
Note: since the exam problems are not locally Lipschitz, their solutions are not necessarily unique. I'm not attempting to list all correct solutions here.

1. Consider the system $\dot{x} = f(x)$ where $x \in \mathbb{R}^n$, f is locally Lipschitz, and $f(0) = 0$. Let us say that the system is *uniformly attractive* (with respect to the zero equilibrium) if for every pair of positive numbers ε and δ there exists a time T (which depends on ε and δ) such that if $|x(0)| \leq \delta$ then $|x(t)| \leq \varepsilon$ for all $t \geq T$. Prove that if the system is uniformly attractive then it is stable in the sense of Lyapunov (again, with respect to the zero equilibrium).

Solution. Take an arbitrary $\varepsilon > 0$. Pick some $\delta_1 > 0$ and then let $T(\varepsilon, \delta_1)$ be as in the definition of uniform attractivity. Moreover, for the same ε and T, by continuous dependence of solutions on initial conditions on finite intervals, and since 0 is an equilibrium, there exists a $\delta_2 > 0$ such that if $|x(0)| \leq \delta_2$ then $|x(t)| \leq \varepsilon$ for all $t \in [0, T]$. Letting $\delta := \min\{\delta_1, \delta_2\}$ we have that if $|x(0)| \leq \delta$ then $|x(t)| \leq \varepsilon$ for all $t \in [0, T]$ (by continuous dependence) as well as for all $t \geq T$ (by uniform attractivity), hence for all $t \geq 0$, which proves Lyapunov stability.

Note: In class we saw Vinograd's example of a system that is attractive but not Lyapunov stable; that system is not uniformly attractive because for $x(0)$ close to 0 it takes arbitrarily long time to traverse an arc and return to 0.

2. Show that if the system $\dot{x} = f(x)$, with $x \in \mathbb{R}^n$ and f is locally Lipschitz, has two or more (but finitely many) locally asymptotically stable equilibria, then it must have some solution trajectories that do not converge to any of these equilibria.

Solution. First, the domain of attraction of each locally AS equilibrium is open. This follows from continuous dependence of solutions on initial conditions on finite intervals. Indeed, let $\bar{x}$ be a locally AS equilibrium. Then there is an ε-ball around $\bar{x}$, for ε small enough, from where all solutions converge to $\bar{x}$. Now take any x_0 in the domain of attraction of $\bar{x}$. The solution starting from x_0 converges to $\bar{x}$, hence after some time T it will be inside the above ε-ball around $\bar{x}$. By continuous dependence, for x close enough to x_0, the solution starting from x will also be in the above ε-ball around $\bar{x}$ after time T, and so after that it will continue to converge to $\bar{x}$. This confirms that the domain of attraction of $\bar{x}$ is open.

Second, if all trajectories were to converge to one of these equilibria, then the union of their domains of attraction would be the whole $\mathbb{R}^n$. We just showed that these domains of attraction are open sets, and also clearly they are disjoint (have empty intersections). But it is well known that such a disjoint union of finitely many open sets cannot be the whole $\mathbb{R}^n$. This is easy to show: subtract the first open set from $\mathbb{R}^n$, we are left with a closed and nonempty set (it's closed because $\mathbb{R}^n$ is closed and if we subtract an open set we get a closed set; it's nonempty because the domain of attraction of each equilibrium is not the whole $\mathbb{R}^n$). Subtract the second open set, what remains is again closed and nonempty. Each time we subtract the next open set, we are left with a nonempty closed set, so we would not be able to arrive at the empty set after finitely many such subtractions. The resulting contradiction proves the claim.

3. Is the scalar system

$$\dot{x} = -\frac{1}{1 + d^2} x$$

input-to-state stable (ISS) with respect to the scalar input d? Prove or disprove.

Solution. Yes, this system is ISS. One way to prove this is as follows:

$$|x(t)| = e^{-\int_0^t \frac{1}{1+d^2(s)}ds}|x_0| \leq e^{-\frac{1}{1+\|d\|^2}t}|x_0| \leq \frac{1}{1+\frac{t}{1+\|d\|^2}}|x_0| \leq \frac{1+\|d\|^2}{1+t}|x_0| \leq \frac{1}{1+t}|x_0| + \frac{1}{2(1+t)^2}|x_0|^2 + \frac{1}{2}\|d\|^4$$

where the first equality follows from solving the system, the next inequality uses the notation $\|d\|$ for the supremum norm of d (either on $[0,\infty)$ or on $[0,t]$, by causality there is no difference between the two options), the next inequality uses the fact that $e^{-a}(1+a) \leq 1$ for all $a \geq 0$ (it's easy to check that the left-hand side equals 1 at $a = 0$ and is monotonically decreasing), and the last inequality follows from square completion. This proves ISS with $\beta(r,t) := \frac{1}{1+t}r + \frac{1}{2(1+t)^2}r^2$ and $\gamma(r) := \frac{1}{2}r^4$.

Another way to prove ISS is to use the ISS-Lyapunov function candidate $V(x) = \frac{1}{2}x^2$ which gives

$$\dot{V} = -\frac{1}{1+d^2}x^2 \leq -\frac{1}{1+\|d\|^2}x^2 = -\frac{2}{1+\|d\|^2}V$$

Treating $\tau = \frac{2}{1+\|d\|^2}t$ as rescaled time, we know that this implies the existence of a class $\mathcal{KL}$ function β_0 such that $|x(t)| \leq \beta_0(|x_0|, \frac{2}{1+\|d\|^2}t)$. This can be upper-bounded as follows, by considering the two cases $|x_0| \leq \|d\|$ and $|x_0| \geq \|d\|$:

$$\beta_0(|x_0|, \tfrac{2}{1+\|d\|^2}t) \leq \beta_0(|x_0|, \tfrac{2}{1+|x_0|^2}t) + \beta_0(\|d\|, \tfrac{2}{1+\|d\|^2}t) \leq \beta_0(|x_0|, \tfrac{2}{1+|x_0|^2}t) + \beta_0(\|d\|, 0)$$

and this proves ISS with $\beta(r,t) := \beta_0(r, \frac{2}{1+r^2}t)$ (it's not difficult to see that this is of class $\mathcal{KL}$) and $\gamma(r) := \beta_0(\|d\|, 0)$.

Yet another way—and this is probably the easiest way—is to notice that with the same Lyapunov function $V(x) = \frac{1}{2}x^2$ we have

$$|x| \geq \rho(|d|) \;\Rightarrow\; \dot{V} \leq -\alpha(|x|)$$

where $\rho(r) = r^2$ and $\alpha(r) = r^2/(1+r)$. It's obvious that ρ is of class $\mathcal{K}_\infty$, and it is not hard to show that α is also of class $\mathcal{K}_\infty$.

4. Let ℓ^2 be the space of infinite sequences $x = (x_1, x_2, \dots)$ with bounded ℓ^2-norm $\|x\|_{\ell^2} := \sqrt{\sum_{k=1}^\infty x_k^2}$. On this space, consider the system defined componentwise by

$$\dot{x}_k(t) = -\big(1 - 3\sqrt[k]{|x_k(t)|}\,\big)x_k(t), \qquad k = 1, 2, \dots$$

Show that for this system Lyapunov's first (indirect) method fails; explain in what sense. (Of course this doesn't contradict what we learned in class because the above system evolves on an infinite-dimensional state space.)

Solution. The above system has an equilibrium at 0 (the identically zero sequence). Its linearization around this equilibrium is

$$\dot{x}_k(t) = -x_k(t), \qquad k = 1, 2, \dots$$

which is exponentially stable. However, for the original nonlinear system the dynamics of each component x_k also have an equilibrium at $x_k = \pm 1/(3^k)$. Thus the vectors x which have 0 in all components except a single k-th component which equals $1/(3^k)$ are equilibria of the system. For k large, these equilibria are arbitrarily close (in the sense of the ℓ^2-norm) to 0. Therefore, the zero equilibrium is not locally exponentially stable, and so the standard statement of Lyapunov's first method (exponential stability of the linearization implies local exponential stability of the nonlinear system) does not apply to this system.

Source: Hans Zwart, Linearization and exponential stability, arXiv:1404.3475, 2014.