Compensating Drift Vector Fields With Gradient Vector Fields for Asymptotic Submanifold Stabilization

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***Abstract*—We derive sufficient conditions on a drift vector field to let an asymptotically stable invariant submanifold of an input-affine system without drift remain asymptotically stable for the system with drift. In doing so, we use the same feedback laws modulo control gain tuning, such that no new feedback laws need to be designed for the system with drift. Our main assumption is that the vector field of the input-affine system without drift assumes the form of a gradient vector field for given feedback laws. We show how one can assess the performance of the system with drift only via knowledge about the system without drift. Finally, we find that our results are relevant in synchronization problems and backstepping controllers for mechanical systems.**

***Index Terms*—Lyapunov methods, nonlinear control.**

I. Motivating Example

Given the exemplary control system

$$\dot{x} = (\|x\| - 1)\, x + \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} x + \frac{1}{\|x\|}\, x u(x) \tag{1}$$

and the task to find $u : \mathbb{R}^2 \to \mathbb{R}$ such that some singleton on the submanifold $\mathbb{S}^1 = \{x \in \mathbb{R}^2 \mid \|x\| = 1\}$ (the unit circle) becomes asymptotically stable, one will find that this is an impossible task; this is due to the drift vector field

$$f(x) = (\|x\| - 1)\, x + \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} x \tag{2}$$

in (1), which is plotted in Fig. 1, and due to the lack of a second control input. If one wishes, however, to find u such that the submanifold $\mathbb{S}^1$ itself becomes asymptotically stable, this would be a feasible task.

One possible solution to this latter problem is to find a function $P : \mathbb{R}^2 \to \mathbb{R}$ whose restriction to any normal space

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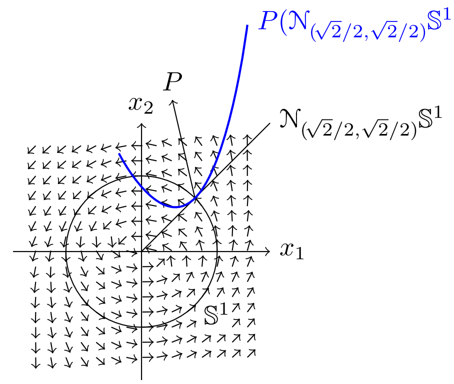

Fig. 1. Illustration of the drift vector field f as in (2) from the right-hand side of (1) (↗), the submanifold $\mathbb{S}^1$ (i.e., the unit circle) (—), one of its normal spaces (—), and the restriction of P to this normal space (—).

$\mathcal{N}_{(x/\|x\|)}\mathbb{S}^1$ of $\mathbb{S}^1$ is strongly convex (at least within a tubular neighborhood of $\mathbb{S}^1$), for instance

$$P(x) = -\frac{1}{2}\|x\|^2 + \frac{1}{3}\|x\|^3 + \frac{1}{6}. \tag{3}$$

The restriction of P to a normal space of $\mathbb{S}^1$ is depicted in Fig. 1 in order to illustrate this strong convexity property. As an Ansatz, equate the control vector field with $-k\nabla P$, where k is a control gain, i.e.,

$$g(x)u(x) = \frac{1}{\|x\|}\, xu(x) = -k\nabla P(x) = -k(\|x\| - 1)x.$$

Solving for u yields

$$u(x) = -k\|x\| \left(\|x\| - 1\right). \tag{4}$$

One finds that for large enough k, in particular for every k greater than $k_0 = 1$, $\mathbb{S}^1$ becomes asymptotically stable. This is, in turn, verified by finding that the squared norm of the $\mathcal{N}_{(x/\|x\|)}\mathbb{S}^1$-coordinate of x, which is

$$V(x) = \frac{1}{2}\left\|\left(1 - \frac{1}{\|x\|}\right)x\right\|^2 = \frac{1}{2}\left(\|x\| - 1\right)^2 \tag{5}$$

has Lie derivative

$$\mathrm{L}_f V(x) + \mathrm{L}_{gu} V(x) = 2(1-k)\|x\|V(x) \tag{6}$$

along $f + gu$, which is, for k greater than $k_0 = 1$, nonpositive and zero if and only if $x \in \mathbb{S}^1$ or $x = 0$. This, in addition, proves that any tubular neighborhood of $\mathbb{S}^1$ (0 is not included in any tubular neighborhood of $\mathbb{S}^1$) is a subset of the region of asymptotic stability of $\mathbb{S}^1$.

This example is, of course, academic; it is simplified by the fact that f splits nicely into normal and tangent coordinates of $\mathbb{S}^1$ as well as by the simplicity of $\mathbb{S}^1$ (for instance, the control problem can easily be solved in polar coordinates).

However, the ideas that came into play (strong convexity on normal spaces, gain tuning, Lie derivatives of squared norms of normal coordinates, restriction to tubular neighborhoods) offer to asymptotically stabilize rather general embedded submanifolds of $\mathbb{R}^n$ for rather general drift vector fields.

This generalization is the scope of this work and in particular, we present a systematic procedure to apply these concepts.

II. Introduction

We study control problems in which a compact, smoothly embedded submanifold M is ought to be stabilized asymptotically for an input-affine system with drift (cf. [1]–[4]). These asymptotic submanifold stabilization problems include setpoint regulation (in which case M is a singleton), pattern generation (in which case M is a circle, cf. [5]), path following (in which case M is the image of a curve, cf. [6]), and synchronization (in which case M is the span of the vector of ones, cf. [7]). The controllability properties of input-affine systems have been characterized completely [8], [9] but asymptotic stabilization by feedback is recognized to be harder [10]. While there are known methods for finding asymptotically stabilizing feedback laws for input-affine systems without drift, the same task turns out to be more involved for input-affine systems with drift [3], [4], [10]–[14]. In this spirit, we propose a two-step design procedure for feedback laws for input-affine systems with drift:

(i) Design feedback laws that asymptotically stabilize a desired submanifold for an input-affine system without drift.

(ii) Find feedback laws that asymptotically stabilize the desired submanifold for the input-affine system with drift by adapting the feedback laws for the input-affine system without drift.

Assuming that the input-affine system without drift assumes the form of a gradient system under the feedback laws derived in step (i), we derive sufficient conditions under which only scalar adaption (control gain tuning) of the feedback laws for the input-affine system without drift is required to asymptotically stabilize a desired submanifold for the input-affine system with drift. Moreover, we propose a way to assess the performance of the feedback laws for the input-affine system with drift only via knowledge about the input-affine system without drift. We refer to the former property as compensability and to the latter property as assessability.

Structure and Contribution: We formalize our problem in Section III. In Section IV-A, we review sufficient conditions for asymptotic stability in gradient systems. Thereafter, in Section IV-B, we consider continuous drift vector fields and find that for such, only practical asymptotic stability can be achieved. In Section V, we derive sufficient conditions for asymptotic stability given different types of drift vector fields, considering one-sided Lipschitz continuous drift in Section V-A, contracting drift in Section V-B, and locally Lipschitz continuous drift in Section V-C. In Section VI, we derive sufficient conditions to assess the performance of the system with drift only through knowledge about the system without drift. We illustrate the relevance of our results on two distinct problems in Section VII that are relevant on their own, considering synchronization problems in Section VII-A (for which we recover the well-known diffusive couplings with the conditions from our main result) and backstepping controllers for mechanical systems with application to robot navigation problems in Section VII-B (for which we explain how our framework offers to guarantee obstacle avoidance despite drift). We conclude the paper with Section VIII.

Notation: For U, $U' \subset \mathbb{R}^n$, $d(U,U')$ is the infimum over all Euclidean distances between a point in U and a point in U' and if one of the sets, or both, are substituted by an element of $\mathbb{R}^n$, then we mean the respective singleton. For a curve $\gamma : [0,1] \to \mathbb{R}^n$, $\ell(\gamma)$ denotes the length of the curve. Given a function $P : \mathbb{R}^n \to \mathbb{R}$ and a subset $U \in \mathbb{R}^n$, we mean that for all $y \in U$, $P(y) < 0$ $(\leq 0, > 0, \geq 0)$ when we write $P(U) < 0$ $(\leq 0, > 0, \geq 0)$. When P is differentiable, $\nabla P : \mathbb{R}^n \to \mathbb{R}^n$ denotes the unique vector field satisfying $\nabla P(x) \cdot y = \lim_{h\to 0}((P(x+hy) - P(x))/h)$ for all $y \in \mathbb{R}^n$, where $\cdot$ is the inner product (we abbreviate $y \cdot y$ by y^2). Accordingly, by $\nabla^2 P : \mathbb{R}^n \to \mathbb{R}^{n\times n}$, we refer to the function which has the result of the application of ∇ to the ith element of ∇P as its ith row. When ∇ acts on a function $f : \mathbb{R}^n \to \mathbb{R}^n$, then we mean an object with ∇ applied to the ith entry of f as its ith row. A sublevel set of P is denoted by $U_P^\alpha = \{y \in \mathbb{R}^n | P(y) \leq \alpha\}$. The Lie derivative of P along a vector field $X : \mathbb{R}^n \to \mathbb{R}^n$ is denoted by $\mathrm{L}_X P$. For the definitions of asymptotic stability, attractivity, and region of asymptotic stability, as well as for some fundamental results employed within our proofs, we refer to Bhatia and Szegõ [15]. When we say that $\varphi_y : \mathbb{R}^n \times (-\epsilon, \epsilon)$, $(y_0, t) \mapsto \varphi_y(y_0, t)$ solves $\dot{y} = Y(y)$, then we mean that it uniquely satisfies $(\mathrm{d}/\mathrm{d}t)\varphi_y(y_0,t) = Y(\varphi_y(y_0,t))$ at least on some interval of existence $(-\epsilon, \epsilon)$. For a set $M \subset \mathbb{R}^n$, denote $B_M^\epsilon = \{y \in \mathbb{R}^n | d(M,y) \leq \epsilon\}$. ∂U is the boundary and $\operatorname{int} U$ the interior of U. $\otimes$ denotes the Kronecker product.

III. Problem Statement

We consider control problems in which a compact, smoothly embedded submanifold $M \subset \mathbb{R}^n$ is to be asymptotically stabilized for systems of the for

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x) u_i(x) \tag{7}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is the so-called drift vector field, $g_i : \mathbb{R}^n \to \mathbb{R}^n$ are control vector fields, and $u_i : \mathbb{R}^n \to \mathbb{R}$ are the feedback laws sought. Such systems are said to be input-affine. While there are known methods for finding asymptotically stabilizing feedback laws for the system without drift (cf. e.g., [13], [14]), i.e., for

$$\dot{z} = \sum_{i=1}^{m} g_i(z) u_i'(z) \tag{8}$$

the same task turns out to be more involved for the system with drift (7) (cf. [3], [4] or [16, Sec. 4]). We are concerned with this latter problem, although relying on solutions to the former.

In this spirit, we propose a two-step design procedure for the feedback laws u_i for (7):

(i) Design feedback laws u_i' that asymptotically stabilize a desired submanifold M for (8).

(ii) Find feedback laws u_i that asymptotically stabilize the desired submanifold M for (7) by adapting u_i'.

We are herein only concerned with step (ii), as step (i) has been investigated thoroughly in previous work [3], [4], [10]–[14].

With the same reason, and for simplicity, we presume that the first step of the design procedure has been completed *a priori*, i.e., that appropriate u_i' have been chosen such that the submanifold M which is ought to be stabilized asymptotically for (7) is an asymptotically stable invariant set of (8). In particular, we assume that there is a continuously differentiable nonnegative potential function $P : \mathbb{R}^n \to \mathbb{R}$ such that

$$\sum_{i=1}^{m} g_i u_i' = -\nabla P \tag{9}$$

and such that P and ∇P vanish on M. Please note that this assumption is meant to simplify the presented results and that it, as such, only improves presentational conciseness. However, for the sake of completeness, we have added a discussion at the end of Section V-A which elaborates how the results would read if (9) would not hold true.

We derive conditions on the drift vector field f to require only scalar adaption of u_i' to render M asymptotically stable for (7), i.e., conditions on f under which it suffices to merely set

$$u_i = k u_i' \tag{10}$$

for some control gain $k \in (0, \infty)$ while maintaining the designed feedback laws u_i'. In particular, we derive sufficient conditions on f for an asymptotically stable invariant set of

$$\dot{z} = -\nabla P(z) =: Z(z) \tag{11}$$

to remain an asymptotically stable invariant set of

$$\dot{x} = f(x) - k\nabla P(x) =: X(x) \tag{12}$$

modulo tuning of the control gain k. We refer to such drift vector fields as compensable drift vector fields.

Definition 1: Given $M \subset \mathbb{R}^n$ and $U \subset \mathbb{R}^n$ such that M is an asymptotically stable invariant set of (11) and U is a subset of its region of asymptotic stability, f is said to be compensable on U, if there exists $k_0 \in [0, \infty)$ such that for every $k \in (k_0, \infty)$, M is an asymptotically stable invariant set of (12) and U is a subset of its region of asymptotic stability.

The above is indeed in the spirit of our motivating example. In particular, if M is an asymptotically stable invariant set of (11), then the gradient vector field will vanish on M and the remaining drift f will prohibit us from asymptotically stabilizing singletons on M.

We wish to remark that the definition resembles the problem statement from high-gain control [17]–[19].

In the remainder, let

$$\varphi_z : \mathbb{R}^n \times (-\epsilon, \epsilon) \to \mathbb{R}^n,\ (z_0, t) \mapsto \varphi_z(z_0, t) \tag{13}$$
$$\varphi_x : \mathbb{R}^n \times (-\epsilon, \epsilon) \to \mathbb{R}^n,\ (x_0, t) \mapsto \varphi_x(x_0, t) \tag{14}$$

denote the solutions to (11) and (12) initialized at z_0 and x_0, respectively; here and throughout the manuscript, we assume these solutions to be unique and to exist at least on some interval $(-\epsilon, \epsilon)$ for all initial conditions under consideration.

Apart from compensability (and asymptotic stabilization of M in general), we are also concerned with the performance of the feedback laws u_i.

We assume that the performance of the feedback laws u_i is measured by a performance output $J : \mathbb{R}^n \to \mathbb{R}$ and is, as such, determined by the signal $J(\varphi_x(x_0, \cdot))$, as long as the solution to (12) exists. As M is the target manifold, here and henceforth, we assume J to be continuously differentiable, to vanish on M and to be positive and regular elsewhere.

We are concerned with the possibility to assess the performance of the feedback laws u_i through knowledge of $J(\varphi_z(z_0, \cdot))$. This is of particular relevance in a situation where f is not known but a numerical integration of (11) can be performed. In particular, when $J(\varphi_x(x_0, \cdot))$ can be overestimated by $J(\varphi_z(z_0, \cdot))$ on any compact interval of time for some control gain k, we say that f is assessable.

Definition 2: Given $U \subset \mathbb{R}^n$ and $U' \subset \mathbb{R}^n$, f is said to be assessable on $U \times U'$, if, for every compact $T \subset (0, \infty)$, for every $z_0 \in U'$, there exists $k_0 \in [0, \infty)$ such that for all $k \in [k_0, \infty)$, for all $x_0 \in U$, for all $t \in T$, $J(\varphi_x(x_0, t)) \leq J(\varphi_z(z_0, t))$.

Under the circumstance that f is compensable, no new feedback laws have to be designed for the system with drift, but merely a parameter k has to be changed in order to asymptotically stabilize a desired submanifold M. If, moreover, f is assessable, we can overestimate the performance of the system with drift only through knowledge of the system without drift.

We derive sufficient conditions for the compensability of f as well as for the assessability of f.

IV. Asymptotic Stability Without Drift and Practical Stability

In this section, before stating our main results, we briefly review sufficient conditions on P for a set M to be an asymptotically stable invariant set of (11). In the spirit of perturbation theory (cf. [20] and other publications of the author for details), we then explain why only practical asymptotic stability of M can be achieved under the circumstance that f is merely continuous.

A. Asymptotic Stability Without Drift

For a set M to be an asymptotically stable set of equilibria of (11), it is necessary that ∇P vanishes on M (as gradient systems admit no periodic orbits). A sufficient condition is that P is positive definite with respect to M.

Definition 3: A continuously differentiable function $P : \mathbb{R}^n \to \mathbb{R}$ is said to be positive definite with respect to M on U, if $U \subset \mathbb{R}^n$ is a neighborhood of $M \subset \mathbb{R}^n$, $P(U \setminus M) > 0$, $P(M) = 0$, $\nabla P(U \setminus M) \neq 0$ and $\nabla P(M) = 0$.

With this definition at hand, we are ready to repeat a well-known result that we frequently use in the remainder.

Lemma 1 (cf. [21, Sec. 9.2 and 9.3]): If P is positive definite with respect to M on U and M is compact, then M is an asymptotically stable invariant set of (11) and for every $\alpha \in (0, \infty)$ such that $U_P^\alpha \subset U$ and U_P^α is compact, U_P^α is a subset of the region of asymptotic stability of M.

Proof: As P is positive definite with respect to M on U, we have $Z(M) = 0$ such that M is a set of equilibria of (11), hence making M an invariant set of (11). The Lie derivative of P along Z is given by $\mathrm{L}_Z P(z) = -\nabla P(z)^2$. As $\nabla P(M) = 0$ and $\nabla P(U \setminus M) \neq 0$, we have $\mathrm{L}_Z P(U \setminus M) < 0$ and $\mathrm{L}_Z P(M) = 0$. Since U is a neighborhood of M, it follows from Lyapunov's direct method [15, Sec. V.2] that M is an asymptotically stable invariant set. As $\mathrm{L}_Z P(U) \leq 0$, we conclude that for every $\alpha \in (0, \infty)$ such that $U_P^\alpha \subset U$ and U_P^α is compact, U_P^α is an invariant set of (11). It follows from LaSalle's invariance principle [22] that every such U_P^α is a subset of the region of asymptotic stability of M. ∎

B. Practical Stability: Continuous Drift Vector Fields

Continuity of f is a mild assumption which is often imposed only to guarantee local existence of solutions to $\dot{y} = f(y)$, for instance by means of the Peano existence theorem. Under the assumption that f is merely continuous, f is not compensable in general. However, it is possible to render M practically asymptotically stable.

Definition 4: A set M is said to be a k-practically asymptotically stable set of $\dot{y} = Y_k(y)$, where $k \in (0, \infty)$ is a parameter, if there exists $\eta \in (0, \infty)$ such that for every $\epsilon \in (0, \eta)$, there exists a $k_0 \in [0, \infty)$ such that for every $k \in (k_0, \infty)$, B_M^ϵ contains an asymptotically stable invariant set of $\dot{y} = Y_k(y)$ whose region of asymptotic stability contains B_M^ϵ. The union of all regions of asymptotic stability of all asymptotically stable invariant sets contained in B_M^ϵ for all $\epsilon \in (0, \eta)$ is the region of k-practical asymptotic stability of M.

Loosely speaking, when a set is practically asymptotically stable, although one is not in the position to render it asymptotically stable, one may find a sufficiently large parameter such that a sufficiently small neighborhood of the set is attractive and contains a stable invariant set. Note that this interpretation closely resembles the definition of practical asymptotic stability from the literature on averaging [23]. In the spirit of Definition 1, we can define practical compensability.

Definition 5: Given $M \subset \mathbb{R}^n$ and $U \subset \mathbb{R}^n$ such that M is an asymptotically stable invariant set of (11) and U is a subset of the region of asymptotic stability of M, f is said to be practically compensable on U, if M is a k-practically asymptotically stable set of (12) and U is a subset of the region of k-practical asymptotic stability of M.

Continuity of f suffices for practical compensability of f, which can be thought of as an application of classical results from perturbation theory (again cf. [20]): In perturbation theory, $f = \epsilon$ is assumed and it is proven that φ_x and φ_z remain close to each other. Although we have f nonconstant, the situation is similar for continuous f when restricted to compacta.

Proposition 1: If P is positive definite with respect to M on U, M is compact, and f is continuous on U, then, for every $\alpha \in (0, \infty)$ such that $U_P^\alpha \subset U$ and U_P^α is compact, f is practically compensable on U_P^α.

Proof: From Lemma 1, we know that M is an asymptotically stable invariant set of (11) and for every $\alpha \in (0, \infty)$ such that $U_P^\alpha \subset U$ and U_P^α is compact, U_P^α is a subset of the region of asymptotic stability of M. The Lie derivative of P along X is given by $\mathrm{L}_X P(x) = \nabla P(x) \cdot f(x) - k \nabla P(x)^2$. Choose any $\alpha \in (0, \infty)$ such that U_P^α is compact and as M is compact, set $\eta = d(M, \partial U_P^\alpha)$. As P is positive definite with respect to M on U, for every $\alpha \in (0, \infty)$ such that $U_P^\alpha \subset U$ and U_P^α is compact, for any $\epsilon \in (0, \eta)$, there exists $\delta \in (0, \alpha)$ such that U_P^δ is a subset of B_M^ϵ. It is then true that $U_P^\alpha \setminus \operatorname{int} U_P^\delta$ is a compact, nonempty set. As f is continuous and P is continuously differentiable, $\nabla P(x) \cdot f(x)$ assumes its maximum on $U_P^\alpha \setminus \operatorname{int} U_P^\delta$, which we denote by f_δ^α. It follows that for all $x \in U_P^\alpha \setminus \operatorname{int} U_P^\delta$, $\mathrm{L}_X P(x) \leq f_\delta^\alpha - k \nabla P(x)^2$. As P is continuously differentiable and positive definite with respect to M on U, for every $\alpha \in (0, \infty)$ such that $U_P^\alpha \subset U$ and U_P^α is compact, $\nabla P(x)^2$ assumes its positive minimum on $U_P^\alpha \setminus \operatorname{int} U_P^\delta$, which we denote by p_δ^α. It follows that $\mathrm{L}_X P(U_P^\alpha \setminus \operatorname{int} U_P^\delta) \leq f_\delta^\alpha - k p_\delta^\alpha$. Setting $k_0 = f_\delta^\alpha / p_\delta^\alpha$, we have that for any $k \in (k_0, \infty)$, $\mathrm{L}_X P(U_P^\alpha \setminus \operatorname{int} U_P^\delta) < 0$, letting us conclude that U_P^δ is an invariant set of (12). Now define a function as being $P - \delta$ outside U_P^δ and to be zero inside U_P^δ. This function is continuous and its Lie derivative along X outside U_P^δ equals $\mathrm{L}_X P$. By Lyapunov's direct method [15, Sec. V.2], it follows that U_P^δ is an asymptotically stable invariant set of (12). Moreover, as we have $\mathrm{L}_X P(U_P^\alpha \setminus \operatorname{int} U_P^\delta) < 0$, we know that U_P^α is an invariant set. It follows from LaSalle's invariance principle [22] that U_P^α is a subset of the region of asymptotic stability of U_P^δ. This concludes the proof. ∎

It follows from Proposition 1 that if $U = \mathbb{R}^n$ and for every $\eta \in (0, \infty)$, there exists an $\alpha \in (0, \infty)$ such that $B_M^\eta \subset U_P^\alpha$ (radial unboundedness of P), then for every $\epsilon \in (0, \infty)$, for every $\eta \in (\epsilon, \infty)$, there exists $k_0 \in [0, \infty)$ such that for every $k \in (k_0, \infty)$, B_M^ϵ contains an asymptotically stable invariant set of (12) whose region of asymptotic stability contains B_M^η. This property is referred to as semi-global practical asymptotic stability of M (cf. [24]).

The statement of Proposition 1 is trivial if no compact U_P^α exists. However, positive definiteness of P with respect to M, continuity of P and compactness of M ensure existence of some compact U_P^α (also cf. [25]).

V. Asymptotic Stability With Drift: Compensability

To render an asymptotically stable invariant set M of (11) an asymptotically stable invariant set of (12), which is, loosely speaking, the definition of compensability (cf. Definition 1), we require M to be a smoothly embedded submanifold of $\mathbb{R}^n$. Under this condition, denote its tangent space at $y \in M$ by $\mathcal{T}_y M$, its normal space at $y \in M$ by $\mathcal{N}_y M$, and its normal bundle by $\mathcal{N}M = \{(y_1, y_2) \in M \times \mathbb{R}^n | y_2 \in \mathcal{N}_{y_1} M\}$. We define the subset $B_{\mathcal{N}M}^\epsilon$ of $\mathcal{N}M$ by $B_{\mathcal{N}M}^\epsilon = \{(y_1, y_2) \in \mathcal{N}M \mid \|y_2\| \leq \epsilon\}$, the bundle projection π from $\mathcal{N}M$ to M by $\pi(y_1, y_2) = y_1$, and the map $\rho : \operatorname{int} B_{\mathcal{N}M}^\epsilon \to \mathbb{R}^n$, $(y_1, y_2) \mapsto y_1 + y_2$.

Definition 6: The neighborhood $\operatorname{int} B_M^\epsilon$ of M is said to be a tubular neighborhood of M if $\operatorname{int} B_M^\epsilon$ is the diffeomorphic image of $\rho : \operatorname{int} B_{\mathcal{N}M}^\epsilon \to \mathbb{R}^n$.

Theorem (Tubular Neighborhood Theorem (cf. [26, Sec. II.11] or [27, Ch. 10])): If M is a compact, smoothly embedded submanifold, then there exists $\epsilon > 0$ such that $\operatorname{int} B_M^\epsilon$ is a tubular neighborhood of M.

As a consequence of the tubular neighborhood theorem

$$r = \pi \circ \rho^{-1} : \rho\left(\operatorname{int} B_{\mathcal{N}M}^\epsilon\right) \to M \tag{15}$$

is a smooth retraction of the tubular neighborhood onto M (again cf. [26, Sec. II.11] or [27, Ch. 10]). Simply said, r brings points from tubular neighborhoods onto M along normal spaces of M in a smooth fashion, i.e., it works like a surjective orthogonal projection, only with restricted domain but with the mentioned uniqueness and smoothness properties.

Within tubular neighborhoods, due to its smoothness, here and throughout the manuscript, r defines the "natural" Lyapunov function candidate

$$V(x) := \frac{1}{2} d\left(x, r(x)\right)^2. \tag{16}$$

Apart from this additional assumption on M, stricter smoothness assumptions will have to be imposed on f when compared to the mere continuity assumption from the previous section. Different such smoothness assumptions will be elaborated within this section.

A. One-Sided Lipschitz Continuous Drift Vector Fields

A particular vector field property, which was coined one-sided Lipschitz continuity, will be utilized in the following. This property was initially used as a sufficient condition for forward (not backward, in contrast to Picard-Lindelöf) uniqueness of solutions to differential equations (cf. [28, Sec. 1.11]).

Definition 7: $f : \mathbb{R}^n \to \mathbb{R}^n$ is said to be one-sided Lipschitz continuous on $U \subset \mathbb{R}^n$, if there exists $q \in \mathbb{R}$ such that for every $y_1, y_2 \in U$, $(f(y_1) - f(y_2)) \cdot (y_1 - y_2) \leq q d(y_1, y_2)^2$.

Moreover, we require a particular convexity property of P, namely, that P is strongly convex when restricted to a normal spaces of M; we refer to this property as strong convexity of P normally with respect to M.

Definition 8: A continuously differentiable function $P : \mathbb{R}^n \to \mathbb{R}$ is said to be strongly convex on U normally with respect to M, if there exists $\lambda \in (0, \infty)$ such that for every $y_1 \in M$, for every $y_2 \in U$ such that $y_2 - y_1 \in \mathcal{N}_{y_1} M$, $(y_2 - y_1) \cdot (\nabla P(y_2) - \nabla P(y_1)) \geq \lambda d(y_2, y_1)^2$.

Lemma 2: Let M be an invariant set of (12). If P is positive definite with respect to M on U, M is a compact, smoothly embedded manifold, f is one-sided Lipschitz continuous on U, and P is strongly convex on U normally with respect to M; then, for every $\alpha \in (0, \infty)$ such that $U_P^\alpha \subset U$ and U_P^α is compact, for every tubular neighborhood $\operatorname{int} B_M^\epsilon$ of M such that $\operatorname{int} B_M^\epsilon \subset U_P^\alpha$, for every $\delta \in [0, \epsilon)$, f is compensable on B_M^δ.

Proof: From Lemma 1, we know that M is an asymptotically stable invariant set of (11) and for every $\alpha \in (0, \infty)$ such that $U_P^\alpha \subset U$ and U_P^α is compact, U_P^α is a subset of the region of asymptotic stability of M. Hence, for every tubular neighborhood $\operatorname{int} B_M^\epsilon$ of M such that $\operatorname{int} B_M^\epsilon \subset U_P^\alpha$, $\operatorname{int} B_M^\epsilon$ is a subset of the region of asymptotic stability of M. As a consequence of the tubular neighborhood theorem, r is smooth on $\operatorname{int} B_M^\epsilon$, and hence the Lie derivative of V along X exists on $\operatorname{int} B_M^\epsilon$ and is given by $\mathrm{L}_X V(x) = \nabla V(x) \cdot f(x) - k \nabla V(x) \cdot \nabla P(x)$. We next show that $\nabla r(x)(x - r(x)) = 0$. To do so, remember that the tangent space of a regular level set of a continuously differentiable function at a point is contained in the nullspace of its gradient at this very point. Here, as M is the image of r, for all $y \in M$, for all $v \in \mathcal{T}_x r^{-1}(y)$ for a given $x \in r^{-1}(y)$, we have that $\nabla r(x) v = 0$. Now choose y to be $r(x)$. We then have that $r^{-1}(r(x)) = \{r(x)\} + \mathcal{N}_{r(x)} M$, an affine space, whose tangent space is thus $\mathcal{N}_{r(x)} M$. As, by its very definition, we have that $x - r(x) \in \mathcal{N}_{r(x)} M$, it thus follows that $\nabla r(x)(x - r(x)) = 0$. With this result at hand, as a consequence of the product rule, we find that the expressions containing ∇r in ∇V vanish. It follows that $\nabla V(x) = x - r(x) \in \mathcal{N}_{r(x)} M$. As M is an invariant set of (12) and as P is positive definite with respect to M on U, it follows that for all $x \in M$, $f(x) \in \mathcal{T}_x M$. As for all $x \in M$, $\mathcal{T}_x M$ is the orthogonal complement of $\mathcal{N}_x M$, we thus have that $\nabla V(x) \cdot f(r(x)) = 0$. Since f is one-sided Lipschitz continuous on U, we conclude that $\mathrm{L}_X V(x) \leq 2qV(x) - k \nabla V(x) \cdot \nabla P(x)$. As P is positive definite with respect to M on U, we have that for all $x \in M$, $\nabla P(x) = 0$. Hence, as P is strongly convex on U normally with respect to M, $\mathrm{L}_X V(x) \leq 2(q - k\lambda) V(x)$. Now set k_0 to q/λ if q is nonnegative or to zero if q is negative. By the very definition of the distance function, we have $V(M) = 0$ and $V(U \setminus M) > 0$. Thus, for any $k \in (k_0, \infty)$, as a consequence of Lyapunov's direct method [15, Sec. V.2], M is an asymptotically stable invariant set of (12). Moreover, $\mathrm{L}_X V(\operatorname{int} B_M^\epsilon) \leq 0$, such that hence, for every $\delta \in [0, \epsilon)$, B_M^δ is an invariant set of (12). It follows from LaSalle's invariance principle [22] that B_M^δ is a subset of the region of asymptotic stability of M, which was to be proven. ■

The proof revealed that finding k_0 is actually constructive; namely, $k_0 = q/\lambda$, i.e., the ratio of the constant defining the one-sided Lipschitz continuity of f and the constant defining the strong convexity that P has on normal spaces of M, or $k_0 = 0$ if q should be negative.

The main ideas from the proof and the sufficient conditions for compensability stated in the lemma are illustrated in Fig. 2. The figure is also meant to illustrate the geometric notions which we introduced in the beginning of the section.

It follows from Lemma 2 that if $U = \mathbb{R}^n$ and for every $\eta \in [0, \infty)$, $\operatorname{int} B_M^\eta$ is a tubular neighborhood of M and there exists an $\alpha \in (0, \infty)$ such that $B_M^\eta \subset U_P^\alpha$ (which is often referred to as radial unboundedness of P), then there exists $k_0 \in [0, \infty)$ such that for every $k \in (k_0, \infty)$, M is an asymptotically stable invariant set of (12) whose region of asymptotic stability is $\mathbb{R}^n$. This property is referred to as global asymptotic stability of M.

One is tempted to ask for relaxations of the assumptions imposed on M in Lemma 2, e.g., when M is not a manifold, or when M is not compact (for instance when M is a vector space).

Considering sets M which are not manifolds is possible, if we insist that the boundary ∂M of M must be a manifold. Under this assumption, define r to be the retraction to ∂M and define a new Lyapunov function candidate which is the above V outside M and on ∂M, but that is zero on the interior of M. Then, ask for P to be strongly convex normally with respect to ∂M but only on those subsets of the normal spaces of ∂M which are not in the interior of M. Then, follow the proof of Lemma 2 as usual, but use Nagumo's Theorem [29]

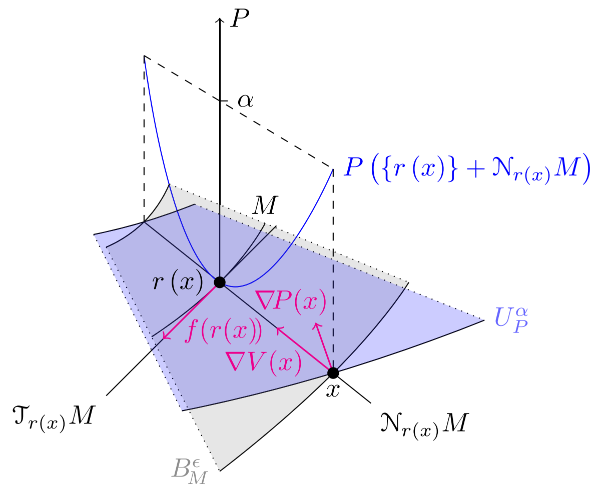

Fig. 2. Sufficient conditions for compensability of f from Lemma 2 illustrated in $\mathbb{R}^n \times \mathbb{R}$ on the example of $n = 2$, i.e., strong convexity of P normally with respect to M, invariance of M, positive definiteness of P with respect to M, and existence of a tubular neighborhood B_M^ϵ. The plot of P is restricted to $\{r(x)\} + \mathcal{N}_{r(x)}M$ for the sake of clarity, as this particular restriction is strongly convex (plot of P on $\{r(x)\} + \mathcal{N}_{r(x)}M$: (—); exemplary point x: (●); retraction $r(x)$: (●); tangent space $\mathcal{T}_{r(x)}M$ at $r(x)$: (—); normal space $\mathcal{N}_{r(x)}M$ at $r(x)$: (—); asymptotically stable invariant submanifold M: (—); drift vector field f at $r(x)$: (↗); gradient vector field ∇P of P at $r(x)$: (↗); gradient vector field ∇V of V at $r(x)$: (↗); sublevel set U_P^α of P: (▢); tubular neighborhood B_M^ϵ of M: (▢)).

to arrive at the inequality $\nabla V(x) \cdot f(r(x)) \leq 0$ (instead of the equality $\nabla V(x) \cdot f(r(x)) = 0$ which was previously used). Consequently, conclude the proof in the above fashion.

Considering noncompact manifolds M is possible by means of two distinct approaches. One option is to impose an additional boundedness assumption on the solutions φ_x. In particular, ask for a compact subset $M' \subset M$ with the property that all solutions φ_x initialized in a closed subset U' of $\rho(\mathcal{N}M')$ remain in U' for all times. Under this assumption, the above proof suffices to show that M' is an asymptotically stable invariant set relative to U' (for the notion of asymptotic stability relative to a set, cf. [15, Sec. V.5]) with $B_M^\delta \cap U'$ being a subset of its region of asymptotic stability. Another option is two assume existence of φ_x on $[0, \infty)$ for all initial conditions in B_M^δ. Under this assumption, the above proof still suffices to show that M is an asymptotically stable invariant set, as V can be underestimated and overestimated by Kamke functions (cf. [15, Sec. V.4]).

A further natural relaxation to ask for would be to neglect (9), i.e., not to restrict oneself to control vector fields assuming the form of gradient vector fields. Indeed, such a relaxation is possible. However, all sufficient conditions from Lemma 2 that were "natural" when expressed in terms of P, become rather cumbersome. In particular, it is sufficient to replace (9) and P positive definite with respect to M on U, P strongly convex on U normally with respect to M with the conditions that for all $y \in M$, $\sum_{i=1}^m g_i(y)u_i'(y) = 0$, and that there exists $\lambda \in (0, \infty)$ such that for every $y_1 \in M$, for every $y_2 \in U$ such that $y_2 - y_1 \in \mathcal{N}_{y_1}M$, $(y_2 - y_1) \cdot (\sum_{i=1}^m g_i(y_2)u_i'(y_2) - g_i(y_1)u_i'(y_1)) \geq \lambda d(y_2, y_1)^2$. The proof of Lemma 2 concludes just as before, but we prefer expressing the sufficient conditions for compensability in terms of the more "natural" assumptions on P rather than in terms of the latter, more general conditions on u_i'.

Last, we wish to remark that the statement of Lemma 2 is trivial if no tubular neighborhood exists. However, as M is a smoothly embedded submanifold, the tubular neighborhood theorem ensures existence of a tubular neighborhood of M (which is also why we restrict ourselves to such submanifolds).

B. Contracting Drift Vector Fields

Among various smoothness properties, contracting vector fields have recurrently received attention in various fields of control theory [30], [31].

Definition 9: $f : \mathbb{R}^n \to \mathbb{R}^n$ is said to be contracting on $U \in \mathbb{R}^n$, if there exists $c \in (0, \infty)$ such that, for all $x \in U$, the symmetric part of $\nabla f(x)$ has its largest eigenvalue smaller than or equal to $-c$.

Lemma 3 (cf. [32]): If f is contracting on U, then f is one-sided Lipschitz continuous on U.

Proof: Let $\varphi_y : \mathbb{R}^n \times (-\epsilon, \epsilon) \to \mathbb{R}^n$ denote the solution to $\dot{y} = f(y)$, let $y_1, y_2 \in U$ and let

$$\gamma : (-\epsilon, \epsilon) \times [0, 1] \to \mathbb{R}^n, \quad (t, s) \mapsto \varphi_y(y_1, t) + s\left(\varphi_y(y_2, t) - \varphi_y(y_1, t)\right). \tag{17}$$

When f is contracting on U, then for the length ℓ of $\gamma(t, \cdot)$, it holds true that $(\mathrm{d}/\mathrm{d}t)\ell(\gamma(t, \cdot)) \leq -c\ell(\gamma(t, \cdot))$ (cf. [30], [31]). By the chain rule, $\ell(\gamma(t, \cdot))(\mathrm{d}/\mathrm{d}t)\ell(\gamma(t, \cdot))$ equals $(\mathrm{d}/\mathrm{d}t)(1/2)\ell(\gamma(t, \cdot))^2$. Hence, by multiplication with $\ell(\gamma(t, \cdot))$, we have $(\mathrm{d}/\mathrm{d}t)(1/2)\ell(\gamma(t, \cdot))^2 \leq -c\ell(\gamma(t, \cdot))^2$. As $\gamma(t, [0, 1])$ is a straight line, $\ell(\gamma(t, \cdot))$ equals $d(\varphi_y(y_2, t), \varphi_y(y_1, t))$. Again by the chain rule, $(\mathrm{d}/\mathrm{d}t)d(\varphi_y(y_2, t), \varphi_y(y_1, t))^2$ equals $2(\varphi_y(y_2, t) - \varphi_y(y_1, t)) \cdot (\mathrm{d}/\mathrm{d}t)(\varphi_y(y_2, t) - \varphi_y(y_1, t))$. When substituting the identity $(\mathrm{d}/\mathrm{d}t)\varphi_y(y_0, t) = f(\varphi_y(y_0, t))$, which holds true on $(-\epsilon, \epsilon)$ for both y_0 being y_1 or y_2, we arrive at $(\varphi_y(y_2, t) - \varphi_y(y_1, t)) \cdot (f(\varphi_y(y_2, t)) - f(\varphi_y(y_1, t)))$ being smaller than or equal to $-cd(\varphi_y(y_2, t), \varphi_y(y_1, t))^2$. At $t = 0$, this is just $(y_2 - y_1) \cdot (f(y_2) - f(y_1)) \leq -cd(y_2, y_1)^2$. With $q = -c$, we recover the definition of f being one-sided Lipschitz continuous on U. ■

With this relation between one-sided Lipschitz continuity and contractivity, we arrive at another sufficient condition for compensability.

Proposition 2: Let M be an invariant set of (12). If P is positive definite with respect to M on U, M is a compact, smoothly embedded submanifold, f is contracting on U, and P is strongly convex on U normally with respect to M, then, for every $\alpha \in (0, \infty)$ such that $U_P^\alpha \subset U$ and U_P^α is compact, for every tubular neighborhood $\operatorname{int} B_M^\epsilon$ of M such that $\operatorname{int} B_M^\epsilon \subset U_P^\alpha$, for every $\delta \in [0, \epsilon)$, f is compensable on B_M^δ.

Proof: The claim is a consequence of Lemma 2 and 3. ■

It is easy to infer from the proofs of Lemmata 2 and 3 that contractivity of f ensures compensability of f with $k_0 = 0$, i.e., that any positive k is sufficient for asymptotic stability of M. Therefore, for contracting drift, it suffices to design feedback laws for the system without drift and to then apply them to the system with drift (i.e., with $k = 1$), with no additional tuning of k necessitated.

C. Locally Lipschitz Continuous Drift Vector Fields

A well-known and comparatively mild assumption on vector fields is local Lipschitz continuity, as this property is sufficient for uniqueness and local existence of solutions to $\dot{y} = f(y)$ by means of the Picard–Lindelöf theorem.

Definition 10: $f : \mathbb{R}^n \to \mathbb{R}^n$ is said to be locally Lipschitz continuous on $U \subset \mathbb{R}^n$, if for every $y_1 \in U$, there exists an open neighborhood U_{y_1} of y_1 such that there exists $L_{y_1} \in [0, \infty)$, such that for every $y_2 \in U_{y_1}$, $d(f(y_1), f(y_2)) \le L_{y_1} d(y_1, y_2)$.

Lemma 4: If f is locally Lipschitz continuous on U, then f is one-sided Lipschitz continuous on every compact subset of U.

Proof: Choose some compact $U' \subset U$. Then $\cup_{y \in U'} U_y$ (with the neighborhoods U_y being as in Definition 10) is an open cover of U', i.e., $U' \subset \cup_{y \in U'} U_y$. As U' is compact, the open cover $\cup_{y \in U'} U_y$ has finite subcover $\cup_{y \in U''} U_y$, i.e., U'' is finite, $U'' \subset U'$, and $U' \subset \cup_{y \in U''} U_y$. The latter is the consequence of the Heine–Borel theorem. As U'' is finite, there exists $L = \max_{y \in U''} L_y$ and we hence have that for all y_1, $y_2 \in U'$, $d(f(y_1), f(y_2)) \le L d(y_1, y_2)$. It is always true that $d(f(y_1) - f(y_2), y_1 - y_2)^2 \ge 0$, yielding $2(f(y_1) - f(y_2)) \cdot (y_1 - y_2) \le d(f(y_1), f(y_2))^2 + d(y_1, y_2)^2$. Together with the Lipschitz inequality, we conclude that $(f(y_1) - f(y_2)) \cdot (y_1 - y_2) \le ((L^2 + 1)/2) d(y_1, y_2)^2$, which is the characterization of f being one-sided Lipschitz continuous on U' with $q = (L^2 + 1)/2$. ∎

This brings us in the position to state our main result, i.e., that local Lipschitz continuity of f is sufficient for compensability of f.

Theorem 2: Let M be an invariant set of (12). If P is positive definite with respect to M on U, M is a compact, smoothly embedded submanifold, f is locally Lipschitz continuous on U, and P is strongly convex on U normally with respect to M, then, for every $\alpha \in (0, \infty)$ such that $U_P^\alpha \subset U$ and U_P^α is compact, for every tubular neighborhood $\operatorname{int} B_M^\epsilon$ of M such that $\operatorname{int} B_M^\epsilon \subset U_P^\alpha$, for every $\delta \in [0, \epsilon)$, f is compensable on B_M^δ.

Proof: As f is locally Lipschitz continuous on U, by Lemma 4, for every $\alpha \in (0, \infty)$ such that $U_P^\alpha \subset U$ and U_P^α is compact, it is one-sided Lipschitz continuous on U_P^α. As P is positive definite with respect to M on U and strongly convex on U normally with respect to M, it is also positive definite with respect to M on U_P^α and strongly convex on U_P^α normally with respect to M. Application of Lemma 2 proves the claim. ∎

It follows from Theorem 2 that if $U = \mathbb{R}^n$ and for every $\eta \in [0, \infty)$, $\operatorname{int} B_M^\eta$ is a tubular neighborhood of M and there exists an $\alpha \in (0, \infty)$ such that $B_M^\eta \subset U_P^\alpha$ (radial unboundedness of P), then for every $\eta \in (\epsilon, \infty)$, there exists $k_0 \in [0, \infty)$ such that for every $k \in (k_0, \infty)$, M is an asymptotically stable invariant set of (12) whose region of asymptotic stability contains B_M^η. This property is referred to as semi-global asymptotic stability of M (cf. [33]).

In contrast to the proof of Lemma 2 and the proof of Proposition 2, finding k_0 was not constructive in the foregoing proof. This adds an additional level of conservatism to the result when compared to those two foregoing results. In particular, the maximal Lipschitz constant taken from the finite subcover is not unique, as it depends on the finite subcover chosen to arrive at the estimate. However, the idea of searching the tubular neighborhood for local estimates of k_0 and taking the maximal one allows for an algorithmic implementation of the approach, which we outline in the following subsection.

D. An Algorithmic Approach to Compensability

As a consequence of Proposition 2, if f is contracting, we have $k_0 = 0$, whereas if f is one-sided Lipschitz continuous, we have $k_0 = q/\lambda$, and application of formula (10) for any $k \in (0, \infty)$ allows computation of feedback laws for the input-affine system with drift. If, however, f is locally Lipschitz continuous, Theorem 2 merely ensures existence of some k_0. To apply formula (10) for some $k \in (k_0, \infty)$, it is thus useful to have an algorithmic procedure at hand that at least provides an approximation of k_0. We propose such an algorithmic procedure here as a supplement to our results.

Within the procedure, given the compact, smoothly embedded submanifold M, we require an ϵ_1-net $G_M^{\epsilon_1} \subset M$ of M as well as, for every $y_1 \in G_M^{\epsilon_1}$, an ϵ_2-net $G_{y_1}^{\epsilon_2} \subset \{y_2 \in \mathcal{N}_{y_1} M | y_2 \le \eta\}$ of $\{y_2 \in \mathcal{N}_{y_1} M | y_2 \le \eta\}$ for some $\eta \in (0, \infty)$, i.e., sets such that for every $y_1 \in M$, there exists $\xi_1 \in G_M^{\epsilon_1}$ such that $d(\xi_1, y_1) < \epsilon_1$ and for every $y_2 \in \{y_2 \in \mathcal{N}_{y_1} M | y_2 \le \eta\}$, there exists $\xi_2 \in G_{y_1}^{\epsilon_2}$ such that $d(\xi_2, y_2) < \epsilon_2$. When M is compact, an ϵ_1-net as above exists for any positive ϵ_1 and as $\eta \in (0, \infty)$, an ϵ_2-net as above exists for any positive ϵ_2. Apart from these nets, an appropriate model of the vector field f must be provided. For every $\eta \in (0, \infty)$, we are then in the position to provide an approximation of k_0 that is valid on B_M^η and that approaches the exact value of k_0 as ϵ_1 and ϵ_2 approach zero, as long as f is continuous, P is continuously differentiable, and B_M^η is the subset of a tubular neighborhood of M. In the spirit of the proof of Lemma 2, Algorithm 1 renders the Lie derivative of V along X negative pointwise for every point on the net

$$\bigcup_{y_1 \in G_M^{\epsilon_1}} G_{y_1}^{\epsilon_2}. \tag{18}$$

By means of continuity of f, ∇P (as P is continuously differentiable) and ∇V (as B_M^η is the subset of a tubular neighborhood of M), for small ϵ_1, ϵ_2, this lets the approximation provided by Algorithm 1 hold true uniformly in $\rho(y_1, y_2)$ such that the value of k_0 provided by Algorithm 1 is valid on B_M^η.

Algorithm 1 k_0-Approximation

Require: drift vector field f; potential function P; smoothly embedded compact submanifold M; $\eta \in (0, \infty)$ such that B_M^η is the subset of a tubular neighborhood of M; ϵ_1-net $G_M^{\epsilon_1}$ of M; ϵ_2-net $G_{y_1}^{\epsilon_2}$ of $\{y_2 \in \mathcal{N}_{y_1} M | y_2 \le \eta\}$ for every $y_1 \in G_M^{\epsilon_1}$

$k_0 \leftarrow 1$
for all $y_1 \in G_M^{\epsilon_1}$ **do**
    for all $y_2 \in G_{y_1}^{\epsilon_2}$ **do**
        $y \leftarrow \rho(y_1, y_2)$
        $k_0^y \leftarrow (\nabla V(y) \cdot f(y)) / (\nabla V(y) \cdot \nabla P(y))$
        if $k_0 < k_0^y$ **then**
            $k_0 \leftarrow k_0^y$
        end if
    end for
end for
Return: k_0

VI. Overestimating Performance Along Solutions: Assessability

Within this section, we take the point of view that the performance of the feedback laws is characterized by the performance output $J : \mathbb{R}^n \to \mathbb{R}$, which we assume to be vanishing on M and positive and regular elsewhere, i.e., by the signal

$$t \mapsto J\left(\varphi_x(x_0, t)\right). \tag{19}$$

We show that the performance of the feedback laws can then be assessed only by knowing

$$t \mapsto J\left(\varphi_z(z_0, t)\right) \tag{20}$$

which was, loosely speaking, the definition of assessability (cf. Definition 2). This is particularly appealing when the system without drift is known well and a numerical integration of (11) can be performed for some z_0. In such a situation, it is possible to derive (upper) performance bounds for (12) in terms of k without knowledge of f that are valid on any compact interval of time. This result is much in the spirit of classical comparison lemmata (cf. [34]); more particular, similar to most comparison lemmata, we also employ Grönwall's inequality.

Theorem 3: Let M be an invariant set of (12). If P and J are positive definite with respect to M on U, M is a compact, smoothly embedded submanifold, f is locally Lipschitz continuous on U, and P is strongly convex on U normally with respect to M; then, for every $\alpha \in (0, \infty)$ such that $U_P^\alpha \subset U$ and U_P^α is compact, for every tubular neighborhood $\operatorname{int} B_M^\epsilon$ of M such that $\operatorname{int} B_M^\epsilon \subset U$, for every $\delta \in [0, \epsilon)$, f is assessable on $B_M^\delta \times U_P^\alpha \setminus M$.

Proof: By Lemma 1, for every $\alpha \in (0, \infty)$ such that $U_P^\alpha \subset U$ and U_P^α is compact, for every $z_0 \in U_P^\alpha$, $\varphi_z(z_0, t)$ exists on $[0, \infty)$. By virtue of the same lemma, M is an asymptotically stable invariant set of (11) and U_P^α is a subset of its region of asymptotic stability. Thus, as J is positive definite with respect to M on U and $U_P^\alpha \subset U$, for every $z_0 \in U_P^\alpha \setminus M$, we have that $J(\varphi_z(z_0, \cdot))$ attains some positive value for any finite t. Moreover, as J is continuously differentiable and φ_z is continuous, for every compact $T \subset (0, \infty)$, $J \circ \varphi_z$ attains its maximum and its minimum on T, and we denote $\omega = \min_{t \in T} J(\varphi_z(z_0, t))$. As f is locally Lipschitz continuous on U and M is compact, by Lemma 4, for every tubular neighborhood $\operatorname{int} B_M^\epsilon$ of M such that $\operatorname{int} B_M^\epsilon \subset U$, for every $\delta \in [0, \epsilon)$, we have that f is one-sided Lipschitz continuous on B_M^δ. Under these circumstances, we have shown in the proof of Theorem 2 that the Lie derivative of V along X can be overestimated by $\mathrm{L}_X V(x) \leq 2(q - k\lambda)V(x)$, and that for any $k \in (q/\lambda, \infty)$, M is an asymptotically stable invariant set of (12) with B_M^δ being a subset of its region of asymptotic stability. Therefore, for every $x_0 \in B_M^\delta$, we know that $\varphi_x(x_0, t)$ exists on $[0, \infty)$. It follows that $\mathrm{L}_X V(\varphi_x(x_0, t)) = (\mathrm{d}/\mathrm{d}t)V(\varphi_x(x_0, t))$. As the overestimate $\mathrm{L}_X V(x) \leq 2(q - k\lambda)V(x)$ holds true on B_M^δ and B_M^δ is an invariant set of (12) for all $k \in [q/\lambda, \infty)$, we conclude that for all $t \in [0, \infty)$, for all $x_0 \in B_M^\delta$, $(\mathrm{d}/\mathrm{d}t)V(\varphi_x(x_0, t)) \leq (q - k\lambda)V(\varphi_x(x_0, t))$. As a consequence of Grönwall's inequality [34, Ch. 1], we consequently arrive at $V(\varphi_x(x_0, t)) \leq \exp((q - k\lambda)t)V(x_0)$. It is true that B_M^δ is a sublevel set of V. In fact, V can attain no greater value than $\delta^2/2$ on B_M^δ. For all $x_0 \in B_M^\delta$, we thus have that $\exp((q - k\lambda)t)V(x_0)$ is smaller than or equal to $(1/2)\exp((q - k\lambda)t)\delta^2$. As the function $\exp((q - k\lambda)t)$ decreases strictly for $k \in ((q/\lambda), \infty)$, $(1/2)\exp((q - k\lambda)\min T)\delta^2 \leq \beta$ is sufficient to have that, for all $t \in T$, $(1/2)\exp((q - k\lambda)t)\delta^2 \leq \beta$. Now set $\beta = (1/2)d(M, \partial U_J^\omega)^2$ such that $V(\varphi_x(x_0, t)) \leq \beta$ implies $J(\varphi_x(x_0, t)) \leq J(\varphi_z(z_0, t))$. Consequently, set

$$k_0 = \frac{1}{\lambda}\left(q - \frac{1}{\min T}\log\left(\frac{2\beta}{\delta^2}\right)\right) \tag{21}$$

if $\beta < (1/2)\delta^2$ or to any value greater than q/λ if $\beta \geq (1/2)\delta^2$ to find that for all $k \in [k_0, \infty)$, for all $x_0 \in B_M^\delta$, for all $t \in T$, $J(\varphi_x(x_0, t)) \leq J(\varphi_z(z_0, t))$, which was claimed. ■

In the proof of Theorem 3, it turned out that one has to distinguish between $\beta < (1/2)\delta^2$ and $\beta \geq (1/2)\delta^2$. The reason for this is that the latter corresponds to the case where $d\left(M, \varphi_z(z_0, \max T)\right) \geq \delta$, such that the initial condition x_0 satisfies $J(x_0) \leq J(\varphi_z(z_0, \max T))$ and, loosely speaking, any decrease of J along φ_x is sufficient to guarantee that for all $t \in T$, $J(\varphi_x(x_0, t)) \leq J(\varphi_z(z_0, t))$.

In our definition of assessability (Definition 2), we had insisted that $\min T$ is greater than zero. Getting back to the proof of Theorem 3, we could equivalently let $\min T = 0$ and impose the additional constraint that $x_0 = z_0$, yielding, in turn, that for $t = 0$, $J(\varphi(x_0, t)) \leq J(\varphi_z(z_0, t))$ (with equality).

Similarly, we could insist to arrive at the strict inequality $J(\varphi(x_0, t)) < J(\varphi_z(z_0, t))$ by taking $k \in (k_0, \infty)$ instead of $k \in [k_0, \infty)$.

VII. Examples

The framework presented above applies to many problems in control theory. We selected two particular problems that have their own distinctive features, each, and that are relevant on their own. Within this section, we outline how the proposed framework applies to these examples.

A. Synchronization Problems and Consensus Networks

Given a network of N dynamical systems with states $x_1 \cdots x_N \in \mathbb{R}^m$, we have $n = Nm$. When each of the systems has its individual vector field $f_i : \mathbb{R}^m \to \mathbb{R}^m$ and the systems are diffusively coupled over some weighted, undirected graph $\mathscr{G} = (\mathscr{V}, \mathscr{E}, \mathscr{W})$, i.e., $\mathscr{V} \subset \mathbb{N}$, $\mathscr{E} \subset (\mathscr{V} \times \mathscr{V})$, $\mathscr{W} : \mathscr{E} \to (0, \infty)$ with Laplacian matrix $\mathscr{L}$, then the overall system assumes the form

$$\begin{bmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_N \end{bmatrix} = \begin{bmatrix} f_1(x_1) \\ \vdots \\ f_N(x_N) \end{bmatrix} - (\mathscr{L} \otimes I_m) \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix}. \tag{22}$$

Setting

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix}, f(x) = \begin{bmatrix} f_1(x_1) \\ \vdots \\ f_N(x_N) \end{bmatrix}, P(x) = \frac{1}{2}x^\top(\mathscr{L} \otimes I_m)x \tag{23}$$

we recover the formulation (12), where the control gain k is determined by the weights of the graph $\mathscr{G}$, i.e., by increasing the weights of the graph, we increase k. In a synchronization problem, the so-called synchronization manifold $M = \mathscr{S} = \{x_1 \cdots x_N \in \mathbb{R}^m | x_1 = \cdots = x_N\}$ is ought to be stabilized asymptotically. A mild issue here is that $\mathscr{S}$ is not compact, which can be overcome by the assumption that there exists a compact $\mathscr{S}' \subset \mathscr{S}$ and a $U' \subset \rho(\mathcal{N}\mathscr{S}')$ such that U' is an invariant set of (22). In this setting, the retraction r onto $\mathscr{S}$ assumes the form

$$r\left(\varphi_x(x_0,t)\right) = \begin{bmatrix} \text{mean}\,\varphi_x(x_0,t) \\ \vdots \\ \text{mean}\,\varphi_x(x_0,t) \end{bmatrix} \tag{24}$$

where $\text{mean}\,\varphi_x(x_0,t)$ is the arithmetic mean of the solutions to the individual systems, i.e.,

$$\text{mean}\,\varphi_x(x_0,t) = \frac{1}{N}\sum_{i=1}^{N} \varphi_{x_i}(x_0,t).$$

In contrast, $\varphi_x(x_0,t) - r(\varphi_x(x_0,t))$ is the stack of deviations of the solutions to the individual systems from the arithmetic mean, which is also called the synchronization error. Synchronization is attained when $\varphi_x(x_0,t) - r(\varphi_x(x_0,t))$ tends to zero as time approaches infinity.

For $f = 0$, the problem of finding $\mathscr{L}$ such that $\mathscr{S}$ is attractive is equivalent to the integrator consensus problem [35]. If the graph $\mathscr{G}$ is connected, we have $\ker \mathscr{L} = \mathscr{S}$ and P is positive definite with respect to $\mathscr{S}$ on $\mathbb{R}^n$. Moreover, if the graph $\mathscr{G}$ is connected, P is strongly convex on $\mathbb{R}^n$ normally with respect to $\mathscr{S}$ and the parameter λ from Definition 8 is just the algebraic connectivity of $\mathscr{G}$ (cf. [36]).

Yet, for $f \neq 0$, the situation turns out to be more involved. In the particular case that we have $f_i = f_j$ for all i, j, it is true that for all $x \in \mathscr{S}$, $f(x) \in \mathcal{T}_x\mathscr{S}$ and hence $\mathscr{S}$ is an invariant set of (12). If, in addition, the vector fields f_i are one-sided Lipschitz continuous, which holds true for many vector fields admitting periodic orbits, such as the Lorenz system, the Van der Pol oscillator, and Chua's circuit, then it is always possible to increase the weights of the graph $\mathscr{G}$ in order to render $\mathscr{S}$ attractive [32], [37]. In our framework, this is readily checked. For the definition of asymptotic stability relative to a set, we refer to [15, Sec. V.5].

Corollary 1 (cf. [32], [37]): If for all i, j, $f_i = f_j$, f is one-sided Lipschitz continuous on a neighborhood U of $\mathscr{S}$, the graph $\mathscr{G}$ is connected, and there exists a compact $\mathscr{S}' \subset \mathscr{S}$ and a closed $U' \subset \rho(\mathcal{N}\mathscr{S}')$ such that U' is invariant, then there exists $\mathscr{W}_0 : \mathscr{E} \to (0,\infty)$ such that for all $\mathscr{W} > \mathscr{W}_0$, $\mathscr{S}$ is asymptotically stable relative to $\mathscr{U}'$ and for every $\epsilon \in [0,\infty)$ such that $B^\epsilon_{\mathscr{S}} \subset U$, $B^\epsilon_{\mathscr{S}} \cap U'$ is a subset of its region of asymptotic stability.

Proof: As $\mathbb{R}^n$ is a tubular neighborhood of M, r is smooth everywhere. Thus, the Lie derivative of V along X is given by $\mathrm{L}_X V(x) = (x - r(x)) \cdot (f(x) - (\mathscr{L} \otimes I_m)x)$. As $f_i = f_j$, we have that for all $x \in M$, $f(x) \in \mathcal{T}_x\mathscr{S}$, hence making $f(r(x))$ orthogonal to $x - r(x)$. Thus, subtracting $(x - r(x)) \cdot f(r(x))$ and overestimating $(x - r(x)) \cdot (f(x) - f(r(x)))$ by $2qV(x)$ by means of one-sided Lipschitz continuity, we have that for all $x \in U$, $\mathrm{L}_X V(x) \leq 2qV(x) - (x - r(x)) \cdot (\mathscr{L} \otimes I_m)x$. As $\mathscr{L}$ is a Laplacian matrix, $r(x) \in \ker \mathscr{L} \otimes I_m$ and adding $(x - r(x)) \cdot (\mathscr{L} \otimes I_m) r(x)$ yields $\mathrm{L}_X V(x) \leq 2qV(x) - P(x - r(x))$. Under the circumstance that $\mathscr{G}$ is connected, P is positive definite with respect to M on $\mathbb{R}^n$ and strongly convex on $\mathbb{R}^n$ normally with respect to $\mathscr{S}$ (cf. [36]). In particular, we have that $P(x - r(x)) \geq \lambda d(x, r(x))^2$, where λ is the algebraic connectivity of $\mathscr{G}$. In turn, we are in the position to overestimate the Lie derivative of V along X by $\mathrm{L}_X V(x) \leq 2(q - \lambda)V(x)$. λ can be increased arbitrarily by increasing $\mathscr{W}$ and for every $\lambda \in (q,\infty)$, we have that $\mathrm{L}_X V(U \setminus \mathscr{S}) < 0$. By our very assumption, U' is invariant. For every $\epsilon \in [0,\infty)$, $B^\epsilon_{\mathscr{S}}$ is a sublevel set of V. Thus, for every $\lambda \in (q,\infty)$, for all $\epsilon \in [0,\infty)$ such that $B^\epsilon_{\mathscr{S}} \subset U$, $U' \cap B^\epsilon_{\mathscr{S}}$ is invariant. Application of Lyapunov's direct method and LaSalle's invariance principle concludes the proof. ■

A nice property here is that the graph $\mathscr{G}$ can be designed for the consensus network, i.e., for $f = 0$, to then only increase its weights and solve the synchronization problem.

The above situation, that is when for all i, j, $f_i = f_j$, is quite well-understood. Yet, recently, the case where $f_i \neq f_j$ has gained attention. If this is the case, and, in particular, the synchronization manifold $\mathscr{S}$ is not an invariant set of (12), then synchronization is not possible. However, under the same conditions as in the foregoing corollary, practical synchronization can be achieved, that is that for every $\epsilon \in (0,\infty)$, there exist weights of $\mathscr{G}$ such that $d\left(\varphi_x(x_0,t), r\left(\varphi_x(x_0,t)\right)\right)$ is ultimately bounded by ϵ (i.e., that an ϵ-neighborhood of the synchronization manifold $\mathscr{S}$ is asymptotically stabilized). Practical synchronization has gained attention recently [38].

Corollary 2 (cf. [38]): If f is continuous on a neighborhood U of $\mathscr{S}$, the graph $\mathscr{G}$ is connected, and there exists a compact $\mathscr{S}' \subset \mathscr{S}$ and a closed $U' \subset \rho(\mathcal{N}\mathscr{S}')$ such that U' is invariant, then for every $\eta \in (0,\infty)$ such that $B^\eta_{\mathscr{S}} \subset U$, for every $\epsilon \in (0,\eta)$, there exist $\mathscr{W}_0 : \mathscr{E} \to (0,\infty)$ such that for all $\mathscr{W} > \mathscr{W}_0$, $B^\epsilon_{\mathscr{S}}$ is asymptotically stable relative to $\mathscr{U}'$ and $B^\eta_{\mathscr{S}} \cap U'$ is a subset of its region of asymptotic stability.

Proof: The Lie derivative of P along X is given by $\mathrm{L}_X P(x) = x^\top(\mathscr{L} \otimes I_m)(f(x) - x)$. Choose any $\alpha \in (0,\infty)$ such that $B^\eta_{\mathscr{S}} \subset U^\alpha_P$. Such an alpha exists as P is positive definite with respect to $\mathscr{S}$ on $\mathbb{R}^n$. Next, choose any $\delta \in [0,\infty)$ such that $U^\delta_P \subset B^\epsilon_{\mathscr{S}}$. As $\epsilon \in (\eta,\infty)$, we have that $U^\delta_P \subset U^\alpha_P$. Thus, $U' \cap (U^\alpha_P \setminus \text{int}\, U^\delta_P)$ is a compact set. As $\mathscr{S}$ is not contained in the latter and P is positive definite with respect to $\mathscr{S}$ on $\mathbb{R}^n$, we have that $P(x)$ is greater or equal λx^2 on this set, where λ is the algebraic connectivity of $\mathscr{G}$. As f is continuous on U, $x^\top(\mathscr{L} \otimes I_m) f(x)$ assumes its maximum on $U' \cap (U^\alpha_P \setminus \text{int}\, U^\delta_P)$, which we denote by f^α_δ. Similarly, we denote the positive minimum of x^2 on $U' \cap (U^\alpha_P \setminus \text{int}\, U^\delta_P)$ by x^α_δ. We thus arrive at the overestimate $\mathrm{L}_X P(x) \leq -\lambda x^\alpha_\delta + f^\alpha_\delta$. λ can be increased arbitrarily by increasing $\mathscr{W}$ and for any algebraic connectivity λ that is greater than $f^\alpha_\delta / x^\alpha_\delta$, it follows that $\mathrm{L}_X P$ is strictly negative on the compact set $U' \cap (U^\alpha_P \setminus \text{int}\, U^\delta_P)$. Application of Lyapunov's direct method and LaSalle's invariance principle concludes the proof. ■

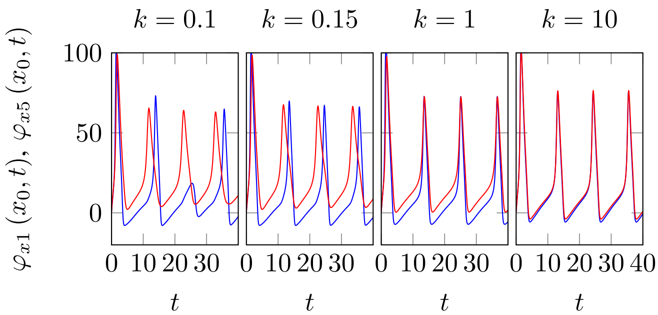

Fig. 3. Solution φ_x to the differential (12) versus time t with f, P as in (22), $N = 2$, and f_i as in the Hodgkin–Huxley action potential models (cf. [39], [40]), i.e., two diffusively coupled Hodgkin–Huxley models with nonidentical conductances, plotted at different gains k. For the sake of clarity, only the cell voltages φ_{x1} and φ_{x5} are plotted (gain k: $k = 0.1, 0.15, 1, 10$ (from left to right); cell voltage $\varphi_{x1}(x_0, t)$ of cell 1: (—); cell voltage $\varphi_{x5}(x_0, t)$ of cell 2: (—)).

We illustrate this latter effect, i.e., practical synchronization, on the example of two Hodgkin–Huxley action potential models (cf. [39, Sec. 2.3.1], [40]) with identical synaptic currents and potentials, but nonidentical conductances, such that convergence to $\mathscr{S}$ is impossible. We however know that with the Laplacian $\mathscr{L} = k[1\ -1]^\top \otimes [1\ -1]$, for any $\epsilon \in (0, \infty)$, there exists a $k \in [0, \infty)$ such that φ_x converges into an ϵ-neighborhood of $\mathscr{S}$. For a random initial condition, we thus simulate the Hodgkin–Huxley action potential model in MATLAB using `ode45` with different control gains $k = 0.1$, 0.15, 1, and 10. Letting φ_{x1} and φ_{x5} denote the cell voltages, we plot the resulting numerical approximations of $\varphi_{x1}(x_0, t)$ and $\varphi_{x5}(x_0, t)$ on the interval [0, 40] in Fig. 3. As expected, the ultimate difference between the solutions decreases as the control gain increases.

B. Backstepping Controllers for Mechanical Systems

Given a fully actuated mechanical system in generalized coordinates

$$\begin{bmatrix} \dot{q} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} v \\ \Delta(q) + F(q, v) \end{bmatrix} \tag{25}$$

where q are the generalized positions, v are the generalized velocities, $F : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$ are the generalized external forces (to be chosen), and $\Delta : \mathbb{R}^m \to \mathbb{R}^m$ is the velocity vector field (i.e., $n = 2m$), one is tempted to apply backstepping in order to design an appropriate feedback law for the system (cf. [41]). In particular, let us consider a situation where one wants to let the generalized positions follow some gradient vector field $v_{\text{ref}} = -k\nabla Q(q)$ for some potential function $Q : \mathbb{R}^r \to \mathbb{R}$ that is positive definite with respect to some target set S, e.g., in a robot navigation problem (cf. [42]).

In the course of the backstepping design procedure, one defines the regulation error $e = v - v_{\text{ref}}$ which suffices the differential equation $\dot{e} = \Delta(q) + F(q, v) - k\nabla^2 Q(q)v$. If one chooses the Lyapunov function candidate

$$P(q, e) = Q(q) + \frac{1}{2}e^2 \tag{26}$$

then a possible Ansatz to render the Lie derivative of P negative is to choose F such that $\dot{e} = -\nabla Q(q) - ke$ holds true. Solving this for F, if one knew Δ, this would yield $-\nabla Q(q) + \nabla^2 Q(q)v - ke - \Delta(q)$ for F. If one however has no knowledge about Δ, then one arrives at $F(q, v) = -\nabla Q(q) + \nabla^2 Q(q)v - ke$. This gives us

$$\begin{bmatrix} \dot{q} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} 0 \\ \Delta(q) \end{bmatrix} + \begin{bmatrix} e - k\nabla Q(q) \\ -\nabla Q(q) - ke \end{bmatrix} \tag{27}$$

in the closed loop. To recover (12), set

$$f(q, e) = \begin{bmatrix} 0 \\ \Delta(q) \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \otimes I_m \nabla P(q, e). \tag{28}$$

Therein, $\left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$ reflects the symplectic/Hamiltonian nature of the system. For $\Delta = 0$, asymptotic stability of $M = S \times \{0\}$ is readily proven [41]. To cope with the effect of Δ, one may employ adaptive control laws that increase k as a function of x (cf. [43], [44]). As a consequence of the framework which we have presented in Section V, it is possible to find a value for k (independent of x) that renders $M = S \times \{0\}$ an asymptotically stable invariant set of (12) with P as in (26) and f as in (28).

Corollary 3 (cf. [41], [43], [44]): Let $M = S \times \{0\}$ be an invariant set of (12). If Q is positive definite with respect to S on U, S is a compact, f is locally Lipschitz continuous on $U \times U'$, and Q is strongly convex on U normally with respect to S, then, for every $\alpha \in (0, \infty)$ such that $U_P^\alpha \subset U \times U'$ and U_P^α is compact, for every tubular neighborhood $\text{int} B_M^\epsilon$ of M such that $\text{int}\, B_M^\epsilon \subset U_P^\alpha$, for every $\delta \in [0, \epsilon)$, there exists $k_0 \in [0, \infty)$ such that for every $k \in (k_0, \infty)$, $S \times \{0\}$ is an asymptotically stable invariant set of (12) and B_M^δ is a subset of its region of asymptotic stability.

Proof: As e^2 is positive definite with respect to $\{0\}$ on $\mathbb{R}^m$, if Q is positive definite with respect to S on U, then P is positive definite with respect to $M = S \times \{0\}$ on $U \times \mathbb{R}^m$. If Q is strongly convex on U normally with respect to S, then P is strongly convex on $U \times \mathbb{R}^n$ normally with respect to $M = S \times \{0\}$, since e^2 is strongly convex on $\mathbb{R}^n$ normally with respect to $\{0\}$. When S is a compact, smoothly embedded submanifold, then $M = S \times \{0\}$ is a compact, smoothly embedded submanifold. The rest of the proof is along the lines of the proof of Theorem 2. ■

Having this result regarding the compensability of (28) at hand, we moreover ask for the assessability of f. In this example, we let $Q(\varphi_{xq}(x_0, t))$ determine the performance of our feedback laws, as it is the case in robot navigation problems, where φ_{xq} denotes the first m entries of φ_x (similarly, let φ_{zq} denote the first m entries of φ_z).

Corollary 4 (cf. [45]): Let $M = S \times \{0\}$ be an invariant set of (12). If Q is positive definite with respect to S on U, S is compact, f is locally Lipschitz continuous on $U \times U'$, and Q is strongly convex on U normally with respect to S, then, for every $\alpha \in (0, \infty)$ such that $U_P^\alpha \subset U \times U'$ and U_P^α is compact, for every tubular neighborhood $\text{int}\, B_M^\epsilon$ of M such that $\text{int}\, B_M^\epsilon \subset U \times U'$, for every $\delta \in [0, \epsilon)$, for every compact $T \subset (0, \infty)$, for every $z_0 \in U_P^\alpha \setminus M$, there exists $k_0 \in [0, \infty)$ such that for all $k \in [k_0, \infty)$, for all $x_0 \in B_M^\delta$, for all $t \in T$, $Q(\varphi_{xq}(x_0, t)) \leq Q(\varphi_{zq}(z_0, t))$.

Proof: The proof is analogous to the proof of Theorem 3, only that we set $\omega = \min_{t \in T} Q(\varphi_{zq}(z_0, t))$. Setting $\beta = (1/2) d(S \times \{0\}, \partial U_P^\omega)^2$ then implies that $P(\varphi_x(x_0, t)) \leq Q(\varphi_{zq}(z_0, t))$. As $e^2 \geq 0$, the latter also implies that $Q(\varphi_{xq}(x_0, t)) \leq Q(\varphi_{zq}(z_0, t))$, concluding the proof. ∎

When knowledge about $Q(\varphi_{zq}(z_0, t))$ is available, say from numerical simulations, then the latter result has application to obstacle avoidance problems. In particular, assuming that Q is a robot navigation function [42], then Q attains its maximum on the boundary of the obstacles which φ_{xq} is ought to avoid. When numerical approximations of φ_{zq} are available, and one has seen that the simulated robot avoids the obstacles as it should, then the foregoing result allows one to merely tune the control gain k in order to avoid the obstacles with φ_{xq} despite the drift vector field f. We want to elaborate this concept on the example of a fully actuated vehicle moving on unknown terrain. Let us assume that the target set S is given by $S = \{q \in \mathbb{R}^m | d(q, \left[\begin{smallmatrix}-\pi \\ -\pi\end{smallmatrix}\right])^2 \leq (1/2)\}$ and the obstacle O is given by $O = \{q \in \mathbb{R}^m | d(q, \left[\begin{smallmatrix}\pi \\ -\pi\end{smallmatrix}\right])^2 \leq (1/2)\}$. The resulting robot navigation function Q is then given by

$$Q(q) = \frac{d(S,q)^2}{\sqrt{\left(d(S,q)^2\right)^2 + d(O,q)^2}} \tag{29}$$

such that we have $Q^{-1}(1) = O$ and $Q^{-1}(0) = S$. Solving the differential (27) with $\Delta = 0$ in MATLAB with `ode45` that φ_{xq} avoids the obstacle O and approaches the target set S asymptotically.

We now assume that the vehicle does indeed not move on plain terrain, but on a landscape with height map

$$H(q) = -\left(1 - \cos(q_1)\right)\left(1 - \cos(q_2)\right) \tag{30}$$

where q_1 and q_2 denote the first and the second entry of q, respectively, such that S and O are sublevel sets of H and H attains its maximum for $q_1 = 0$. As a consequence, the drift vector field

$$\Delta(q) = -a\nabla H(q) \tag{31}$$

is introduced, where a is gravitational constant divided by mass. Without losing generality, we set $a = 1$. If we naively use the same feedback law that we used for $\Delta = 0$ for the case that Δ suffices (31), when solving the differential (27) in MATLAB with `ode45`, we see that φ_{xq} approaches the obstacle O. However, by the foregoing corollary, we can find a k_0 such that for any greater k, the convergence properties of the nominal system are recovered. Thus, setting $k = 50$, when solving the differential (27) in MATLAB with `ode45`, we find that φ_{xq} avoids the obstacle O and approaches the target set S asymptotically, which was claimed. All three numerical approximations of φ_{xq} resulting from $\Delta = 0$ and $k = 1$, Δ as in (31) and $k = 1$, and Δ as in (31) and $k = 50$ are depicted in Fig. 4.

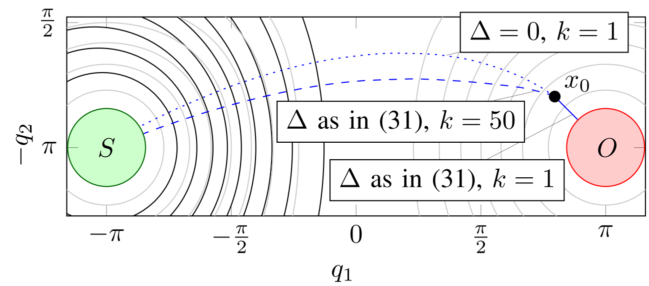

Fig. 4. Solution φ_x to the differential (12) with P, f as in (26) and (28), respectively, and Q as in (29), i.e., a robot navigation problem with target set S and obstacle O, plotted at different gains k for different drift vector fields, i.e., $\Delta = 0$ and Δ as in (31), where the latter corresponds to the effect of a landscape with height map H as in (30). For the sake of clarity, only the two first components of φ_x, i.e., φ_{xq}, are plotted, which are the positions q_1 and q_2 (gain k and drift vector field Δ: $k = 1, 50$, $\Delta = 0$, Δ as in (31) (indicated by annotations); initial condition x_0: (●); positions $\varphi_{xq}(x_0, t)$ for Δ as in (31), $k = 1$: (—) for $\Delta = 0$, $k = 1$: (····) for Δ as in (31), $k = 50$: (- -); obstacle O: (○); target set S: (○); level sets of navigation function Q: (○); level sets of height map H: (○)).

VIII. Conclusion

We proposed a two-step design procedure for feedback laws that render a submanifold asymptotically stable for input-affine systems with drift, in which the first step requires computation of feedback laws that render the desired submanifold asymptotically stable for the input-affine system without drift, whereas the second step merely requires tuning of a control gain. Assuming that the vector field of the input-affine system without drift assumes the form of a gradient vector field for the feedback laws computed in the first step, we derived sufficient conditions for recovering practical asymptotic stability of the submanifold under consideration for continuous drift.

For one-sided Lipschitz continuous, contracting, or locally Lipschitz continuous drift, we could also guarantee asymptotic stability of the submanifold. Moreover, we derived sufficient conditions to overestimate the performance of the input-affine system with drift by the performance of the input-affine system without drift. In both results, an important idea was to assume strong convexity of the restriction of the function defining the gradient vector field to normal spaces of the submanifold which is ought to be stabilized asymptotically (at least on one of its tubular neighborhoods). As for our main result, we pointed out several possible relaxations and extensions. We illustrated the relevance of our proposed approach on synchronization problems, in which we recovered the well-known diffusive couplings, and on backstepping controllers for mechanical systems, in which we found the conditions from our main results satisfied, and could guarantee obstacle avoidance in robot navigation problems despite drift.

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