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Brief paper

Dynamic Lyapunov functions☆

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ABSTRACT

Lyapunov functions are a fundamental tool to investigate stability properties of equilibrium points of linear and nonlinear systems. The existence of Lyapunov functions for asymptotically stable equilibrium points is guaranteed by *converse Lyapunov theorems*. Nevertheless the actual computation (of the analytic expression) of the function may be difficult. Herein we propose an approach to avoid the computation of an explicit solution of the Lyapunov partial differential inequality, introducing the concept of *Dynamic Lyapunov function*. These functions allow to study stability properties of equilibrium points, similarly to standard Lyapunov functions. In the former, however, a positive definite function is combined with a dynamical system that render Dynamic Lyapunov functions easier to construct than Lyapunov functions. Moreover families of standard Lyapunov functions can be obtained from the knowledge of a Dynamic Lyapunov function by rendering invariant a desired submanifold of the extended state-space. The invariance condition is given in terms of a system of partial differential equations similar to the Lyapunov pde. Differently from the latter, however, in the former no constraint is imposed on the sign of the solution or on the sign of the term on the right-hand side of the equation. Several applications and examples conclude the paper.

1. Introduction

Stability analysis of equilibrium points, usually characterized in the sense of Lyapunov, see Lyapunov (1992), is a fundamental topic in systems and control theory. Lyapunov's theory provides a tool to assess (asymptotic) stability of equilibrium points of linear and nonlinear systems, avoiding the explicit computation of the solution of the underlying ordinary differential equation. In particular, it is well-known, see e.g. Bacciotti (1992), Khalil (2001), that the existence of a scalar positive definite function, the time derivative of which along the trajectories of the system is negative definite, is sufficient to guarantee asymptotic stability of an equilibrium point. Conversely, several theorems, the so-called *converse Lyapunov theorems*, implying the existence of a Lyapunov function defined in a neighborhood of an asymptotically stable equilibrium point, have been established, see e.g. Khalil (2001), La Salle (1976), Lin, Sontag, and Wang (1996), Massera (1949) and Meilakhs (1979).

In recent years the development of control techniques requiring the explicit knowledge of a Lyapunov function, such as *backstepping* and *forwarding*, see e.g. Isidori (1995), Khalil (2001), Kokotovic and Arcak (2001), Mazenc and Praly (1996), Praly, Carnevale, and Astolfi (2010) and Sepulchre, Jankovic, and Kokotovic (1997), have conferred a crucial role to the computation of Lyapunov functions. In addition Lyapunov functions are useful to characterize and to estimate the region of attraction of locally asymptotically stable equilibrium points, see for instance Chiang, Hirsch, and Wu (1988) and Vannelli and Vidyasagar (1985), where the notion of *maximal* Lyapunov function has been introduced. However the actual computation of the analytic expression of the function may be difficult. From a practical point of view this is the main drawback of Lyapunov methods (Bacciotti & Rosier, 2005). An alternative approach consists in determining, if it exists, a *weak* Lyapunov function, i.e. a function the time derivative of which is only negative semi-definite along the trajectories of the system, and then prove asymptotic properties by means of *Invariance Principle* arguments (Khalil, 2001). A somewhat more flexible approach is pursued in Malisoff and Mazenc (2009), where the authors give sufficient conditions under which *weak* Lyapunov functions, which may be more easily available, can be employed to construct Lyapunov functions.

The main contribution of the paper consists of the definition of the concept of *Dynamic Lyapunov function*. The stability

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properties of equilibrium points of linear and nonlinear systems are characterized in terms of Dynamic Lyapunov functions and the relation between these functions and Lyapunov functions is explored. The former consist of a positive definite function combined with a dynamical system that render Dynamic Lyapunov functions easier to construct than Lyapunov function. In the second part of the paper, exploiting the ideas developed in Sassano and Astolfi (2012), Sassano and Astolfi (2010a) and Sassano and Astolfi (2010b), a constructive methodology to define Dynamic Lyapunov functions is proposed. In particular, a Dynamic Lyapunov function implicitly includes a *time-varying* term the behavior of which is autonomously adjusted and defined in terms of the solution of a differential equation. The latter, as a matter of fact, automatically enforces negativity of the time derivative of the positive definite function. Moreover, it is shown how to obtain Lyapunov functions from the knowledge of Dynamic Lyapunov functions. Preliminary versions of this work can be found in Sassano and Astolfi (2011) and Sassano and Astolfi (2012).

The rest of the paper is organized as follows. The notion of Dynamic Lyapunov function is introduced in Section 2. The topic of Section 3 is the construction of Dynamic Lyapunov functions for linear time-invariant systems. The extension to nonlinear systems is dealt with in Section 4. Section 5 deals with the problem of constructing Lyapunov functions from the knowledge of a Dynamic Lyapunov function for linear and nonlinear systems. The paper is concluded with numerical examples and some comments in Sections 6 and 7, respectively.

2. Dynamic Lyapunov functions

Consider the autonomous nonlinear system described by equations of the form

$$\dot{x} = f(x), \tag{1}$$

where $x(t) \in \mathbb{R}^n$ denotes the state of the system and $f : \mathcal{D} \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable, with $\mathcal{D}$ containing the origin of $\mathbb{R}^n$.

Definition 1 (*Dynamic Lyapunov Function*)**.** Consider the nonlinear autonomous system (1) and suppose that the origin of the state-space is an equilibrium point of (1). A (weak) *Dynamic Lyapunov function* $\mathcal{V}$ is a pair (D_τ, V) defined as follows

- D_τ is the ordinary differential equation $\dot{\xi} = \tau(x, \xi)$, with $\xi(t) \in \mathbb{R}^n$ and $\tau : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ locally Lipschitz, $\tau(0, 0) = 0$;
- $V : \Omega \subseteq \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is positive definite around $(x, \xi) = (0, 0)$ and it is such that $\dot{V}(x, \xi) = V_x f(x) + V_\xi \tau(x, \xi) < 0$, for all $(x, \xi) \in \Omega \setminus \{0\}$. ($\dot{V}(x, \xi) \le 0$ for a weak Dynamic Lyapunov function).

Theorem 1. *Consider the nonlinear autonomous system (1) and suppose that the origin of the state-space is an equilibrium point of (1). Suppose that there exists a Dynamic Lyapunov function for system (1). Then $x = 0$ is an asymptotically stable equilibrium point of the system (1).* □

Proof. Suppose that $\mathcal{V}$ is a Dynamic Lyapunov function for (1). Then the positive definite function V is a Lyapunov function for the *augmented* system

$$\dot{x} = f(x), \qquad \dot{\xi} = \tau(x, \xi), \tag{2}$$

implying, by Lyapunov's theorem, asymptotic stability of the equilibrium point $(x, \xi) = (0, 0)$ of the system (2). By Lemma 4.5 of Khalil (2001), the latter is equivalent to the existence of a class $\mathcal{KL}$ function β such that $\|(x(t), \xi(t))\| \le \beta(\|(x(0), \xi(0))\|, t)$ for all $t \ge 0$ and for any $(x(0), \xi(0)) \in \Omega$. Therefore $\|x(t)\| \le \beta(\|(x(0), 0)\|, t) \triangleq \tilde{\beta}(\|x(0)\|, t)$ proving asymptotic stability of the origin of the system (1), since $x(t)$ does not depend on $\xi(0)$. □

Theorem 1 states that Dynamic Lyapunov functions represent a mathematical tool to investigate Lyapunov stability properties of equilibrium points, alternative to standard Lyapunov functions.

Remark 1. In the following sections it is explained how to select τ to enforce negativity of the time derivative of the function V as in Definition 1. The key aspect consists in introducing a class of positive definite functions and then designing τ such that each element of this class becomes a Lyapunov function for the augmented system (2). Therefore, the importance of the following theorem lies in the fact that its proof is not carried out by augmenting system (1) with an asymptotically stable autonomous system and then resorting to arguments similar to those in the proofs of standard converse Lyapunov theorems. On the contrary, the proof provides a systematic methodology to construct Dynamic Lyapunov functions without assuming knowledge of the solution of the underlying differential equation and without involving any partial differential equation. In practical situations this aspect represents an advantage of Dynamic Lyapunov functions over Lyapunov functions. □

The following theorem establishes a *converse* result that guarantees the existence of a Dynamic Lyapunov function in a neighborhood of an exponentially stable equilibrium point.

Theorem 2. *Consider the nonlinear autonomous system (1) and suppose that the origin of the state-space is a locally exponentially stable equilibrium point of (1). Then there exists a Dynamic Lyapunov function for system (1).* □

The proof of Theorem 2 is constructive and it is given in Sections 3 and 4 for linear and nonlinear systems, respectively. It is worth underlining that, even if the proofs of the Lyapunov converse theorems are usually performed by constructing Lyapunov functions in a neighborhood of the equilibrium point, in most cases this construction requires the knowledge of the explicit solutions of the differential equation, which is a serious drawback from the practical point of view. Likewise, the statement of Theorem 2 guarantees the existence of a Dynamic Lyapunov function in a neighborhood of an exponentially stable equilibrium point, while the proof provides explicitly a Dynamic Lyapunov function $\mathcal{V} = (D_\tau, V)$.

We conclude this section showing that the knowledge of Dynamic Lyapunov functions can be exploited to *construct* standard Lyapunov functions for the system (1).

Theorem 3. *Consider the nonlinear autonomous system (1). Suppose that $\mathcal{V} = (D_\tau, V)$ is a Dynamic Lyapunov function for (1) and that there exists a $\mathcal{C}^1$ mapping $h : \mathbb{R}^n \to \mathbb{R}^n$, $h(0) = 0$, such that*

$$h_x(x) f(x) = \tau(x, h(x)). \tag{3}$$

Then $V_{\mathcal{M}}(x) \triangleq V(x, h(x))$ is a Lyapunov function for the system (1). □

Proof. The condition (3) implies that the manifold $\mathcal{M} = \{(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n : \xi = h(x)\}$ is invariant for the dynamics of the *augmented* system (2). The restriction of the system (2) to the invariant manifold is a copy of the dynamics of the system (1). Note that, by definition of Dynamic Lyapunov function, $V(x, \xi) > 0$ and $\dot{V}(x, \xi) < 0$ for all $(x, \xi) \in \Omega \subset \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\}$. Moreover

$$\begin{aligned}
\dot{V}_{\mathcal{M}} &= V_x(x, \lambda)|_{\lambda = h(x)} f(x) + V_\lambda(x, \lambda)|_{\lambda = h(x)} h_x(x) f(x) \\
&= V_x(x, \lambda)|_{\lambda = h(x)} f(x) + V_\lambda(x, \lambda)|_{\lambda = h(x)} \tau(x, h(x)) \\
&= \dot{V}(x, h(x)) < 0,
\end{aligned}$$

where the second equality is obtained considering the (3). The function $V_{\mathcal{M}}$ depends only on x, is positive definite around $x = 0$ and its time derivative is negative definite, which proves the claim. □

It is interesting to note that for nonlinear systems the condition to achieve the invariance of the desired submanifold is given in terms of a system of first-order partial differential equations similar to the Lyapunov pde, namely $V_x f(x) = -\nu(V)$ where ν is a class $\mathcal{K}$ function. However, as explained in Section 5, differently from the Lyapunov pde the solution of which must be positive definite around the origin, no constraint is imposed on the solution h or on the mapping τ, which is an advantage of the latter approach with respect to the former.

Finally, the closed-form solution h of (3) can be replaced by an approximation as clarified in the following statement.

Theorem 4. *Consider the nonlinear autonomous system (1). Suppose that $\mathcal{V} = (D_\tau, V)$ is a Dynamic Lyapunov function for (1) and that there exists a $\mathcal{C}^1$ mapping $\hat{h} : \mathbb{R}^n \to \mathbb{R}^n$ such that*[2]

$$\|\hat{h}_x(x)f(x) - \tau(x, \hat{h}(x))\| < \frac{\|\dot{V}(x, \hat{h}(x))\|}{\|\kappa(x, \hat{h}(x))\|}, \tag{4}$$

for all $x \in \Omega \setminus \{0\}$, where $\kappa(x, \hat{h}(x)) = V_\lambda(x, \lambda)|_{\lambda=\hat{h}(x)}$. Then $\hat{V}_{\mathcal{M}}(x) \triangleq V(x, \hat{h}(x))$ is a Lyapunov function for the system (1). □

Proof. To begin with, note that $\hat{V}_{\mathcal{M}}$ is positive definite around the origin. The time derivative of $\hat{V}_{\mathcal{M}}$ along the trajectories of the system (1) is

$$\begin{aligned}\dot{\hat{V}}_{\mathcal{M}} &= V_x(x, \lambda)|_{\lambda=\hat{h}(x)} f(x) + V_\lambda(x, \lambda)|_{\lambda=\hat{h}(x)} \hat{h}_x(x) f(x) \\ &= V_x(x, \lambda)|_{\lambda=\hat{h}(x)} f(x) + V_\lambda(x, \lambda)|_{\lambda=\hat{h}(x)} \tau(x, \hat{h}(x)) \\ &\quad + V_\lambda(x, \lambda)|_{\lambda=\hat{h}(x)} \left[\hat{h}_x(x) f(x) - \tau(x, \hat{h}(x))\right] \\ &\le \dot{V}(x, \hat{h}(x)) + \|\kappa(x, \hat{h}(x))\| \, \|\hat{h}_x(x) f(x) - \tau(x, \hat{h}(x))\| \\ &< 0,\end{aligned}$$

for all $x \in \Omega \setminus \{0\}$, where the last strict inequality is derived considering the condition (4). □

Remark 2. Every mapping h that solves the partial differential equation (3) is also a solution of (4) since in this case the left-hand side of (4) is equal to zero for all $x \in \mathbb{R}^n$. □

3. Construction of Dynamic Lyapunov functions for linear systems

Consider a linear, time-invariant, autonomous system described by equations of the form

$$\dot{x} = Ax, \tag{5}$$

with $x(t) \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n\times n}$. Consider the system (5) and suppose that there exists a mapping $x^\top P$, with $P = P^\top > 0$ such that

$$\frac{1}{2}x^\top PAx + \frac{1}{2}x^\top A^\top Px = -x^\top Qx, \tag{6}$$

for some given $Q = Q^\top > 0$ and for all $x \in \mathbb{R}^n$. Note that the mapping $x^\top P$ is an exact differential. However to present the main ideas of the proposed approach and to prove Theorem 2 for linear systems suppose that, instead of integrating the mapping $x^\top P$ obtaining the quadratic function

$$V(x) = \frac{1}{2}x^\top Px = \int_0^1 (\zeta(\sigma)^\top P) d\sigma, \tag{7}$$

for any state trajectory such that $\zeta(0) = 0$ and $\zeta(1) = x$, we exploit the mapping $P(x) = x^\top P$ to construct an *auxiliary* function defined in an *extended* space, namely

$$V(x, \xi) = \xi^\top Px + \frac{1}{2}\|x - \xi\|_R^2, \tag{8}$$

with $\xi \in \mathbb{R}^n$ and $R = R^\top > 0$ to be determined. A Schur complement argument shows that the function V is globally positive definite provided $R > \frac{1}{2}P$. Define now the *augmented* linear system in triangular form described by the equations

$$\dot{x} = Ax, \qquad \dot{\xi} = F\xi + Gx, \tag{9}$$

with F and G to be determined, and consider the problem of studying the stability properties of the origin of the system (9) using the function V, defined in (8), as a *candidate* Lyapunov function.

Lemma 1. *Consider the linear, time-invariant, system (5) and suppose that the origin is an asymptotically stable equilibrium point. Let $P = P^\top > 0$ be the solution of (6) for some positive definite matrix Q. Let the matrices F and G be defined as*

$$F = -kR, \qquad G = k(R - P). \tag{10}$$

Suppose that[3]

$$\underline{\sigma}(R) > \frac{1}{2}\bar{\sigma}(P)\left[\frac{\bar{\sigma}(PA)}{\underline{\sigma}(Q)}\right]. \tag{11}$$

Then, V in (8) is positive definite and there exists $\bar{k} \ge 0$ such that for all $k > \bar{k}$ the time derivative of V along the trajectories of the system (9) is negative definite. □

Proof. To prove that V is globally positive definite it is sufficient to show that the condition (11) implies $R > \frac{1}{2}P$. The latter follows immediately noting that, by (6) and the inequality $\bar{\sigma}(B_1 + B_2) \le \bar{\sigma}(B_1) + \bar{\sigma}(B_2)$, the condition $\bar{\sigma}(PA) \ge \underline{\sigma}(Q)$ holds. Note that the partial derivatives of the function V in (8) are given by

$$\begin{aligned}V_x &= x^\top P + (x - \xi)^\top (R - P), \\ V_\xi &= x^\top P - (x - \xi)^\top R.\end{aligned} \tag{12}$$

Therefore, the time derivative of the function V along the trajectories of the augmented system (9) is $\dot{V} = V_x Ax + V_\xi(F\xi + Gx)$. Setting the matrices F and G as in (10) yields $\dot{\xi} = -kV_\xi^\top$. Consequently,

$$\begin{aligned}\dot{V}(x, \xi) &= x^\top PAx + x^\top A^\top (R - P)(x - \xi) \\ &\quad - k(x^\top P - (x - \xi)^\top R)(Px - R(x - \xi)) \\ &= -x^\top Qx + x^\top A^\top (R - P)(x - \xi) \\ &\quad - k[x^\top \ (x - \xi)^\top] C^\top C [x^\top \ (x - \xi)^\top]^\top,\end{aligned} \tag{13}$$

with $C = [P \ \ -R]$, where the second equality is obtained using the condition (6). Note that the time derivative (13) can be rewritten as a quadratic form in x and $(x - \xi)$, i.e.

$$\dot{V}(x, \xi) = -[x^\top \ (x - \xi)^\top][M + kC^\top C][x^\top \ (x - \xi)^\top]^\top,$$

where the matrix M is defined as

$$M = \begin{bmatrix} Q & -\frac{1}{2}A^\top (R - P) \\ -\frac{1}{2}(R - P)A & 0_n \end{bmatrix}.$$

[2] A similar condition is considered in Mazenc and Praly (1996).

[3] $\bar{\sigma}(B)$ and $\underline{\sigma}(B)$ denote the maximum and the minimum singular value of the matrix B, respectively.

The kernel of C is spanned by the columns of the matrix $Z = [I\ PR^{-1}]^\top$. As a result, the condition of positive definiteness of the matrix M restricted to Z reduces to the condition (see Anstreicher & Wright, 2000 for more details)

$$\frac{1}{2}PR^{-1}(R-P)A + \frac{1}{2}A^\top(R-P)R^{-1}P < Q. \tag{14}$$

The left-hand side of the inequality (14) can be rewritten as

$$\begin{aligned}&\frac{1}{2}PR^{-1}(R-P)A + \frac{1}{2}A^\top(R-P)R^{-1}P\\ &\quad = -Q - \frac{1}{2}PR^{-1}PA - \frac{1}{2}A^\top PR^{-1}P.\end{aligned}$$

Therefore, the condition (14) is equivalent to

$$-\frac{1}{2}PR^{-1}PA - \frac{1}{2}A^\top PR^{-1}P < 2Q. \tag{15}$$

Moreover, recalling that $\bar{\sigma}(B^{-1}) = 1/\underline{\sigma}(B)$ yields

$$\begin{aligned}&-\frac{1}{2}PR^{-1}PA - \frac{1}{2}A^\top PR^{-1}P\\ &\quad \le \frac{1}{2}\|P\|\,\|R^{-1}\|\,\|PA\| + \frac{1}{2}\|P\|\,\|R^{-1}\|\,\|A^\top P\|\\ &\quad = \frac{1}{2}\frac{\bar{\sigma}(P)}{\underline{\sigma}(R)}(\|PA\| + \|A^\top P\|),\end{aligned}$$

where $\|B\|$ denotes the induced 2-norm of the matrix B. Hence, by condition (11), the inequality (15) holds. Therefore, by Anstreicher and Wright (2000), there exists a value $\bar{k} \ge 0$ such that for all $k > \bar{k}$ the time derivative of V in (8) is negative definite along the trajectories of the augmented system (9) for all $(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n$. □

Remark 3. The condition (11) can be satisfied selecting the matrix *R sufficiently large*. □

As a consequence of Lemma 1 consider the following statement, which proves Theorem 2 for linear time-invariant systems, not just establishing existence of a Dynamic Lyapunov function but also constructing a class of Dynamic Lyapunov functions.

Proposition 1. *Consider the system (5) and suppose that the origin is an asymptotically stable equilibrium point. Let P and Q be such that (6) holds. Let $R = R^\top > 0$ be such that (11) holds. Then there exists $\bar{k} \ge 0$ such that $\mathcal{V} = (D_\tau, V)$, where D_τ is the differential equation*

$$\dot{\xi} = -kR\xi + k(R-P)x \tag{16}$$

and V is defined in (8), is a Dynamic Lyapunov function for the system (5) for all $k > \bar{k}$. □

Proof. The claim follows immediately from Lemma 1, since V in (8) is positive definite around the origin, provided R satisfies condition (11), and moreover $V_x Ax + V_\xi \tau(x,\xi)$ is negative definite with the choice of τ given in (16) and k sufficiently large. □

4. Construction of Dynamic Lyapunov functions for nonlinear systems

Consider the nonlinear autonomous system (1) and suppose that the origin of the state-space is an equilibrium point, i.e. $f(0) = 0$. Hence, there exists a continuous matrix-valued function $F : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ such that $f(x) = F(x)x$ for all $x \in \mathbb{R}^n$.

Assumption 1. The equilibrium point $x = 0$ of the system (1) is locally exponentially stable, i.e. there exists a matrix $\bar{P} = \bar{P}^\top > 0$ such that

$$\frac{1}{2}\bar{P}A + \frac{1}{2}A^\top\bar{P} = -Q, \tag{17}$$

where $Q = Q^\top > 0$ and $A = \frac{\partial f}{\partial x}|_{x=0} = F(0)$.

Clearly, by Eq. (17), the quadratic function

$$V_l(x) = \frac{1}{2}x^\top\bar{P}x, \tag{18}$$

is a local (around the origin) Lyapunov function for the nonlinear system (1). In this section – mimicking the results and the construction in Section 3 for linear systems – we present a constructive methodology to obtain a Dynamic Lyapunov function for the system (1) thus providing a constructive proof of Theorem 2. Interestingly, the constructive methodology proposed in the proof does not suffer the common drawbacks intrinsic in the construction of a Lyapunov function. Specifically, the approach does not require knowledge of the solution of the differential equation (1) and does not involve any partial differential equation.

Consider the Lyapunov partial differential inequality

$$V_x f(x) < 0, \tag{19}$$

for all $x \in \mathbb{R}^n \setminus \{0\}$ and the following notion of solution.

Definition 2. Let $\Gamma(x) = \Gamma(x)^\top > 0$ for all $x \in \mathbb{R}^n$. A $\mathcal{X}$-*algebraic $\bar{P}$ solution* of the inequality (19) is a continuously differentiable mapping $P : \mathbb{R}^n \to \mathbb{R}^{1\times n}$, $P(0) = 0$, such that

$$P(x)f(x) \le -x^\top\Gamma(x)x, \tag{20}$$

for all $x \in \mathcal{X} \subset \mathbb{R}^n$, with $\mathcal{X}$ containing the origin, and such that P is *tangent* at the origin to $\bar{P}$, namely $P_x(x)|_{x=0} = \bar{P}$. If $\mathcal{X} = \mathbb{R}^n$ then P is called an *algebraic $\bar{P}$ solution*.

In what follows we assume the existence of an *algebraic $\bar{P}$ solution*, i.e. we assume $\mathcal{X} = \mathbb{R}^n$. All the statements can be modified accordingly if $\mathcal{X} \subset \mathbb{R}^n$. Note that (20) implies that $\Gamma(0) \le Q$. The mapping P does not need to be the *gradient vector* of any scalar function. Hence the condition (20) may be interpreted as the algebraic equivalent of (19), since in the former the *integrability* and (partly) the *positivity* constraints are relaxed. Using the mapping P define, similarly to (8), the function

$$V(x,\xi) = P(\xi)x + \frac{1}{2}\|x-\xi\|_R^2, \tag{21}$$

with $\xi \in \mathbb{R}^n$ and $R = R^\top \in \mathbb{R}^{n\times n}$ positive definite.

Remark 4. Consider V in (21) and note that there exist a non-empty compact set $\Omega_1 \subseteq \mathbb{R}^n \times \mathbb{R}^n$ containing the origin of $\mathbb{R}^n \times \mathbb{R}^n$ and a positive definite matrix $\bar{R}$ such that for all $R > \bar{R}$ the function V in (21) is positive definite for all $(x,\xi) \in \Omega_1 \subseteq \mathbb{R}^n \times \mathbb{R}^n$. In fact, since P is tangent at $x = 0$ to the solution of the Lyapunov (17), the function $P(x)x : \mathbb{R}^n \to \mathbb{R}$ is, locally around the origin, quadratic and moreover has a local minimum for $x = 0$. Hence the function $P(\xi)x$ is (locally) quadratic in (x,ξ) and, restricted to the manifold $\mathcal{E} = \{(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n : \xi = x\}$, is positive definite in Ω_1. □

The partial derivatives of the function V defined in (21) are given by

$$\begin{aligned}V_x &= P(x) + (x-\xi)^\top(R - \Phi(x,\xi))^\top,\\ V_\xi &= x^\top P_\xi(\xi) - (x-\xi)^\top R,\end{aligned} \tag{22}$$

where $\Phi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n\times n}$ is a continuous matrix-valued function such that $P(x) - P(\xi) = (x-\xi)^\top\Phi(x,\xi)^\top$. As in the linear setting, define an *augmented* nonlinear system described by equations of the form

$$\dot{x} = f(x),\ \dot{\xi} = -k(P_\xi(\xi) - R)^\top x - kR\xi \triangleq g(\xi)x - kR\xi, \qquad (23)$$

and let the function (21) be a *candidate* Lyapunov function to investigate the stability properties of the equilibrium point $(x, \xi) = (0, 0)$ of the system (23). To streamline the presentation of the following result – providing conditions on the choice of the parameter k such that V in (21) is indeed a Lyapunov function for the augmented system (23) – define the continuous matrix-valued function $\Delta(x, \xi) = (R - \Phi(x, \xi))R^{-1}P_\xi(\xi)^\top$.

Lemma 2. *Consider the system (1). Suppose Assumption 1 holds. There exist a set $\Omega \subset \mathbb{R}^n \times \mathbb{R}^n$ and a constant $\bar{k} \geq 0$ such that V, defined in (21), is positive definite in Ω and its time derivative along the trajectories of the system (23) is negative definite for all $k > \bar{k}$ if and only if*

$$\frac{1}{2}F(x)^\top \Delta(x, \xi) + \frac{1}{2}\Delta(x, \xi)^\top F(x) < \Gamma(x), \qquad (24)$$

for all $(x, \xi) \in \Omega \setminus \{0\}$. □

Proof. The time derivative of the function V defined in (21) is

$$\begin{aligned}\dot{V} &= P(x)f(x) + x^\top F(x)^\top (R - \Phi(x, \xi))(x - \xi) \\ &\quad - k[x^\top \ (x - \xi)^\top]C(\xi)^\top C(\xi)[x^\top \ (x - \xi)^\top]^\top,\end{aligned}$$

with $C(\xi) = [P_\xi(\xi)^\top \ -R]$. Note that the matrix $C : \mathbb{R}^n \to \mathbb{R}^{n\times 2n}$ has constant rank n for all $\xi \in \mathbb{R}^n$, since R is non-singular. The columns of the matrix $Z(\xi) \triangleq [I \ \ P_\xi(\xi)R^{-1}]^\top$, which has constant rank, span the kernel of the matrix $C(\xi)$ for all $\xi \in \mathbb{R}^n$. Consider now the restriction of the matrix

$$M(x, \xi) \triangleq \begin{bmatrix} \Gamma & -\frac{1}{2}F^\top (R - \Phi) \\ -\frac{1}{2}(R - \Phi)^\top F & 0 \end{bmatrix}$$

to the set $\mathcal{P} = \{(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n : P_\xi(\xi)^\top x - R(x - \xi) = 0\}$, namely $Z(\xi)^\top M(x, \xi)Z(\xi)$. Condition (24) implies that the matrix $Z(\xi)^\top M(x, \xi)Z(\xi)$ is positive definite for all $(x, \xi) \in \Omega$. Therefore, by Anstreicher and Wright (2000), condition (24) guarantees the existence of a constant $\bar{k} \geq 0$ and of a non-empty subset $\Omega \subset \mathbb{R}^n$ such that, for all $k > \bar{k}$, $\dot{V}(x, \xi) < 0$ for all $(x, \xi) \in \Omega \subset \mathbb{R}^n \times \mathbb{R}^n$ and $(x, \xi) \neq (0, 0)$. □

Remark 5. If the *algebraic $\bar{P}$ solution* of the inequality (19) is linear in x, i.e. $P(x) = x^\top \bar{P}$, then $\Phi(x, \xi) = \bar{P}$. Moreover the choice $R = \bar{P}$ is such that in Eq. (23) $g(\xi) \equiv 0$ and the condition (24) is satisfied for all $(x, \xi) \in \mathbb{R}^{2n} \setminus \{0\}$. □

Remark 6. The *gain* k in (23) may be defined as a function of x and ξ. □

The following result provides a constructive proof of the Theorem 2 for nonlinear systems.

Proposition 2. *Consider the nonlinear system (1) and suppose that the origin is a locally exponentially stable equilibrium point of (1). Let P be an* algebraic $\bar{P}$ solution *of the inequality (19). Let $R = \Phi(0, 0) = \bar{P}$. Then there exist a constant $\bar{k} \geq 0$ and a non-empty set $\Omega \subseteq \mathbb{R}^n \times \mathbb{R}^n$ such that $\mathcal{V} = (D_\tau, V)$, where D_τ is the differential equation $\dot{\xi} = g(\xi)x - kR\xi$ and V is defined in (21), is a Dynamic Lyapunov function for the system (1) in Ω, for all $k > \bar{k}$.* □

Proof. Since $\Phi(0, 0) = \bar{P}$ and recalling Remark 4, there exists a set $\Omega_1 \subseteq \mathbb{R}^n \times \mathbb{R}^n$ containing the origin in which the function V defined in (21) is positive definite. Therefore, to prove the claim it is sufficient to show that the condition (24) of Lemma 2 is, at least locally, satisfied. Note that the choice $R = \Phi(0, 0)$ implies that the left-hand side of the inequality (24) is zero at the origin, whereas the right-hand side, i.e. $\Gamma(0)$, is positive definite. Hence, by continuity, there exists a non-empty set $\Omega_2 \subseteq \mathbb{R}^n \times \mathbb{R}^n$ containing the origin in which the condition (24) holds, which proves the claim with $\Omega = \Omega_1 \cap \Omega_2$. □

5. From Dynamic Lyapunov functions to Lyapunov functions

5.1. Linear systems

The result in Lemma 1 can be exploited to *construct* a Lyapunov function for the linear system (5), as detailed in the following result, which is an application of Theorem 3 to linear time-invariant systems.

Corollary 1. *Consider the linear time-invariant system (5). Suppose that the conditions (6) and (11) are satisfied, fix $k > \bar{k}$ and let $Y \in \mathbb{R}^{n\times n}$ be the solution of the equation*

$$k(R - P) - kRY = YA. \qquad (25)$$

Then the subspace $\mathcal{L} = \{(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n : \xi = Yx\}$ is invariant and the restriction of the function V in (8) to $\mathcal{L}$, defined as

$$V_{\mathcal{L}} = V(x, Yx) = \frac{1}{2}x^\top [Y^\top P + PY + (I - Y)^\top R(I - Y)]x, \qquad (26)$$

depends only on the variable x, it is positive definite and its time derivative along the trajectories of the system (5) is negative definite, hence $V_{\mathcal{L}}$ is a Lyapunov function for the system (5). □

Proof. The condition $\sigma(A) \cap \sigma(-kR) = \emptyset$ guarantees existence and unicity of the matrix Y and therefore the existence of the invariant subspace $\mathcal{L}$. The claim is proved showing that the assumptions of Theorem 3 are satisfied. To begin with, by Proposition 1, $\mathcal{V} = (D_\tau, V)$, where D_τ is the differential equation (16) and V is defined in (8), is a Dynamic Lyapunov function for the system (5) for all $k > \bar{k}$. Moreover, by (25), the mapping $h(x) = Yx$ is a solution of the partial differential equation (3), which reduces in the linear case to the equation

$$\begin{bmatrix} A & 0_n \\ k(R - P) & -kR \end{bmatrix}\begin{bmatrix} I_n \\ Y \end{bmatrix} = \begin{bmatrix} I_n \\ Y \end{bmatrix}A. \quad \square$$

Remark 7. The condition (25) implies the existence of the linear subspace $\mathcal{L}$ parameterized in x, which is invariant with respect to the dynamics of the augmented system (9) and such that the flow of the system (9) restricted to $\mathcal{L}$ is a copy of the flow of the system (5). Moreover, $V_{\mathcal{L}}$ describes a family of Lyapunov functions for the system parameterized by the matrix $R > \frac{1}{2}P$ and $k > \bar{k}$. □

Corollary 2. *Suppose that Y is a common solution of the Sylvester equation (25) and of the algebraic Riccati equation*

$$Y^\top (P - R) + (P - R)Y + Y^\top RY - (P - R) = 0. \qquad (27)$$

Then $V_{\mathcal{L}}$ coincides with the original *quadratic Lyapunov function (7), i.e. $V(x, Yx) = \frac{1}{2}x^\top Px$.* □

Remark 8. The Lyapunov function V defined as in (7) does not necessarily belong to the family parameterized by $V_{\mathcal{L}}$, hence the need for condition (27). Recall in fact that the matrix P is defined together with the matrix Q, i.e. the pair (P, Q) is such that V in (7) is a quadratic positive definite function and $\dot{V} = -x^\top Qx$ along the trajectories of the linear system (5). Therefore the function V in (7) belongs to the family of Lyapunov functions $V_{\mathcal{L}}$ if and only if there exists $k > \bar{k}$ and R such that (11) holds and such that $\dot{V}_{\mathcal{L}} = -x^\top Qx$. □

5.2. Nonlinear systems

As in the linear case a Dynamic Lyapunov function can be employed to construct a family of Lyapunov functions for the system (1). The following proposition represents a particularization of Theorem 3 to the dynamics of the *augmented* system defined as in (23).

Corollary 3. *Consider the system (1). Suppose Assumption 1 holds. Let $k > \bar{k}$. Suppose that the condition (24) is satisfied and that there exists a smooth mapping $h : \mathbb{R}^n \to \mathbb{R}^n$ such that the manifold $\mathcal{M} = \{(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n : \xi = h(x)\}$ is invariant with respect to the dynamics of the* augmented *system (23), i.e.*

$$g(h(x))x - kRh(x) = \frac{\partial h}{\partial x} f(x). \tag{28}$$

Then the restriction of the function V in (21) to the manifold $\mathcal{M}$, namely

$$V_{\mathcal{M}}(x) = P(h(x))x + \frac{1}{2}\|x - h(x)\|_R^2, \tag{29}$$

yields a family of Lyapunov functions for the nonlinear system (1). □

Remark 9. The family of Lyapunov functions (29) is parameterized by R and $k > \bar{k}$. □

Note that, by (28), $\mathcal{M}$ is invariant under the flow of the system (23) and moreover the restriction of the flow of the *augmented* system (23) to the manifold $\mathcal{M}$ is a *copy* of the flow of the nonlinear system (1). The condition (28) is a system of partial differential equations similar to (19), without any sign constraint on the solution, i.e. the mapping h, or on the sign of term on the right-hand side of the equation.

Remark 10. The arguments in Remark 4 imply the existence of a neighborhood $\mathcal{N}$ of the origin such that, for any $x \in \mathcal{N}$ there exists a continuously differentiable function $\varpi : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$P_\xi(\varpi(x))x + R(\varpi(x) - x) = 0. \tag{30}$$

Then, $V(x, \varpi(x))$ is a Lyapunov function for the system (1). The explicit computation of the mapping ϖ may be difficult in general. Note, however, that the solution of (30) may be approximated by the system $\dot{\xi} = -k\left(P_\xi(\xi)x + R(\xi - x)\right)$ – hence recovering precisely the dynamics of the extension ξ in (23) – which allows to obtain the same result, with k sufficiently large, by invoking *singular perturbation* theory. □

5.3. Approximate solution of the invariance pde

An explicit solution of the partial differential equation (28) may still be difficult to determine even without the sign constraint. Therefore, consider the following *algebraic* condition which allows to approximate, with an arbitrary degree of accuracy, the closed-form solution of the partial differential equation (28). Suppose that there exists a mapping $H_{k,R} : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ such that

$$H_{k,R}(x)f(x) + kRH_{k,R}(x)x - g(H_{k,R}(x)x)x = 0. \tag{31}$$

Note that the solution of the condition (31), which is merely an algebraic equation, is parameterized by k and R. Let now $\hat{h}(x) = H_{k,R}(x)x$ and consider the submanifold $\mathcal{M}_\eta \triangleq \{(x, \xi) \in \mathbb{R}^{2n} : \xi = \hat{h}(x)\}$.

Lemma 3. *Let $\mathcal{W} \subset \mathbb{R}^n \times \mathbb{R}^n$ be a compact set containing the origin. Suppose that the condition (31) is satisfied and that*

(i) *there exists a function $\phi_R : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$\|H_{k,R}(x)\| < \phi_R(\|x\|), \tag{32}$$

for all $k > \bar{k}$, with $\bar{k}$ defined in Proposition 2;

(ii) *there exists $R = R^\top > \bar{R}$ such that $\underline{\sigma}(R) > \|G(x, \xi)\|/(1-\mu)$, for some $\mu \in (0, 1)$, for all $(x, \xi) \in \mathcal{W}$, where $G : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n\times n}$ is such that $P_\xi(H_{k,R}(x)x)^\top x - P_\xi(\xi)^\top x = G(x, \xi)(\xi - H_{k,R}(x)x)$.*

Then there exists $\tilde{k} > \bar{k}$ such that the submanifold $\mathcal{M}_\eta \cap \mathcal{W}$ is almost-invariant[4] *for the system (23) for all $k \geq \tilde{k}$.* □

Proof. Define the error variable $\eta = \xi - H_{k,R}(x)x$. The dynamics of η are given by

$$\dot{\eta} = -kR\xi + g(\xi)x - H_{k,R}(x)f(x) - \theta(x)f(x), \tag{33}$$

where $\theta(x) = \frac{\partial(H_{k,R}(x)\lambda)}{\partial x}|_{\lambda=x}$. Letting $\xi = \eta + H_{k,R}(x)x$, adding to the (33) the term in the left-hand side of (31) which is equal to zero and recalling the definition of the mapping g in (23), the Eq. (33) yields

$$\begin{aligned}\dot{\eta} &= -kR\eta - \theta(x)f(x) + k[P_\xi(H_{k,R}(x)x) \\ &\quad - P_\xi(\eta + H_{k,R}(x)x)]^\top x \\ &\triangleq -kR\eta - \theta(x)f(x) + kG(x, \eta + H_{k,R}(x))\eta.\end{aligned} \tag{34}$$

Therefore the system (34) can be rewritten as

$$\dot{\eta} = -k(R - G(x, \eta + H_{k,R}(x)))\eta - \theta(x)f(x). \tag{35}$$

Consider now the Lyapunov function $V = \eta^\top R^{-1}\eta$ the time derivative of which along the trajectories of system (35) is

$$\begin{aligned}\dot{V} &= -2k\eta^\top\eta + 2k\eta^\top R^{-1}G(x, \xi)\eta - 2\eta^\top R^{-1}\theta(x)f(x) \\ &\leq -2k\|\eta\|^2 + 2k\bar{\sigma}(R^{-1})\|G(x, \xi)\|\,\|\eta\|^2 \\ &\quad + 2\bar{\sigma}(R^{-1})\|\theta(x)f(x)\|\,\|\eta\| \\ &= -2\|\eta\|\left[k\left(1 - \frac{\|G(x, \xi)\|}{\underline{\sigma}(R)}\right)\|\eta\| - \frac{\|\theta(x)f(x)\|}{\underline{\sigma}(R)}\right]\end{aligned}$$

which is negative definite in the set $\{(x, \xi) \in \mathcal{W} : \|\eta\| > \|\theta(x)f(x)\|/(k\mu\underline{\sigma}(R))\}$. Let $\tilde{k} = \|\theta(x)f(x)\|/\bar{\varepsilon}\mu\underline{\sigma}(R)$, then the Lyapunov function V is such that $\dot{V} < 0$ when $\|\eta\| > \bar{\varepsilon}$ for all $k \geq \tilde{k}$. For any $\varepsilon > 0$ let $\bar{\varepsilon} = \sqrt{\bar{\sigma}(R)/\underline{\sigma}(R)}\varepsilon$, then the set $\{\eta \in \mathbb{R}^n : \|\eta\| < \varepsilon\}$ is attractive and positively invariant hence the submanifold $\mathcal{M}_\eta \cap \mathcal{W}$ is almost-invariant. □

Remark 11. If the *algebraic* $\bar{P}$ *solution* of the inequality (19) is linear in x, i.e. $P(x) = x^\top\bar{P}$, then the condition (ii) of Lemma 3 is satisfied for any constant $\mu \in (0, 1)$ since $G \equiv 0$. □

Note that the result of Lemma 3 implies the existence of a continuously differentiable mapping $\pi : \mathbb{R}^n \to \mathbb{R}^n$ such that $\xi(t) = \hat{h}(x(t)) + \pi(x(t))$ and $\|\pi(x(t))\| \leq \varepsilon$ for all $t \geq 0$.

[4] A submanifold $\mathcal{F}$ is said to be *almost-invariant* with respect to the system (1) if, for any given $\varepsilon > 0$, $\mathrm{dist}(x(0), \mathcal{F}) \leq \varepsilon$ implies $\mathrm{dist}(x(t), \mathcal{F}) \leq \varepsilon$ for all $t \geq 0$, where $\mathrm{dist}(x(t), \mathcal{F})$ denotes the distance of $x(t)$ from the submanifold $\mathcal{F}$.

Corollary 4. *Suppose that the conditions of Lemma 3 are satisfied. Suppose that*

$$\left\| \frac{\partial \pi}{\partial x} \right\| < \frac{\mu \|\dot{V}(x, \hat{h}(x))\|}{\|\kappa(x, \hat{h}(x)) f(x)\|} \tag{36}$$

with $\mu \in (0, 1)$ *and* $\kappa(x, \hat{h}(x)) = V_\lambda(x, \lambda)|_{\lambda=\hat{h}(x)}$. *Then there exist a matrix* $R > \bar{R}$ *and a constant* $k > \bar{k}$ *such that each element of the family of functions* $V_{\mathcal{M}_\eta}(x) = V(x, \hat{h}(x))$ *with*

$$V(x, \hat{h}(x)) = P(H_{k,R}(x)x)x + \frac{1}{2}\|x - H_{k,R}(x)x\|^2_R, \tag{37}$$

parameterized by R and k, is a Lyapunov function for the nonlinear system (1). □

Proof. Since the subset $\mathcal{M}_\eta \cap \mathcal{W}$ is almost-invariant, the time derivative of the function $V_{\mathcal{M}_\eta}$ as in (37) yields the Eqs. (38),

$$\begin{aligned}
\dot{V}_{\mathcal{M}_\eta} &= V_x(x, \lambda)|_{\lambda=\hat{h}(x)+\pi(x)} f(x) \\
&\quad + V_\lambda(x, \lambda)|_{\lambda=\hat{h}(x)+\pi(x)} \left(\hat{h}_x(x) f(x) + \pi_x(x) f(x)\right) \\
&= V_x(x, \lambda)|_{\lambda=\hat{h}(x)} f(x) \\
&\quad + \left(V_x(x, \lambda)|_{\lambda=\hat{h}(x)+\pi(x)} - V_x(x, \lambda)|_{\lambda=\hat{h}(x)}\right) f(x) \\
&\quad + V_\lambda(x, \lambda)|_{\lambda=\hat{h}(x)} \hat{h}_x(x) f(x) \\
&\quad + \left(V_\lambda(x, \lambda)|_{\lambda=\hat{h}(x)+\pi(x)} - V_\lambda(x, \lambda)|_{\lambda=\hat{h}(x)}\right) \hat{h}_x(x) f(x) \\
&\quad + V_\lambda(x, \lambda)|_{\lambda=\hat{h}(x)} \pi_x(x) f(x) \\
&\quad + \left(V_\lambda(x, \lambda)|_{\lambda=\hat{h}(x)+\pi(x)} - V_\lambda(x, \lambda)|_{\lambda=\hat{h}(x)}\right) \pi_x(x) f(x) \\
&\le \dot{V}(x, \hat{h}(x)) + \|\pi(x)\| \left(L_x + L_\lambda \|\hat{h}_x\| + L_\lambda \|\pi_x\|\right) \|f(x)\| \\
&\quad + \|\pi_x(x)\| \, \|\kappa(x, \hat{h}(x)) f(x)\| \\
&\le \mu \dot{V}(x, \hat{h}(x)) + \|\pi_x(x)\| \, \|\kappa(x, \hat{h}(x)) f(x)\| < 0
\end{aligned} \tag{38}$$

with $\mu \in (0, 1)$, $L_x > 0$ and $L_\lambda > 0$, where the last three inequalities are obtained considering that V_x and V_λ are continuous functions and $x \in \mathcal{W}$, recalling that $\|\pi(x)\| \le \varepsilon$, with $\varepsilon > 0$ arbitrarily small, and by the condition (36), respectively. □

It is interesting to note that *almost-invariance* of the subset $\mathcal{M}_\eta \cap \mathcal{W}$ is not enough to ensure that the restriction of the function $V(x, \xi)$ to the subset $\mathcal{M}_\eta \cap \mathcal{W}$ is a Lyapunov function for the system (1), hence the need for the condition (36). The latter condition entails the fact that the time derivative of the distance of the trajectory $(x(t), \xi(t))$ from the subset must be *sufficiently* small.

6. Numerical examples

Example 1. Consider the nonlinear system described by the equations

$$\dot{x}_1 = -x_1, \qquad \dot{x}_2 = x_1^2 - x_2, \tag{39}$$

with $x(t) = (x_1(t), x_2(t)) \in \mathbb{R}^2$. Note that the zero equilibrium of the system (39) is globally asymptotically stable and locally exponentially stable. A choice of a Lyapunov function for the linearization around the origin of the system (39) is provided by $V_l = \frac{1}{2}(x_1^2 + x_2^2)$, i.e. $\bar{P} = I$ which is a solution of the Eq. (17) with $Q = I$. The quadratic function V_l may be employed to estimate the region of attraction, $\mathcal{R}_0$, of the zero equilibrium of the nonlinear system (39). The estimate is given by the largest connected component, containing the origin of the state-space, of the level set of the considered Lyapunov function entirely contained in the set $\mathcal{N} \triangleq \{x \in \mathbb{R}^2 : \dot{V} < 0\}$. Note that $\mathcal{N} \subset \mathbb{R}^2$ and consequently $\mathcal{R}_0 \subset \mathbb{R}^2$. Moreover note that the use of any quadratic function $V_q = \frac{1}{2}x^\top \bar{P} x$, $\bar{P} = \bar{P}^\top$ with $p_{12} \neq 0$, does not allow to obtain $\mathcal{N} = \mathbb{R}^2$. In fact, the time derivative of V_q along the trajectories of the system (39) yields $\dot{V}_q = -p_{11}x_1^2 - 2p_{12}x_1x_2 - p_{22}x_2^2 + p_{12}x_1^3 + p_{22}x_2x_1^2$, which, if evaluated along $x_2 = 0$, is equal to $\dot{V}_q|_{x_2=0} = -x_1^2(p_{11} - p_{12}x_1)$. Therefore, $\dot{V}_q > 0$ for $x_2 = 0$ and $\mathrm{sign}(p_{12})x_1 > \frac{p_{11}}{|p_{12}|}$, hence $\mathcal{R}_0 \neq \mathbb{R}^2$. In what follows we show that the notion of Dynamic Lyapunov function allows to construct a Lyapunov function proving global asymptotic stability of the zero equilibrium of system (39). Note finally that a (global) Lyapunov function for system (39) can be constructed noting that the system has a cascaded structure and exploiting *forwarding* arguments. The mapping $P : \mathbb{R}^2 \to \mathbb{R}^2$ defined as the gradient vector of the quadratic function V_l is an *algebraic* $\bar{P}$ *solution* of the inequality (19) for the nonlinear system (39). To begin with note that the choice $R = \bar{P}$ guarantees that $g(\xi)$ is identically equal to zero and that the condition (24) is trivially satisfied for all $(x, \xi) \in \mathbb{R}^4 \setminus \{0\}$. In the following two different approaches to construct a Lyapunov function for the system (39) are proposed.

To construct the Lyapunov function V_d defined in Corollary 3 one needs to determine mappings $h_1 : \mathbb{R}^2 \to \mathbb{R}$ and $h_2 : \mathbb{R}^2 \to \mathbb{R}$ such that the manifold $\{(x_1, x_2, \xi_1, \xi_2) \in \mathbb{R}^4 : \xi_1 = h_1(x_1, x_2), \xi_2 = h_2(x_1, x_2)\}$ is invariant for the dynamics of the *augmented* system (23). Note that the system of partial differential equation (28) reduces to two identical (decoupled) partial differential equations given by

$$-\frac{\partial h_i}{\partial x_1}(x_1, x_2)x_1 + \frac{\partial h_i}{\partial x_2}(x_1, x_2)(x_1^2 - x_2) + kh_i(x_1, x_2) = 0, \tag{40}$$

for $i = 1, 2$. The solutions h_1 and h_2 are defined as $h_1(x) = h_2(x) = L\left(\frac{x_2 + x_1^2}{x_1}\right) x_1^k$, $k \ge 1$, where $L : \mathbb{R} \to \mathbb{R}$ is any differentiable function. Let, for example, $L(a) = a$ and construct the family of Lyapunov functions

$$V_d(x) = \frac{1}{2}(x_1^2 + x_2^2) + (x_2 + x_1^2)^2 \left(x_1^{k-1}\right)^2, \tag{41}$$

with $h(x) = [h_1(x)\, h_2(x)]^\top$, and $k \ge 1$. Letting $k = 1$ yields $V_d^1 = \frac{1}{2}(x_1^2 + x_2^2) + (x_2 + x_1^2)^2$ the time derivative of which along the trajectories of the system (39), namely $\dot{V}_d^1 = -x_1^2 - 3x_2^2 - 3x_2x_1^2 - 2x_1^4$, is negative definite for all $(x_1, x_2) \in \mathbb{R}^2$, hence $\mathcal{N} = \mathbb{R}^2$, which proves global asymptotic stability of the zero equilibrium. Finally, note that $\dot{V}_d^k$ is negative definite for all $(x_1, x_2) \in \mathbb{R}^2$ and for all $k \ge 1$. Interestingly, the pde (40) has a structure similar to the Eq. (19) but the solution obtained is not positive definite, hence it does not qualify as a Lyapunov function. Fig. 1 shows the phase portraits of the trajectories of the system (39) together with the level lines of the Lyapunov function V_d.

We now show how Corollary 4 can be used to construct a Lyapunov function. Let $H : \mathbb{R}^2 \to \mathbb{R}^{2\times 2}$ and consider the condition (31), which reduces to two identical conditions on $\bar{h}_{11}, \bar{h}_{12}$ and $\bar{h}_{21}, \bar{h}_{22}$, elements of H, namely

$$\begin{aligned}
&-x_1\bar{h}_{11} + \bar{h}_{12}(x_1^2 - x_2) + k\bar{h}_{11}x_1 + k\bar{h}_{12}x_2 = 0, \\
&-x_1\bar{h}_{21} + \bar{h}_{22}(x_1^2 - x_2) + k\bar{h}_{21}x_1 + k\bar{h}_{22}x_2 = 0.
\end{aligned} \tag{42}$$

A solution of (42) is given by $\bar{h}_{11}(x) = \bar{h}_{21}(x) = -x_2 - x_1^2(k-1)^{-1}$ and $\bar{h}_{12}(x) = \bar{h}_{22}(x) = x_1$ and, since the condition (32) holds, the manifold $\{(x, \xi) \in \mathbb{R}^4 : \xi_1 = \xi_2 = -x_1^3(k-1)^{-1}\}$ is *almost*

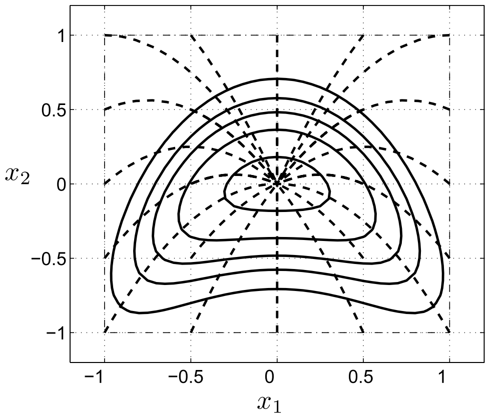

Fig. 1. Phase portraits (dashed) of the system (39) together with the level lines (solid) of the Lyapunov function V_d.

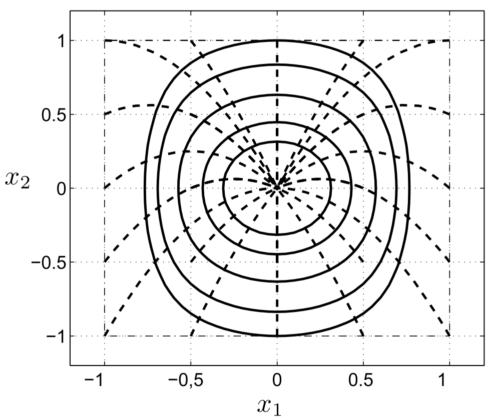

Fig. 2. Phase portraits (dashed) of the system (39) together with the level lines (solid) of the Lyapunov function V_{d_a}.

invariant for $k > 1$. Moreover note that the mapping $\delta(x) \triangleq [\bar{h}_{11}(x)\ \bar{h}_{12}(x)]$ is not the gradient of a scalar function, since the Jacobian $\delta_x(x)$ is not a symmetric matrix. Setting $k = 2$ and using Corollary 4 yield the Lyapunov function $V_{d_a} = \frac{1}{2}(x_1^2 + x_2^2) + x_1^6$ the time derivative of which along the trajectories of the system (39) is $\dot{V}_{d_a} = x_2 x_1^2 - x_1^2 - x_2^2 - 6x_1^6$ which is negative definite for all $(x_1, x_2) \in \mathbb{R}^2$. Fig. 2 displays the phase portraits of the trajectories of the system (39) together with the level lines of the Lyapunov function V_{d_a}.

Example 2. A Polynomial System without Polynomial Lyapunov function. Consider the nonlinear system described by equations (Ahmadi, Krstic, & Parrilo, 2011)

$$\dot{x}_1 = -x_1 + x_1 x_2, \qquad \dot{x}_2 = -x_2, \tag{43}$$

with $x_1(t) \in \mathbb{R}$ and $x_2(t) \in \mathbb{R}$ and note that the zero equilibrium of the system (43) is globally asymptotically stable. In Ahmadi et al. (2011) it has been shown that the system (43) does not admit a polynomial Lyapunov function. In the same paper the authors have shown that $V = \ln(1 + x_1^2) + x_2^2$ is a Lyapunov function for the system (43) such that $\dot{V} < 0$ for all $(x_1, x_2) \in \mathbb{R}^2$. The quadratic function $V_l = 1/2(x_1^2 + x_2^2)$, i.e. $V_l = 1/2x^\top \bar{P} x$ with $\bar{P} = I$, is a Lyapunov function for the linearized system. Note that the mapping $[x_1\, x_2]^\top \bar{P}$ is an *algebraic* $\bar{P}$ *solution*. Let $R = \bar{P}$, then $\mathcal{V} = (D_\tau, V)$ with $\dot{\xi}_i = -k\xi_i$, $i = 1, 2$, with $k > 0$, and $V(x_1, x_2, \xi_1, \xi_2) = \frac{1}{2}(x_1^2 + x_2^2 + \xi_1^2 + \xi_2^2)$ is a Dynamic Lyapunov function, the time derivative of which is negative definite in a neighborhood of the origin of the *extended* state space. The *invariance* partial differential equation (28) reduces to two identical equations for h_1 and h_2, namely

$$\frac{\partial h_i}{\partial x_1}(x_1, x_2)(x_1 x_2 - x_1) - \frac{\partial h_i}{\partial x_2} x_2 + k h_i(x_1, x_2) = 0, \tag{44}$$

for $i = 1, 2$. The functions $h_1(x_1, x_2) = h_2(x_1, x_2) \triangleq x_1 x_2^{k-1} e^{x_2}$, with $k \geq 1$ are solutions of the partial differential equation (44). Note that the function h is not positive definite, hence it is not a Lyapunov function. Letting, for instance, $k = 1$ the restriction of the Dynamic Lyapunov function to the invariant submanifold $\mathcal{M} \triangleq \{(x_1, x_2, \xi_1, \xi_2) \in \mathbb{R}^4 : \xi_1 = h_1(x_1, x_2), \xi_2 = h_2(x_1, x_2)\}$ is $V_{\mathcal{M}}(x_1, x_2) = \frac{1}{2}(x_1^2 + x_2^2) + x_1^2 e^{2x_2}$, which is positive definite for all $(x_1, x_2) \in \mathbb{R}^2$. Finally note that the time derivative of the function $V_{\mathcal{M}}$ along the trajectories of the system (43) is $\dot{V}_{\mathcal{M}} = -x_1^2 - x_2^2 + x_1^2 x_2 - 2x_1^2 e^{2x_2}$ which is negative definite for all $(x_1, x_2) \in \mathbb{R}^2 \setminus \{0\}$.

Example 3 (*Synchronous Generator*)**.** The problem of estimating, as precisely as possible, the region of attraction of an equilibrium point is crucial in power system analysis (Genesio, Tartaglia, & Vicino, 1985). In fact, when a fault occurs in the system, the operating condition is moved to a different state. The possibility of assessing whether the equilibrium is recovered after the fault critically depends on the estimate of the region of attraction of the equilibrium point itself, hence enlargements of the estimate of the basin of attraction may be of paramount importance in case of failure occurrence. Consider the model of a synchronous generator described by the equations (Genesio et al., 1985)

$$\frac{d^2\delta}{dt^2} + d\frac{d\delta}{dt} + \sin(\delta + \delta_0) - \sin(\delta_0), \tag{45}$$

where $d > 0$ is the damping factor, δ_0 is the power angle and $\delta(t) \in \mathbb{R}$ is the power angle variation. Defining $x_1 = \delta$ and $x_2 = \dot{\delta}$, the model (45) is described by

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -dx_2 - \sin(x_1 + \delta_0) + \sin(\delta_0), \tag{46}$$

with $x(t) = (x_1(t), x_2(t)) \in \mathbb{R}^2$. The origin of the state-space is a locally exponentially stable equilibrium point of the system (46) provided $|\delta_0| < \frac{\pi}{2}$. Consider the linearization of the system (46) around the equilibrium point $(x_1, x_2) = (0, 0)$, namely

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -\cos(\delta_0) & -d \end{bmatrix} x. \tag{47}$$

Select, as in Genesio et al. (1985), $d = 0.5$ and $\delta_0 = 0.412$ and define the quadratic Lyapunov function $V_l(x) = \frac{1}{2} x^\top \bar{P} x$, where $\bar{P} = \bar{P}^\top > 0$ is the solution of the Lyapunov equation (17) with $Q = I$, namely

$$V_l = \frac{1}{2} x^\top \begin{bmatrix} 4.3783 & 1.0913 \\ 1.0913 & 4.1826 \end{bmatrix} x.$$

To construct a Lyapunov function note that the mapping $P(x) = x^\top \bar{P}$ is an *algebraic* $\bar{P}$ *solution* of the Eq. (20). Letting $R = \bar{P}$ and noting that the partial differential equation (28) does not admit a closed-form solution, consider the algebraic equivalent of the invariance pde, namely the Eq. (31). Partitioning the mapping H as above yields

$$\begin{aligned} 0 &= \bar{h}_1 x_2 + \bar{h}_2(-0.5x_2 - (\sin(x_1 + \delta_0) - \sin(\delta_0))) \\ &\quad + 4.3783k(\bar{h}_1 x_1 + \bar{h}_2 x_2) + 1.0913k(\bar{h}_3 x_1 + \bar{h}_4 x_2), \\ 0 &= \bar{h}_3 x_2 + \bar{h}_4(-0.5x_2 - (\sin(x_1 + \delta_0) - \sin(\delta_0))) \\ &\quad + 1.0913k(\bar{h}_1 x_1 + \bar{h}_2 x_2) + 4.1826k(\bar{h}_3 x_1 + \bar{h}_4 x_2). \end{aligned} \tag{48}$$

The solution H of (48) is exploited to construct the function $V_m(x) = x^\top H(x)^\top \bar{P} x + \frac{1}{2}\|x - H(x)x\|^2$. The estimates of the basin

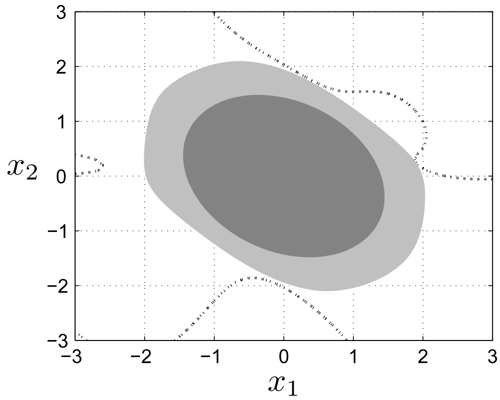

Fig. 3. The estimates of the basin of attraction given by the quadratic Lyapunov function V_l (dark-gray region) and by the Dynamic Lyapunov function V_m (light-gray region), together with the level line corresponding to $\dot{V}_m = 0$, dash-dotted line.

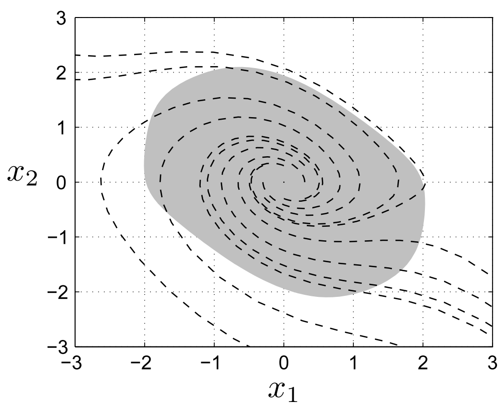

Fig. 4. Phase-portrait of the system (46), with $\delta_0 = 0.412$, together with the estimate of the basin of attraction obtained using the function V_m, gray region.

of attraction obtained with the quadratic function V_l and with the function V_m are displayed in Fig. 3, together with the level line corresponding to $\dot{V}_m = 0$, dash-dotted line. Fig. 4 shows the phase-portrait of the system (46), with $\delta_0 = 0.412$ and $d = 0.5$ together with the estimate of the basin of attraction given by the function V_m.

7. Conclusions

The notion of Dynamic Lyapunov function has been introduced. Similarly to the classical notion of Lyapunov function, this notion allows to study stability properties of (equilibrium points of) linear and nonlinear systems. Unlike Lyapunov functions, Dynamic Lyapunov functions may be constructed without the knowledge of the explicit solution of the ordinary differential equation and involving the solution of any partial differential equation or inequality. In addition Dynamic Lyapunov functions allow to construct families of Lyapunov functions. Implications of Dynamic Lyapunov functions have been discussed and examples have been used to illustrate the advantages of Dynamic Lyapunov functions.

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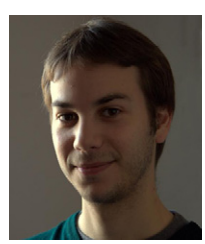

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