Time and place. In accordance with the university's final exam schedule, the final exam will be held on Tuesday, Dec 13, from 7:00pm to 10:00pm, in our usual classroom. There will be no conflict/make-up exam given at any other time.

Topics covered. The exam will be "cumulative", in other words, it will cover all the material discussed in class this semester. Proportionally to time spent in class on various topics, you can expect to see somewhat bigger emphasis placed on the topics discussed after the midterm exam (as well as their connections to earlier topics). Like the midterm, the final will focus on your understanding of the key concepts and your ability to decide which theory from class is relevant for solving a problem, not on calculations. Compared to the midterm, some problems on the final will test your general intuitive understanding of the subject and might call for more creativity.

What to bring. The exam is closed-book, closed-notes. You may bring two (double-sided) sheets of notes.

Tips for preparing. Make sure to follow up on all lecture material, readings, and homework problems and solutions (including the last problem set, which is now posted).

Lectures and office hours:

- We will have regular class on Tuesday Nov 29 and Thursday Dec 1, and I will hold regular office hours that week.
- *During the week of Dec 5 I will be attending the CDC (IEEE Conference on Decision and Control) in Cancun, and will not be reachable.*
- The TA, Emre Eraslan, will cover the last lecture on Tuesday Dec 6, which will be a review, and will hold office hours while I'm away.
- I plan to hold office hours on Monday Dec 12 (most likely on Zoom) and on Tuesday Dec 13 (most likely in person). I will send an email on Sunday or Monday (Dec 11–12) with more info.

Practice exam. See next page for a practice final exam. For solutions, please see the TA.

Problem 1 For an LTI system with output (and no input) $\dot{x} = Ax$, $y = Cx$, consider the following 4 properties:

$$1)\ y(t) \equiv 0 \ \Rightarrow\ x(t) \equiv 0, \quad 2)\ y(t) \equiv 0 \ \Rightarrow\ x(t) \to 0, \quad 3)\ y(t) \to 0 \ \Rightarrow\ x(t) \to 0, \quad 4)\ y(t) \to 0 \ \Rightarrow\ x(t) \equiv 0$$

where " $\equiv 0$" means "equals 0 for all t", " $\to 0$" means "converges to 0 as $t \to \infty$", and the implication is understood to hold for all solutions, so that property 2, for example, means "every state trajectory that produces the identically zero output must converge to 0".

a) Suppose the system is observable. Which of the properties 1)–4) are satisfied for it? Justify your answers.

b) Answer the same question for the case when the system is detectable.

Problem 2 Consider the scalar system

$$\dot{x} = ax + u$$

with $a > 0$, and a linear feedback law

$$u = -kx.$$

Suppose that we have to implement a *sample-and-hold* version of this feedback law: For a given sampling period $T > 0$ and sampling times $\tau_i := iT$, $i = 0, 1, 2, \dots$, the actual control input is

$$u(t) = -kx(\tau_i) \qquad \text{for } t \in [\tau_i, \tau_{i+1}),\ i = 0, 1, 2, \dots$$

Verbally, we compute the control at each sampling time and apply this constant control value over the next sampling interval.

Find the range of values for the feedback gain k (in terms of given a and T) for which the closed-loop system is asymptotically stable.

Problem 3 Consider an LTV system $\dot{x} = A(t)x$, and suppose that for every *fixed* t the eigenvalues of the matrix $A(t)$ have negative real parts. Does this imply asymptotic stability of the LTV system? If yes, explain why; if not, give a counterexample.

Problem 4 Consider the system

$$\dot{x}_1 = u$$
$$\dot{x}_2 = x_2$$

with $x = (x_1, x_2) \in \mathbb{R}^2$ and $u \in \mathbb{R}$. Let the initial state be $x_1(0) = 0$, $x_2(0) = 1$. Describe the set of all states that can be reached at time $t = 1$, starting from the above initial state and applying all possible controls. Justify your answer.

Problem 5 Consider the system

$$\dot{x}_1 = x_1 + u$$
$$\dot{x}_2 = x_1^2 + u^2$$

with some given initial condition $(x_1(t_0), x_2(t_0))$. Suppose that we want to minimize the cost

$$J(u) = x_1^2(t_1) + x_2(t_1)$$

where t_1 is a given final time.

Show that the optimal control takes the form of linear time-varying feedback $u^* = k(t)x^*$ (here u^* is the optimal control and x^* is the optimal state trajectory). Provide a differential equation and a boundary condition that the gain $k(t)$ must satisfy.

(Hint: try to reduce the above optimal control problem to a more familiar one, for which we know the solution to have the required form.)