(x)$, in which a Lyapunov function candidate $v(x) > 0$ was used for $\dot{v}(x) \leq 0$ and the LaSalle invariance principle was used to prove asymptotic stability of a zero equilibrium state [2]. Recently, the situations of $v(x) > 0$ and $\dot{v}(x) \leq 0$ were used in a different way [3], using the higher order time derivatives of $v(x)$ instead of the LaSalle invariance principle. It is shown if $\dot{v}(x) \leq 0$ anywhere and $v^{(2k+1)}(x) < 0$ for some $k \geq 1$ while $\dot{v}(x)$ vanishes, then the zero equilibrium state of a system is AS. The result of this work has been used to investigate the hub-appendage problem [4].

Kudaev [5] and Yorke [6,7] investigated conditions such as $\min\{\dot{v}(x), h\ddot{v}(x)\} < 0$ (where $h > 0$) in some region about the origin to prove asymptotic stability of the zero equilibrium state of $\dot{x} = f(x)$. However Butz [8] showed such a condition is trivial, because it implies $\dot{v}(x) < 0$ in some region around the origin. Then he augmented the role of $\dddot{v}(x)$ in the stability analysis and showed if $a_3\dddot{v}(x) + a_2\ddot{v}(x) + \dot{v}(x) < 0,\ \forall x \neq 0$ ($a_i \geq 0$ for $i = 2, 3$), then the zero equilibrium state of $\dot{x} = f(x)$ is AS. Heinen and Vidyasagar [9] used this result in the Lagrange stability analysis of nonlinear systems.

Ahmadi and Parrilo [10] used a non-monotone Lyapunov function candidate $v(x)$ for stability analysis of a given discrete time system $x_{k+1} = f(x_k)$. They showed if $[v(x_{k+1}) - v(x_k)] + \sum_{j=2}^{m} a_j[v(x_{k+j}) - v(x_k)] < 0$ for some $a_j \geq 0,\ j = 2, \ldots, m$ then the zero equilibrium state of $x_{k+1} = f(x_k)$ is globally asymptotically stable.

Others have used a non-monotone Lyapunov function candidate $v(t, x)$ to investigate stability of a given non-autonomous system (1). Gunderson [11] assumes $v^{(m)}(t, x) \leq g_m(v, \dot{v}, \ldots, v^{(m-1)})$ along the trajectories of (1) for some $m \in \mathbb{N}$, and compares it with a nonlinear co-system $u^{(m)}(t) = g_m(u, \dot{u}, \ldots, u^{(m-1)})$. If the nonlinear co-system has an asymptotically stable zero solution $u(t) \equiv 0$ and $g_m(\cdot)$ satisfies some more conditions then the zero equilibrium state of (1) would also be AS. This method cannot use a linear co-system. Thus the stability analysis of the co-system is a new complicated problem [11].

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doi:10.1016/j.sysconle.2012.07.005

Nomenclature

$\|\cdot\|$	A given norm on $\mathbb{R}^n$
$V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^m$	Vector function of dimension m
$\phi \in \mathcal{K}(\mathcal{K}_\infty)$	ϕ is a function of class K (K infinity) [2]
$a \le b$	Component-wise inequality, if a and b are vectors in $\mathbb{R}^n$
C^i	The class of functions with continuous i'th order partial derivatives
(U)(G)AS	(Uniformly) (Globally) Asymptotically Stable
ODVF	Over Derivatives Vector Function

Our previous results [12,13] used the inequality $\sum_{i=0}^{m} a_i v^{(i)}(t,x) \le 0$ for a locally positive definite function $v(t,x)$ to conclude the zero equilibrium state of (1) being UAS. The authors used a linear time invariant co-system $\sum_{i=0}^{m} a_i u^{(i)}(t) = 0$ with a Hurwitz characteristic equation $\sum_{i=0}^{m} a_i s^i = 0$.

The Hurwitz condition for $\sum_{i=0}^{m} a_i s^i = 0$ was the main limitation of [12,13], and a result of this paper reduces this condition to $a_i \ge 0$ for $i = 0, 1, \ldots, m$. However, the proof here is more complicated, due to the possibility of the linear time invariant co-system $\sum_{i=0}^{m} a_i u^{(i)}(t) = 0$ being unstable; hence the comparison method is not useful.

The first theorem in this paper shows if $\sum_{i=1}^{m} a_i v^{(i)}(t,x)$ is a negative definite function and $a_i \ge 0$ for $i = 1, \ldots, m$, then the zero equilibrium state of (1) is UAS. This generalizes the result of [8] to the case of higher order time derivatives ($m > 3$) and for time-varying systems. However, the proving scheme of [8] fails for $m > 3$ because it uses a marginally stable co-system $a_3 \dddot{u}(t) + a_2 \ddot{u}(t) + \dot{u}(t) = 0$, which may be unstable in the case of $m > 3$. The second theorem in this paper uses the higher order time derivatives of a Lyapunov function candidate satisfying the conditions of the first theorem to obtain a suitable Lyapunov function for stability analysis. The details are presented in [14].

This paper is organized as follows. The definitions are given in Section 2. The theorems for the stability analysis of nonlinear non-autonomous systems and some corollaries are given in Section 3, and the proof of the main theorem (Theorem 1) is left to Section 4. Some examples for verifying the method are presented in Section 5, and concluding remarks are given in Section 6.

2. The preliminaries

Some definitions and technical material are necessary here. Any arbitrary $v(t,x)$ function is called *decrescent* if there exists $\psi \in \mathcal{K}$ such that $v(t,x) < \psi(\|x\|)$ for all $x \in \mathbb{R}^n$ and $t \ge 0$. In this paper $v_1(t,x)$ is a C^1 positive definite function, i.e. $v_1(t,\mathbf{0}) = 0,\ \forall t \ge 0$ and there is $\phi_1 \in \mathcal{K}_\infty$ such that

$$v_1(t,x) \ge \phi_1(\|x\|) \quad \forall x \in \mathbb{R}^n,\ \forall t \ge 0 \tag{2}$$

holds. The total time derivative of $v_1(t,x)$ along the solutions of (1) is given as follows:

$$\dot{v}_1(t,x) \triangleq [\partial v_1(t,x)/\partial x]^T f(t,x) + \partial v_1(t,x)/\partial t. \tag{3}$$

Also if $v_1(t,x) \in C^m$ and $f(t,x) \in C^{m-1}$ then the m higher order time derivatives of $v_1(t,x)$ are computed iteratively using the following formula:

$$\begin{aligned} v_1^{(j)}(t,x) &= \frac{d}{dt}[v_1^{(j-1)}(t,x)] \\ &= [\partial v_1^{(j-1)}(t,x)/\partial x]^T f(t,x) + \partial v_1^{(j-1)}(t,x)/\partial t, \\ & \quad j = 1, \ldots, m. \end{aligned} \tag{4}$$

Hence defining $v_{j+1}(t,x) \triangleq \dot{v}_j(t,x)$ for $j = 1, \ldots, m-1$, we call the C^1 vector function $V(t,x) \triangleq [v_1(t,x), v_2(t,x), \ldots, v_m(t,x)]^T$ a *derivatives vector function*. Notice that the $v_2, \ldots, v_m$ components of $V(t,x)$ may be positive or negative, so it differs from a *vector Lyapunov function* introduced in the literature.

While computing a derivatives vector function, it sometimes happens that $\dot{v}_j(t,x) \notin C^1$ for some $j \in \mathbb{N}$, thus the equation $v_{j+1}(t,x) \triangleq \dot{v}_j(t,x)$ cannot be used to obtain a C^1 function $v_{j+1}(t,x)$. However, we may find a C^1 upper bound function $v_{j+1}(t,x)$ for the given non C^1 function $\dot{v}_j(t,x)$, i.e. $v_{j+1}(t,x) \ge \dot{v}_j(t,x)$, then the level jumps to $j+1$. Hence the C^1 vector function $V(t,x) \triangleq [v_1(t,x), v_2(t,x), \ldots, v_m(t,x)]^T$ may be called an *over derivatives vector function* (ODVF).

The length m of an ODVF is not pre-determined. In fact we start from a C^1 positive definite function $v_1(t,x)$, then we iterate $v_{j+1}(t,x) \ge \dot{v}_j(t,x)$ for each $j = 1, 2, \ldots$ to find a C^1 function $v_{j+1}(t,x)$ as an upper bound for each $\dot{v}_j(t,x)$. The procedure would be truncated after finding some negative definite function $\dot{v}_m(t,x)$ for some $m \in \mathbb{N}$, i.e. construction of the following differential inequalities using some $\phi_2 \in \mathcal{K}$:

$$\begin{cases} \dot{v}_1(t,x) \le v_2(t,x) \\ \quad\vdots \\ \dot{v}_{m-1}(t,x) \le v_m(t,x) \\ \dot{v}_m(t,x) \le -\phi_2(\|x\|). \end{cases} \tag{5}$$

All the components of a given ODVF $V(t,x) = [v_1(t,x), v_2(t,x), \ldots, v_m(t,x)]^T$ must be decrescent, i.e. there are some $\psi_j \in \mathcal{K}$ for $j = 1, \ldots, m$ such that

$$v_j(t,x) \le \psi_j(\|x\|) \quad \forall x \in \mathbb{R}^n,\ \forall t \ge 0 \tag{6}$$

holds true. Then the relations (2), (5) and (6) will be used to prove asymptotic stability of a zero equilibrium state of (1). However the main theorem will generalize the inequalities in (5).

3. The main theorem

Let us state the main theorem.

Theorem 1 (*Main Theorem*). *Suppose an m-vector C^1 function $V(t,x) = [v_1(t,x), v_2(t,x), \ldots, v_m(t,x)]^T$ with the following properties:*

i. *The first component $v_1(t,x)$ of $V(t,x)$ is a positive definite function, i.e. (2) is satisfied.*
ii. *All the $v_j(t,x)$ components are decrescent, i.e. (6) is true.*
   (a) *If there exist a $\phi_2 \in \mathcal{K}$ and a lower triangular matrix $A = [a_{ij}]_{m\times m}$ with the following property:*

$$a_{ij} \begin{cases} = 0, & \text{if } i < j \\ > 0, & \text{if } i = j \\ \ge 0, & \text{if } i > j \end{cases} \tag{7}$$

such that the following differential inequality is satisfied along the trajectories of (1),

$$\begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & 0 & \vdots \\ \vdots & a_{ij} & \ddots & 0 & 0 \\ a_{m-1,1} & \cdots & & a_{m-1,m-1} & 0 \\ a_{m1} & \cdots & & a_{m,m-1} & a_{mm} \end{bmatrix} \begin{bmatrix} \dot{v}_1 \\ \dot{v}_2 \\ \vdots \\ \dot{v}_{m-1} \\ \dot{v}_m \end{bmatrix} \le \begin{bmatrix} v_2 \\ v_3 \\ \vdots \\ v_m \\ -\phi_2(\|x\|) \end{bmatrix} \tag{8}$$

then the zero equilibrium state of (1) is UGAS.

(b) *If the above conditions hold only locally, i.e. for $\|x\| < r$ for a given $r > 0$ then the zero equilibrium state of (1) is UAS.* □

The proof of Theorem 1 is delayed until the next section.

Remark 1. For $m = 1$ the above theorem is reduced to the Lyapunov direct theorem for the stability analysis of the zero equilibrium state of (1).

Remark 2. A special form of Eq. (8) in Theorem 1 is as follows:

$$\dot{v}_j(t,x) = v_{j+1}(t,x), \quad j = 1,\ldots,m-1$$
$$\sum_{j=1}^{m} a_j v_1^{(j)}(t,x) \le -\phi_2(\|x\|). \tag{9}$$

The above relation was the first objective of this paper and clearly generalizes the result of Butz [8] to the case of non-autonomous systems and for $m > 3$. □

Corollary 1. *Application of Theorem 1 with C^1 autonomous functions, i.e. $v_j(x)$ just requires $v_j(\mathbf{0}) = 0$ for $j = 1,\ldots,m$ instead of (6).*

Proof. Any C^1 function $v(x)$ is also continuous, and if $v(\mathbf{0}) = 0$ then $v(x) \le \psi(\|x\|)$, $\forall x \in \mathbb{R}^n$ for $\psi(p) \triangleq \sup_{\|x\|\le p} v(x)$ and $\psi \in \mathcal{K}$. □

The following corollary coincides with the main result of [13]:

Corollary 2 *([13]). Let a C^1 m-vector function satisfy the following differential inequality along the trajectories of (1):*

$$\begin{bmatrix} \dot{v}_1(t,x) \\ \dot{v}_2(t,x) \\ \vdots \\ \dot{v}_m(t,x) \end{bmatrix} \le \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & \cdots & -a_{m-1} \end{bmatrix} \begin{bmatrix} v_1(t,x) \\ v_2(t,x) \\ \vdots \\ v_m(t,x) \end{bmatrix}. \tag{10}$$

If in addition

i. *$v_1(t,x)$ is a positive definite function, i.e. (2) is satisfied.*
ii. *The characteristic equation $s^m + a_{m-1}s^{m-1} + \cdots + a_1 s + a_0 = 0$ is Hurwitz.*
iii. *$|v_j(t,x)| \le \psi_j(\|x\|)$, $\psi_j \in \mathcal{K}$ for $j = 1,\ldots,m$.*

Then the zero equilibrium state of (1) is UGAS. □

Proof. The inequality $|v_j(t,x)| \le \psi_j(\|x\|)$ implies (6), and starting from the last row of (10), and substituting from previous rows and (2) into the row, implies that

$$0 \ge \dot{v}_m(t,x) + \sum_{j=1}^{m-1} a_j v_{j+1}(t,x) + a_0 v_1(t,x)$$
$$\ge \dot{v}_m(t,x) + \sum_{j=1}^{m-1} a_j \dot{v}_j(t,x) + a_0\phi_1(\|x\|) \Rightarrow$$
$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ a_1 & a_2 & \cdots & a_{m-1} & 1 \end{bmatrix} \begin{bmatrix} \dot{v}_1 \\ \dot{v}_2 \\ \vdots \\ \dot{v}_{m-1} \\ \dot{v}_m \end{bmatrix} \le \begin{bmatrix} v_2 \\ v_3 \\ \vdots \\ v_m \\ -a_0\phi_1(\|x\|) \end{bmatrix}.$$

The conditions of Theorem 1 are satisfied and the zero equilibrium state of (1) is UGAS. □

Remark 3. Let us compare Theorem 1 and Corollary 2. Eq. (10) implies a special form of (8), but the converse is not true. On the other hand, Theorem 1 removes the Hurwitz condition of Corollary 2, and relaxes the $|v_j(t,x)| \le \psi_j(\|x\|)$ inequalities to (6).

Theorem 2. *If the conditions of Theorem 1 hold true then the function*

$$s(t,x) \triangleq \sum_{j=1}^{m} a_{mj} v_j(t,x) \tag{11}$$

is a decrescent function, $s(t,\mathbf{0}) = 0$, and $s(t,x) > 0$ for $x \ne 0$. Also $\dot{s}(t,x)$ is a negative definite function. □

Proof. The sum of decrescent functions is also decrescent and $s(t,\mathbf{0}) = \sum_{j=1}^{m} a_{mj} v_j(t,\mathbf{0}) = \sum_{j=1}^{m} a_{mj}\cdot 0 = 0$. The last row of (8) implies $\dot{s}(t,x) = \sum_{j=1}^{m} a_{mj}\dot{v}_j(t,x) \le -\phi_2(\|x\|)$. Integrating both sides of $\phi_2(\|x\|) \le -\dot{s}(t,x)$ along a trajectory $x(t)$ starting at $x(t_0) = x_0$, on its life time, i.e. from t_0 to infinity, yields:

$$\int_{t_0}^{+\infty} \phi_2(\|x(t)\|)dt \le -\int_{t_0}^{+\infty} \dot{s}[t,x(t)]dt$$
$$= s(t_0,x_0) - \lim_{t\to+\infty} s[t,x(t)]. \tag{12}$$

The statement in Theorem 1 makes the zero equilibrium state of (1) AS, and hence $\lim_{t\to+\infty} s[t,x(t)] = 0$. Substituting this for (12) yields that $s(t_0,x_0) \ge \int_{t_0}^{+\infty} \phi_2(\|x(t)\|)dt > 0$ for all $x_0 \ne 0$ concluding the proof. □

Remark 4. The last theorem states that the higher order time derivatives of a given Lyapunov function candidate $v_1(t,x)$ can be used to obtain a suitable Lyapunov function $s(t,x)$ for the stability analysis of (1), at least in the case of autonomous systems. However, without using the attractivity result of Theorem 1, there is no reason for $s(t,x)$ to be a Lyapunov function. Moreover, if we have a suitable Lyapunov function $v_1(t,x)$, then the higher order time derivatives $v_1^{(i)}(t,x)$ can be used to find a new Lyapunov function $s(t,x)$ proving a bigger region of attraction. Example 2 shows this case. □

Remark 5. A difficulty of the method presented here is that there is no general approach to find the number m of higher order derivatives of a Lyapunov function candidate. The inequalities in (8) are constructed row by row. At the i'th row the non-negative a_{ij} weights are searched to make the summation $\sum_{j=1}^{i} a_{ij}\dot{v}_j(t,x)$ a negative definite function. If such weights are found, then the procedure is terminated by the assumption $m \triangleq i$, otherwise some useful upper bounding function $v_{i+1}(t,x) \ge \sum_{j=1}^{i} a_{ij}\dot{v}_j(t,x)$ is sought and the row jumps to $i+1$. However, we need an algorithm to seek correct weights and functions. □

4. The proof of main theorem

Some lemmas are required to prove Theorem 1.

Lemma 1. *Under the assumptions of Theorem 1, a lower triangular matrix $Q = [q_{ij}]_{m\times m}$ with a similar property as A in (7) exists, such that:*

(a) *If $\|x(t)\| \ge b \ge 0$, $\forall t \in [t_0, t_0+T]$ for given t_0, $T < +\infty$ then for $i = 1,\ldots,m$*

$$\sum_{j=1}^{i} q_{ij} v_j(t,x) \le -\phi_2(b)\frac{(t-t_0)^{m+1-i}}{(m+1-i)!} + \sum_{r=i}^{m} \frac{(t-t_0)^{r-i}}{(r-i)!} \sum_{j=1}^{r} q_{rj} v_j(t_0,x_0) \tag{13}$$

is satisfied. In particular the following holds:

$$q_{11}\phi_1(\|x\|) \leq q_{11}v_1(t,x) \leq -\phi_2(b)\frac{(t-t_0)^m}{m!} + \sum_{r=1}^{m}\frac{(t-t_0)^{r-1}}{(r-1)!}\sum_{j=1}^{r}q_{rj}v_j(t_0,x_0) \triangleq A_b(t). \quad (14)$$

(b) *A special application of* (14) *for* $b = 0$ *yields:*

$$q_{11}\phi_1(\|x\|) \leq \sum_{r=1}^{m}\frac{(t-t_0)^{r-1}}{(r-1)!}\sum_{j=1}^{r}q_{rj}v_j(t_0,x_0) = A_0(t),\ \forall t \geq t_0. \quad (15)$$

(c) *No trajectory of* (1) *escapes to infinity in finite time.* □

Proof. The $Q_{m\times m}$ matrix is a lower triangular matrix, i.e. $q_{ij} = 0$ for $i < j$. The other elements of Q are defined recursively, beginning from the last row. Define the following:

$$q_{mj} \triangleq a_{mj}, \quad j = 1,\ldots,m, \quad (16)$$

where the a_{mj}'s were introduced in (8). The other rows of Q are defined recursively in order of bottom to top of the matrix using the following algorithm:

For $i = m-1,\ldots,2,1$
For $j = i,\ldots,1$

$$q_{ij} \triangleq \sum_{r=j}^{i}q_{i+1,r+1}a_{rj} = [q_{i+1,j+1}\cdots q_{i+1,r+1}\cdots q_{i+1,i+1}] \times [a_{jj}\cdots a_{rj}\cdots a_{ij}]^T. \quad (17)$$

Next j
Next i.

It is clear that $q_{ij} \geq 0$ for $i \geq j$, also $q_{ii} = \prod_{r=i}^{m}a_{rr} > 0$ for $i = 1,\ldots,m$.

Part (a) For the proof of this part, let $\|x(t)\| \geq b \geq 0$ for $t \in [t_0, t_0+T]$. The inequality (13) will be proved by a descending induction on i, i.e. for $i = m, m-1,\ldots,1$.

1: For $i = m$, substitute $\|x(t)\| \geq b$ and (16) for the last row of (8), as follows:

$$\sum_{j=1}^{m}q_{mj}\dot{v}_j(t,x) \leq -\phi_2(b) \quad \forall t \in [t_0, t_0+T]. \quad (18)$$

Then integrate both sides of (18) on the time interval $\tau \in [t_0, t]$ for some $t \in [t_0, t_0+T]$:

$$\int_{t_0}^{t}\left[\sum_{j=1}^{m}q_{mj}\dot{v}_j(\tau,x)\right]d\tau \leq \int_{t_0}^{t}-\phi_2(b)\,d\tau \Rightarrow$$
$$\sum_{j=1}^{m}q_{mj}v_j(t,x) \leq -\phi_2(b)(t-t_0) + \sum_{j=1}^{m}q_{mj}v_j(t_0,x_0). \quad (19)$$

This relation coincides with (13) for $i = m$.

2: Suppose (13) holds true for some $2 \leq i \leq m$. Starting from the expression in the left-hand side of (13), using $q_{ir} \geq 0$ and substituting $v_1(t,x) \geq 0$ and also substituting $v_{r+1}(t,x) \geq \sum_{j=1}^{r} a_{rj}\dot{v}_j(t,x)$ from the preceding lines of (8), exchanging the summations, and using (17) yields the following:

$$\sum_{j=1}^{i}q_{ij}v_j(t,x) = q_{i1}v_1(t,x) + \sum_{r=1}^{i-1}q_{i,r+1}v_{r+1}(t,x) \geq \sum_{r=1}^{i-1}q_{i,r+1}v_{r+1}(t,x) \geq \sum_{r=1}^{i-1}q_{i,r+1}\sum_{j=1}^{r}a_{rj}\dot{v}_j(t,x) = \sum_{j=1}^{i-1}\left(\sum_{r=j}^{i-1}q_{i,r+1}a_{rj}\right)\dot{v}_j(t,x) = \sum_{j=1}^{i-1}q_{i-1,j}\dot{v}_j(t,x). \quad (20)$$

Then substituting (20) for (13), and replacing t with the dummy variable τ, and integrating on the time interval $\tau \in [t_0, t]$ for some $t \in [t_0, t_0+T]$, yields:

$$\int_{t_0}^{t}\sum_{j=1}^{i-1}q_{i-1,j}\dot{v}_j(\tau,x)d\tau \leq \int_{t_0}^{t}\left[-\phi_2(b)\frac{(\tau-t_0)^{m+1-i}}{(m+1-i)!} + \sum_{r=i}^{m}\frac{(\tau-t_0)^{r-i}}{(r-i)!}\sum_{j=1}^{r}q_{rj}v_j(t_0,x_0)\right]d\tau. \quad (21)$$

Evaluating the integrals and rearranging some terms from left to right, yields:

$$\sum_{j=1}^{i-1}q_{i-1,j}v_j(t,x) \leq \sum_{j=1}^{i-1}q_{i-1,j}v_j(t_0,x_0) - \phi_2(b)\frac{(t-t_0)^{m+2-i}}{(m+2-i)!} + \sum_{r=i}^{m}\frac{(t-t_0)^{r+1-i}}{(r+1-i)!}\sum_{j=1}^{r}q_{rj}v_j(t_0,x_0). \quad (22)$$

Combining the right-hand summations in (22) yields the appropriate form of (13) for $(i-1)$, i.e.

$$\sum_{j=1}^{i-1}q_{i-1,j}v_j(t,x) \leq -\phi_2(b)\frac{(t-t_0)^{m+2-i}}{(m+2-i)!} + \sum_{r=i-1}^{m}\frac{(t-t_0)^{r+1-i}}{(r+1-i)!}\sum_{j=1}^{r}q_{rj}v_j(t_0,x_0). \quad (23)$$

3: Particular use of (13) for $i = 1$ and then substituting (2) yields (14).

Part (b) $\|x(t)\| \geq 0\ \forall t \geq t_0$, thus repeating part ($a$) with $b = 0$, implies (15).

Part (c) The finiteness of $\|x(t)\|$ for all $t \in [0, +\infty)$ is implied from (15), because $q_{11} > 0$, $\phi_1 \in \mathcal{K}_\infty$, and the right hand side of (15) is a polynomial of time t, therefore the solutions could not have any finite escape time. □

Lemma 2. *Under the assumptions of Theorem 1, the trajectories of the system cannot stay far away from the origin forever, i.e.*

$$\forall b > 0\ \forall x(t_0) = x_0\ \exists t_1 \geq t_0, \quad \|x(t_1)\| \leq b. \quad (24)$$

Proof. Let b, c and T be arbitrary positive values, such that

$$\begin{cases} \|x_0\| \le c \\ \|x(t)\| \ge b > 0, \quad \forall t \in [t_0, t_0 + T] \end{cases} \tag{25}$$

holds true. Then using (6) and Lemma 1(a), i.e. combining $v_j(t_0, x_0) \le \psi_j(\|x_0\|) \le \psi_j(c)$ into (14) yields

$$q_{11}\phi_1(b) \le q_{11}\phi_1(\|x(t)\|) \le A_b^c(\tau), \quad \forall \tau \triangleq t - t_0 \in [0, T], \tag{26}$$

where

$$A_b^c(\tau) \triangleq -\phi_2(b)\frac{\tau^m}{m!} + \sum_{r=1}^{m} \frac{\tau^{r-1}}{(r-1)!} \sum_{j=1}^{r} q_{rj}\psi_j(c) \tag{27}$$

is a polynomial of $\tau \ge 0$ with the following properties:

$$\forall c \ge b > 0, \quad \begin{cases} A_b^c(0) = q_{11}\psi_1(c) \ge q_{11}\phi_1(b) > 0 \\ \left[\dfrac{d}{d\tau}A_b^c(\tau)\right]\Bigg|_{\tau=0} = q_{21}\psi_1(c) + q_{22}\psi_2(c) > 0 \\ \lim\limits_{\tau \to +\infty} A_b^c(\tau) = -\infty. \end{cases} \tag{28}$$

These properties imply the following

$$\exists T_1(b, c) > 0, \ \forall \tau > T_1(b, c) \quad A_b^c(\tau) \le A_b^c(T_1) = q_{11}\phi_1(b). \tag{29}$$

If $T_1(b, c) < T$ then using (26) and (29) implies $q_{11}\phi_1(b) \le A_b^c(\tau) \le q_{11}\phi_1(b)$ (and hence $A_b^c(\tau) \equiv q_{11}\phi_1(b)$) for all $T_1(b, c) < \tau < T$. This is a false result for a non-constant polynomial of τ. Therefore (25) does not hold true for $T > T_1(b, c)$, i.e.

$$\forall \|x(t_0)\| \le c \ \exists t_1 \in [t_0, t_0 + T_1(b, c)], \quad \|x(t_1)\| \le b. \tag{30}$$

This concludes the proof. □

Lemma 3. *Under the assumptions of Theorem 1, there exists a function $\alpha \in \mathcal{K}_\infty$ such that*

$$\|x(t_2)\| \le \alpha(\|x(t_1)\|) \quad \forall t_2 \ge t_1 \tag{31}$$

holds true.

Proof. Let $c = b \ge 0$ in Eqs. (25)–(29). Then define $\tau_1(b) \ge 0$ as the smallest time such that

$$\forall \tau \ge \tau_1(b) \quad A_b^b(\tau) \le A_b^b[\tau_1(b)] = q_{11}\phi_1(b). \tag{32}$$

If $b = 0$ then $A_b^b(\tau) = q_{11}\phi_1(b) \equiv 0$ and hence $\tau_1(0) = 0$. The function $\tau_1(b)$ is also well-defined for $b > 0$, because it is the minimum value for $T_1(b, b)$ that satisfies (29). This value is one of the m roots for the polynomial equation $A_b^b(\tau_1) = q_{11}\phi_1(b)$. Clearly $\tau_1(b)$ is a continuous function of b, because all roots of the polynomial equation $A_b^b(\tau_1) = q_{11}\phi_1(b)$ continuously depend on b. Then using (32) the following is defined for all $b \ge 0$.

$$M(b) \triangleq \sup_{\tau \ge 0} A_b^b(\tau) = \sup_{\tau \in [0, \tau_1(b)]} A_b^b(\tau). \tag{33}$$

The continuity of $\tau_1(b)$ implies that $\tau_1(b)$ has a maximum value $\tau_1^{\max}$ over any compact interval $b \in [0, a]$. Then using (32) and (33) the following is achieved:

$$M(b) = \sup_{\tau \in [0, \tau_1^{\max}]} A_b^b(\tau), \quad \forall b \in [0, a]. \tag{34}$$

The function $M(b)$ is the maximum value of the polynomial $A_b^b(\tau)$ over a compact interval $[0, \tau_1^{\max}]$. Since $A_b^b(\tau)$ is a continuous function of b and τ, then $M(b)$ is also a continuous function of b. This function has another main property:

$$q_{11}\phi_1(\|x(t_2)\|) \le M(\|x(t_1)\|) \quad \forall t_2 \ge t_1. \tag{35}$$

Considering some finite times $t_2 \ge t_1$ we prove (35) in either of the following two cases:

Case (1) $\|x(t_2)\| \le \|x(t_1)\| \triangleq b$: In this case $q_{11}\phi_1(\|x(t_2)\|) \le q_{11}\phi_1(b) = A_b^b[\tau_1(b)] \le M(b) = M(\|x(t_1)\|)$ is trivial using (32) and (33) respectively.

Case (2) $\|x(t_2)\| > \|x(t_1)\| \triangleq b$: In this case, there is some $t_0 \in [t_1, t_2)$ such that

$$\begin{cases} \|x(t_0)\| = \|x(t_1)\| = b \\ \|x(t)\| \ge b, \quad \forall t \in [t_0, t_2] \end{cases} \tag{36}$$

is satisfied. By defining $T \triangleq t_2 - t_0$ the expression (36) coincides with (25) for $c = b$. Now use the derived relation (26) for $t = t_2$ to conclude $q_{11}\phi_1(\|x(t_2)\|) \le A_b^b(t_2 - t_0) \le M(b) = M(\|x(t_1)\|)$.

Now for all $a \ge 0$ defining $\bar{M}(a) \triangleq \sup_{b \in [0,a]} M(b)$, to be a continuous and non-decreasing function and $\bar{M}(0) = M(0) = 0$, then define $\alpha'(b) \triangleq \bar{M}(b) + b$. It is clear that $\alpha' \in \mathcal{K}_\infty$ and $M(b) \le \bar{M}(b) \le \alpha'(b)$. Using (35) we conclude that $q_{11}\phi_1(\|x(t_2)\|) \le \alpha'(\|x(t_1)\|)$ for all $t_2 \ge t_1$. Thus (31) is proved by defining $\alpha(\cdot) \triangleq \phi_1^{-1}[\alpha'(\cdot)/q_{11}] \in \mathcal{K}_\infty$. □

Proof of Theorem 1. *Part* (a) The uniform stability of $x = \mathbf{0}$ in the sense of Lyapunov was proved in Lemma 3. Now (30) and (31) are combined to prove global uniform attractivity of $x = \mathbf{0}$, i.e. to show for each $\eta > 0$ and $c > 0$ there exists $T_2 = T_2(\eta, c) > 0$ such that

$$\forall \|x(t_0)\| \le c \ \forall t \ge t_0 + T_2(\eta, c), \quad \|x(t)\| \le \eta \tag{37}$$

is satisfied. First use (31) to conclude

$$\|x(t_1)\| \le b \Rightarrow (\forall t_2 \ge t_1 \ \|x(t_2)\| \le \alpha(b)). \tag{38}$$

Combining (30) and (38) yields the following relation shown in Fig. 1:

$$\forall \|x(t_0)\| \le c \ \exists t_1 \in [t_0, t_0 + T_1(b, c)] \ \forall t_2 \ge t_1, \quad \|x(t_2)\| \le \alpha(b). \tag{39}$$

By deleting the t_1 variable from (39), the following conclusion is straightforward:

$$\forall \|x(t_0)\| \le c \ \forall t_2 \ge t_0 + T_1(b, c), \quad \|x(t_2)\| \le \alpha(b). \tag{40}$$

Substituting $\alpha(b) = \eta > 0$ for (40) yields:

$$\forall \|x(t_0)\| \le c \ \forall t_2 \ge t_0 + T_1[\alpha^{-1}(\eta), c], \quad \|x(t_2)\| \le \eta. \tag{41}$$

Therefore (37) is satisfied for each $\eta > 0$ and $c > 0$, defining $T_2(\eta, c) \triangleq T_1[\alpha^{-1}(\eta), c]$ and the zero equilibrium state of (1) is UGAS.

Part (b) The UAS of zero equilibrium state is a local property of the nonlinear system (1) and could be proved in a similar manner as above, since (31) guarantees that the trajectories starting very near to the origin do not escape the region of $\|x\| < r$. Note that all class $\mathcal{K}_\infty$ functions in part (a) must be replaced by class $\mathcal{K}$ functions in part (b). □

5. Some examples

The following two examples show the usefulness of Theorems 1 and 2, respectively.

Example 1. A time-varying nonlinear system is given by:

$$\begin{cases} \dot{x}_1 = -g_1(x_1) - 2x_1x_2 + ke^{-pt}x_1 \\ \dot{x}_2 = -g_2(x_2) + x_1^2 + ke^{-pt}x_2 \end{cases} \quad t \ge 0, \ (k, p > 0) \tag{42}$$

$g_1(\sigma)$, and $g_2(\sigma)$ are continuous (but possibly non-smooth) functions and $g_i(0) = 0, g_i(\sigma)\sigma > 0, \ \forall \sigma \ne 0$. An ODVF is constructed using the following functions:

$$v_i(t, x) \triangleq (2ke^{-pt})^{(i-1)}(x_1^2 + 2x_2^2), \quad i = 1, 2, \ldots, m. \tag{43}$$

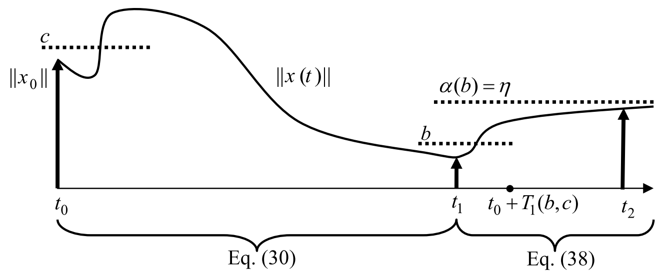

Fig. 1. The combination of (30) and (38) in obtaining (39).

$v_1(t,x) = x_1^2 + 2x_2^2$ is positive definite and $v_i(t,x)$'s are all decrescent functions. By straightforward computation we obtain the following

$$\begin{aligned}\dot{v}_i(t,x) &= -p(i-1)(2ke^{-pt})^{(i-1)}(x_1^2+2x_2^2) \\ &\quad + (2ke^{-pt})^{(i-1)}(2x_1\dot{x}_1 + 4x_2\dot{x}_2) \\ &= -(2ke^{-pt})^{(i-1)}[2x_1g_1(x_1) + 4x_2g_2(x_2) \\ &\quad + (p(i-1) - 2ke^{-pt})(x_1^2+2x_2^2)]\end{aligned} \tag{44}$$

$\dot{v}_1(t,x)$ might be sign-indefinite, since $k > 0$ (e.g. consider $g_1(\sigma) = g_2(\sigma) = \sigma^3$), and we have

$$\begin{aligned}\dot{v}_i(t,x) &\le (2ke^{-pt})^{(i)}(x_1^2+2x_2^2) \\ &= v_{i+1}(t,x) \quad \forall i = 1,2,\ldots,m-1.\end{aligned} \tag{45}$$

Arrange a weighted average $\sum_{i=1}^m a_i\dot{v}_i(t,x)$ with the positive a_i weights:

$$\begin{aligned}&\sum_{i=1}^m a_i\dot{v}_i(t,x) = -[2x_1g_1(x_1) + 4x_2g_2(x_2)]\sum_{i=1}^m a_i(2ke^{-pt})^{(i-1)} \\ &\quad + (x_1^2+2x_2^2)\sum_{i=1}^m a_i(2ke^{-pt})^{(i)} \\ &\quad - (x_1^2+2x_2^2)\sum_{i=2}^m a_i(2ke^{-pt})^{(i-1)}p(i-1) \\ &\Rightarrow \sum_{i=1}^m a_i\dot{v}_i(t,x) = -[2x_1g_1(x_1) + 4x_2g_2(x_2)]\sum_{i=1}^m a_i(2ke^{-pt})^{(i-1)} \\ &\quad + 2ke^{-pt}(x_1^2+2x_2^2)\Bigg[a_m(2ke^{-pt})^{m-1} + a_1 - pa_2 \\ &\quad + \sum_{i=2}^{m-1}(a_i - pa_{i+1}i)(2ke^{-pt})^{i-1}\Bigg].\end{aligned} \tag{46}$$

Supposing

$$\begin{aligned}&a_i - pa_{i+1}i = 0 \quad \text{for } i = 2,3,\ldots,m-1 \\ &\sup_{t\ge 0}[a_m(2ke^{-pt})^{m-1} + a_1 - pa_2] \\ &\quad = a_m(2k)^{m-1} + a_1 - pa_2 = 0\end{aligned} \tag{47}$$

and manipulating (46) yields:

$$\begin{aligned}\sum_{i=1}^m a_i\dot{v}_i(t,x) &\le -[2x_1g_1(x_1) + 4x_2g_2(x_2)]\sum_{i=1}^m a_i(2ke^{-pt})^{(i-1)} \\ &\le -a_1(2x_1g_1(x_1) + 4x_2g_2(x_2)).\end{aligned} \tag{48}$$

The inequalities (45) and (48) together make (8). Using Theorem 1 if $a_1 > 0$ and $a_i \ge 0,\ i = 2,\ldots,m$ then the zero equilibrium state of (42) is UGAS. Setting $a_1 = 1$ and solving (47) for $a_i,\ i = 2,\ldots,m$ yields:

$$\begin{aligned}&a_1 = 1, \qquad a_2 = \frac{1}{p\left[1 - (2k/p)^{m-1}\frac{1}{(m-1)!}\right]}, \\ &a_i = a_2/(i-1)!p^{i-2}, \quad i = 3,\ldots,m.\end{aligned} \tag{49}$$

The positivity of coefficients in (49) is guaranteed by the following condition:

$$(2k/p)^{m-1}\frac{1}{(m-1)!} < 1. \tag{50}$$

The integer $m \ge 2$ is arbitrary in the above procedure and $[(2k/p)^{m-1}\frac{1}{(m-1)!}] \to 0$ as $m \to +\infty$. Therefore for every pair $p, k > 0$, an $m \ge 2$ could be found to satisfy (50). Hence the UGAS of zero equilibrium state of (42) is proved for each $k, p > 0$.

This result can be verified using the Lyapunov direct method. Let $m \to +\infty$ in (49) to imply

$$\begin{cases} a_1 = 1 \\ a_i \to 1/(i-1)!p^{i-1}, \quad i = 2,3,\ldots, \end{cases} \quad \text{as } m \to +\infty \tag{51}$$

and use these limiting weights to construct the following suitable Lyapunov function for (42):

$$\begin{aligned}s(t,x) &\triangleq \sum_{i=1}^{+\infty} a_iv_i(t,x) = (x_1^2+2x_2^2)\sum_{i=1}^{+\infty}\left(\frac{2ke^{-pt}}{p}\right)^{(i-1)}\frac{1}{(i-1)!} \\ &= (x_1^2+2x_2^2)\exp\left(\frac{2ke^{-pt}}{p}\right),\end{aligned} \tag{52}$$

whose time derivative

$$\dot{s}(t,x) = -(2x_1g_1(x_1) + 4x_2g_2(x_2))\exp[2ke^{-pt}/p] \tag{53}$$

is a negative definite function. Therefore the higher order time derivatives of an unsuitable Lyapunov function candidate leads to construction of a suitable one in this example. □

Example 2. Consider the following nonlinear autonomous system:

$$\begin{cases} \dot{x}_1 = -x_1 + x_2^3 \\ \dot{x}_2 = x_1^3 - x_2. \end{cases} \tag{54}$$

The Lyapunov method proves asymptotic stability of a zero equilibrium state of this system, using the suitable Lyapunov function $v_1(x) = x_1^2 + x_2^2$ and its derivative $\dot{v}_1(x) = 2(x_1^2 + x_2^2)(x_1x_2 - 1)$. Fig. 2 shows that the circle $v_1(x) = x_1^2 + x_2^2 = 2$ is the biggest curve $v_1(x) = c$ enclosed in $W = \{x \mid \dot{v}_1(x) < 0\}$. Therefore the region of $R = \{x \mid v_1(x) = x_1^2 + x_2^2 < 2\}$ is the proved region of attraction by this suitable Lyapunov function.

We want to prove a bigger region of attraction using a new suitable Lyapunov function of the form of (11). Choose $m = 2$ and a weighted average $s_a(x) = v_1(x) + a\dot{v}_1(x)$ for some $a \ge 0$. Then $s_a(x)$ would be a suitable Lyapunov function if we show

$$\begin{aligned}&s_a(x) = v_1(x) + a\dot{v}_1(x) > 0 \\ &\dot{s}_a(x) = \dot{v}_1(x) + a\ddot{v}_1(x) < 0\end{aligned} \tag{55}$$

near the origin. Moreover defining

$$R_a \triangleq \{x \mid s_a(x) < 2\} \quad \text{and} \quad W_a \triangleq \{x \mid \dot{s}_a(x) < 0\}, \tag{56}$$

a new region of attraction R_a might be proved if we show $R_a \subset W_a$ for some $a > 0$.

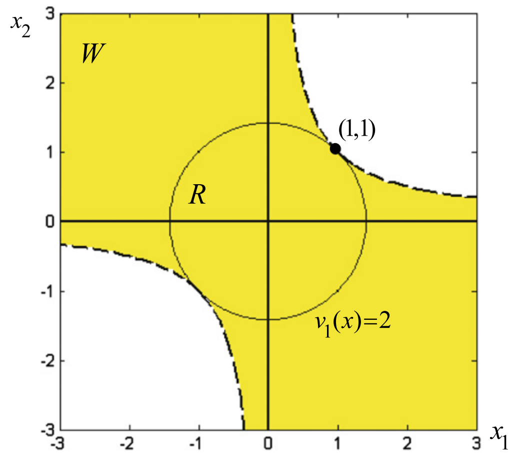

Fig. 2. Finding region of attraction by the Lyapunov method.

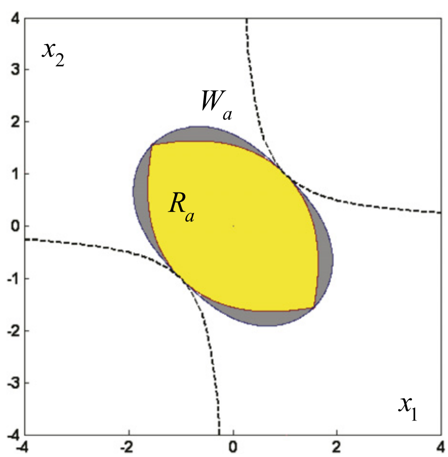

Fig. 3. Finding region of attraction by Theorem 2.

Clearly $R = R_0 \subset W_0 = W$ and $R \subset R_a$ for $a > 0$, thus R_a grows as a increases. It is not very hard to show that $R \subset R_a \subset W_a \subset W$ for very small positive values of a. We have found analytically the maximum value $a_m = 1.5 - \sqrt{2}$ for a such that $R_a \subset W_a$, and the biggest proved region of attraction R_a is shown in Fig. 3. See page 81 of [14] for details. □

6. Conclusion

A new method for investigating the asymptotic stability of a zero equilibrium state of a nonlinear non-autonomous system is presented. At first it is tried to find a suitable Lyapunov function candidate $v_1(t, x)$, i.e. with negative definite first order derivative. If $\dot{v}_1(t, x)$ is not a negative definite function, then Theorem 1 presents negative definiteness of $\sum_{i=1}^{m} a_i v_1^{(i)}(t, x)$ for some $m > 1$ and $a_i \geq 0$ as a sufficient condition for uniform asymptotic stability of a zero equilibrium state. If the higher order time derivatives are not well-defined, then some approximation form is used.

When the sufficient condition of Theorem 1 is satisfied, then Theorem 2 presents a suitable Lyapunov function $s(t, x) = \sum_{i=1}^{m} a_i v_1^{(i-1)}(t, x)$ for stability analysis of a zero equilibrium state. Even if $v_1(t, x)$ is a suitable Lyapunov function, Theorem 1 can be used to prove a bigger region of attraction. Some examples are given to show the usefulness of the approach.

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