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Nonlinear minimum-time control with pre- and post-actuation ☆

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ABSTRACT

This article studies the time-optimal output transition problem to change the system output, from an initial value $y(t) = \underline{y}$ (for all time $t \leq 0$) to a final value $y(t) = \bar{y}$ (for all time $t \geq T$), for invertible nonlinear systems. The main contribution of the article is to show that the use of pre- and post-actuation input outside the transition interval $\mathbb{I}_T = [0, T]$ can reduce the transition time T beyond the standard bang–bang-type inputs for optimal state transition. The advantage of using pre- and post-actuation is demonstrated with an illustrative nonlinear example.

1. Introduction

The minimum-time state transition problem with bounds on the input magnitude leads to the classical bang–bang-type input for the fastest state transition. However, the transition time can be reduced further if only the system output needs to be transitioned from one value to another rather than the entire system state. The time-optimal output transition problem is to change the system output from an initial value $y(t) = \underline{y}$ (for all time $t \leq 0$) to a final value $y(t) = \bar{y}$ (for all time $t \geq T$), as shown in Fig. 1. The contributions of this article are (i) to quantify the additional flexibility with output transition (as opposed to standard state transition from an initial value $x(0)$ to a final value $x(T)$) in terms of geometric conditions on the internal dynamics of the system and (ii) to show that the use of pre- and post-actuation input outside the transition interval $\mathbb{I}_T = [0, T]$ can reduce the transition time T beyond the standard bang–bang-type inputs for optimal state transition. The advantage of using pre- and post-actuation is demonstrated with an illustrative nonlinear example.

As opposed to the optimal output transition (OOT) problem, the time-optimal transition between two states has been well studied in literature (Boscain & Piccoli, 2001; Bryson & Ho, 1975; Frank & Vassilis, 1995; Sussmann, 1990; Wing & Desoer, 1963). Such time-optimal solutions to the standard state transition (SST) from $x(0) = \underline{x}$ to $x(T) = \bar{x}$ can be used to solve the output transition problem. For example, by choosing the boundary states $\underline{x}, \bar{x}$ to be equilibrium states (corresponding to the boundary values of the output $\underline{y}, \bar{y}$), SST approaches can reduce the time needed for the output transition. This choice of boundary states ensures that the output remains at the desired (constant) values outside the output-transition time interval $\mathbb{I}_T$ without the need for pre- and post-actuation. Such time-optimal state transition problems have been studied for a variety of applications such as: (i) endpoint positioning of large-scale manipulators (Farrenkopf, 1979; Singhose, Banerjee, & Seering, 1997); (ii) maneuvering of spacecraft (Ben-Asher, Burns, & Cliff, 1992; Singh, Kabamba, & McClamroch, 1989; Thompson, Junkins, & Vadali, 1989; Tzes & Yurkovich, 1993); (iii) maneuvering of flexible structures (Chen & Desrochers, 1990; Hindle & Singh, 2001; Meckl & Kinceler, 1994; Pao & Frankin, 1990); (iv) positioning of read-write heads in disk drives (Hai, 1997; La-orpacharapan & Pao, 2004; McCormick & Horowitz, 1991) and (v) nano-scale positioning using relatively-smaller piezoactuators (Moallem, Kermani, Patel, & Ostojic, 2004). In the nonlinear setting, computational issues for the state transition problem are studied in Kim, Choi, and Ha (2005). Moreover, previous investigations include the synthesis of feedback-type solutions for the optimal control problem (Boscain & Piccoli, 2001; Sussmann, 1987, 1990), the minimum-time swing-up for the pendulum-and-cart problem (Mason, Broucke, & Piccoli, 2008), as well as the minimal-time transition between different cart positions for the pendulum-and-cart problem (Matthew & Raffaello, 2005). While the SST approach reduces the output transition time T, the present article shows that the minimum transition time with the SST approach can be reduced further by directly solving the minimum-time OOT problem.

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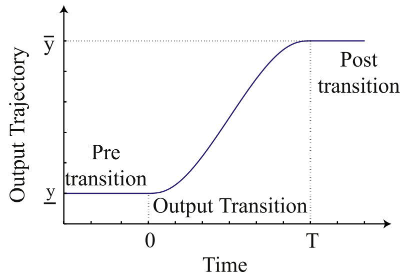


Fig. 1. Minimum-time output transition problem: the output trajectory has to be maintained at a constant initial value $\underline{y}$ before the beginning of the output-transition ($y(t) = \underline{y}, t \leq 0$), and at a constant final value $\bar{y}$ after the completion of the output-transition ($y(t) = \bar{y}, t \geq T$).

The problem of minimizing the time T needed to reach a desired output $y(T) = \bar{y}$ for linear systems was studied in Lewis (1981). A nonlinear version of the problem of changing part of the state (say the output) to a desired value in minimum time has been recently studied in Chang, Petit, and Rouchon (2006). As opposed to achieving the desired output at a particular time instant (as in Chang et al., 2006; Lewis, 1981), the issue of also maintaining the output afterwards (i.e., $t \geq T$) at the desired value $\bar{y}$ was studied in Emami-Naeini (1992) for linear systems. Computational challenges in the nonlinear extensions of this output transition problem are addressed in Nesic and Mareels (1998). The current work extends previous work on the nonlinear output transition problem to include both pre- and post-actuation. In particular, the proposed approach allows the system to evolve in the internal dynamics (outside of the transition interval $\mathbb{I}_T$). Thus, it ensures that the output is kept at the desired values by using pre-actuation input (for $t \leq 0$) and post-actuation input (for $t \geq T$). Note that the use of pre- and post-actuation effectively increases the time available to apply the input, without an increase in the time T needed for the output transition. The resulting availability of additional time to apply inputs (during pre- and post-actuation) tends to lower the required output-transition time T.

The problem of optimal output transition for linear systems, with pre- and post-actuation, was posed in Dowd and Thanos (2000), Piazzi and Visioli (2000, 2001), which find the minimum-time solution from a pre-specified class of trajectories. For example, polynomials were used to pre-specify a set of output trajectories from which a minimal-time solution was obtained in Piazzi and Visioli (2000). However, the output and input trajectories are not intuitive for solutions to typical minimum-time problems, and therefore, it is challenging to include them in the initial set of pre-specified trajectories. In contrast to choosing the output trajectory from a pre-specified set, the input and output trajectories are found as part of the optimization process with the minimum-time OOT approach. For the linear case, previous works have shown that the use of pre- and post-actuation can be used to reduce cost functions involving the input energy (Iamratanakul, Jordan, Leang, & Devasia, 2008; Perez & Devasia, 2003). The current article extends such pre- and post-actuation in two aspects: (a) minimization of the transition time rather than a cost function that involves the input energy, and (b) generalization to the nonlinear setting. Although the current approach is not formulated to handle output constraints such as reducing overshoot in minimum-time problems (the handling of such output constraints was studied recently in Consolini and Piazzi (2009)), the additional flexibility with the potential use of pre- and post-actuation could lead to the reduction of the transition time even with such output constraints.

2. Problem formulation

The output transition problem is posed for invertible nonlinear systems.

2.1. System description

Consider a system described by

$$\begin{aligned} \dot{x}(t) &= f[x(t)] + g[x(t)]u(t) \\ y(t) &= h[x(t)] \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state, $y(t) = [y_1(t), y_2(t), \ldots, y_p(t)]^T$ is the output, with the same number of inputs as outputs, i.e., $u(t), y(t) \in \mathbb{R}^p$, and the input is bounded as

$$\|u(t)\|_\infty \leq U_{\max} \quad \forall -\infty < t < \infty. \tag{2}$$

2.2. Optimal output transition (OOT) problem

Let $\underline{x}$ and $\bar{x}$ be controlled equilibrium points of the system (Eq. (1)) corresponding to inputs $\underline{u}$ and $\bar{u}$ and outputs $\underline{y}$ and $\bar{y}$, i.e.,

Definition 1 (*Delimiting States for Transition*).

$$\begin{aligned} f[\underline{x}] + g[\underline{x}]\underline{u} = 0, &\quad \underline{y} = h[\underline{x}] \\ f[\bar{x}] + g[\bar{x}]\bar{u} = 0, &\quad \bar{y} = h[\bar{x}]. \end{aligned} \tag{3}$$

The output transition problem is formally stated next.

Definition 2 (*Output Transition Problem*). Given the delimiting states and a transition time interval $[0, T]$, find a bounded input-state trajectory $[u_{ref}(\cdot), x_{ref}(\cdot)]$ that satisfy the system equations (1), (2) for all time ($-\infty < t < \infty$)

$$\begin{aligned} &\dot{x}_{ref}(t) = f[x_{ref}(t)] + g[x_{ref}(t)]u_{ref}(t) \\ &y_{ref}(t) = h[x_{ref}(t)] \\ &\|u_{ref}(t)\|_\infty \leq U_{\max} \end{aligned} \tag{4}$$

and the following two conditions.

[I. *The output transition condition*] The output transitions in the time interval $\mathbb{I}_T = [0, T]$ and is maintained at the desired value outside the time interval $\mathbb{I}_T$, i.e.,

$$\begin{aligned} &\text{from } y_{ref}(t) = \underline{y} \quad \text{for all time } t \leq 0 \\ &\text{to } y_{ref}(t) = \bar{y} \quad \text{for all time } t \geq T. \end{aligned} \tag{5}$$

[II. *The delimiting state condition*] The system state approaches the delimiting states as time goes to (plus or minus) infinity,

$$\begin{aligned} x(t) \to \underline{x} &\quad \text{as } t \to -\infty \\ x(t) \to \bar{x} &\quad \text{as } t \to \infty. \end{aligned} \tag{6}$$

The time-optimal output transition seeks to minimize the transition time T with constraints on the input.

Definition 3 (*OOT*). The minimum-time optimal output transition (OOT) problem is to find the bounded input-state trajectory $[u^*(\cdot), x^*(\cdot)]$ that satisfies the output transition problem (in Definition 2), and minimizes the transition time T with the cost function

$$J = \int_0^T 1 \, dt = T. \tag{7}$$

3. Solution

The section begins with the standard approach based on time-optimal state transition, followed by the solution to the time-optimal output transition problem for invertible systems.

3.1. Standard state transition (SST) solution

The output-transition problem can be solved using the time-optimal, state transition approach defined below.

Definition 4 (*Time-optimal State Transition*). Given an initial state x_i and a final state x_f, find a bounded input-state trajectory $[u_{ref}(\cdot), x_{ref}(\cdot)]$ that satisfies the system equations and input constraints (Eqs. (1), (2)) in the transition interval $\mathbb{I}_T$ (as in Eq. (4)) and achieves the state transition

$$\text{from } x_{ref}(0) = x_i \text{ to } x_{ref}(T) = x_f \tag{8}$$

while minimizing the transition time T.

Definition 5 (*SST Approach to Output Transition*). Solve the time-optimal state transition problem (in Definition 4) with initial and final states as

$$\text{from } x_{ref}(0) = \underline{x} \text{ to } x_{ref}(T) = \overline{x}. \tag{9}$$

3.2. Minimum-time OOT solution

The OOT approach is presented, in this subsection, for invertible systems.

3.2.1. Invertibility assumption

In the following, it is assumed that the system (in Eq. (1)) is invertible, i.e., the system equation (1) can be rewritten, through a coordinate transformation $\mathbb{T}$

$$x(t) = \mathbb{T}[\zeta(t), \eta(t)] \tag{10}$$

in the following form

$$\begin{bmatrix} \dfrac{d^{r_1}}{dt^{r_1}} y_1(t) \\ \dfrac{d^{r_2}}{dt^{r_2}} y_2(t) \\ \vdots \\ \dfrac{d^{r_p}}{dt^{r_p}} y_p(t) \end{bmatrix} = \alpha[\eta(t), \zeta(t)] + \beta[\eta(t), \zeta(t)]u(t) \tag{11}$$

$$\dot{\eta}(t) = s_1[\eta(t), \zeta(t)] + s_2[\eta(t), \zeta(t)]u(t) \tag{12}$$

where the matrix $\beta[\eta(t), \zeta(t)]$ is invertible in the region of interest

$x(t) \in \mathbb{B} \subset \mathbb{R}^n$.

The state component $\zeta(t)$ represents the output and its time derivatives

$$\begin{aligned} \zeta' &= \Big[y_1 \ \frac{dy_1}{dt} \ \ldots \ \frac{d^{r_1-1}}{dt^{r_1-1}} y_1 \ y_2 \ \frac{dy_2}{dt} \ \ldots \ \frac{d^{r_2-1}}{dt^{r_2-1}} y_2 \ \ldots \\ &\qquad \ldots \ y_p \ \frac{dy_p}{dt} \ \ldots \ \frac{d^{r_p-1}}{dt^{r_p-1}} y_p \Big] \\ \underline{\zeta}' &= \left[\underline{y}_1 \ 0 \ \ldots \ 0 \ \underline{y}_2 \ 0 \ \ldots \ 0 \ \ldots \underline{y}_p \ 0 \ \ldots \ 0 \right] \\ \overline{\zeta}' &= \left[\overline{y}_1 \ 0 \ \ldots \ 0 \ \overline{y}_2 \ 0 \ \ldots \ 0 \ \ldots \ \overline{y}_p \ 0 \ \ldots \ 0 \right] \end{aligned} \tag{13}$$

where $'$ indicates transpose of a matrix.

Remark 6. The invertibility assumption is satisfied if the system (in Eq. (1)) has a well defined relative degree (Isidori, 1995) $r = [r_1 \, r_2 \, \ldots \, r_p]$.

3.2.2. Pre- and post-actuation

During pre and post-actuation, the output is constant, therefore, the state component ζ is known in terms of the desired output, $\underline{\zeta}$, $\overline{\zeta}$ as in Eq. (13), i.e.,

$$\begin{aligned} \zeta(t) &= \underline{\zeta} \quad \forall\, t \le 0 \\ \zeta(t) &= \overline{\zeta} \quad \forall\, t \ge T. \end{aligned} \tag{14}$$

Moreover, the input to maintain the desired constant output, during pre- and post-actuation, can be found from Eqs. (11) and (13) as

$$\begin{aligned} u(t) &= \underline{u}[\eta(t)] = -[\beta(\eta(t), \underline{\zeta})]^{-1} \alpha[\eta(t), \underline{\zeta}] \quad \forall\, t \le 0 \\ u(t) &= \overline{u}[\eta(t)] = -[\beta(\eta(t), \overline{\zeta})]^{-1} \alpha[\eta(t), \overline{\zeta}] \quad \forall\, t \ge T. \end{aligned} \tag{15}$$

The state dynamics during pre- and post-actuation reduces to the following time-invariant internal dynamics (obtained by rewriting Eq. (12))

$$\begin{aligned} \dot{\eta}(t) &= s_1[\eta(t), \underline{\zeta}] - s_2[\eta(t), \underline{\zeta}][\beta(\eta(t), \underline{\zeta})]^{-1} \alpha[\eta(t), \underline{\zeta}] \\ &= s[\eta(t), \underline{\zeta}] \quad \forall\, t \le 0 \\ \dot{\eta}(t) &= s_1[\eta(t), \overline{\zeta}] - s_2[\eta(t), \overline{\zeta}][\beta(\eta(t), \overline{\zeta})]^{-1} \alpha[\eta(t), \overline{\zeta}] \\ &= s[\eta(t), \overline{\zeta}] \quad \forall\, t \ge T. \end{aligned} \tag{16}$$

For a specified output transition, the evolution of the internal dynamics (Eq. (16)) as well as the inverse input (Eq. (15)) needed to maintain a fixed output (during pre- and post-actuation) are both determined by the boundary values ψ of the internal state

$$\psi = \begin{bmatrix} \eta(0) \\ \eta(T) \end{bmatrix}. \tag{17}$$

3.2.3. Acceptable boundary values

The set Ψ of acceptable boundary values ψ (as in Eq. (17)) should satisfy the following three conditions

[I. *Delimiting condition on internal state*] The internal states should meet the delimiting state condition in Eq. (6). This is satisfied provided the initial boundary condition $\eta(0)$ is on the local unstable manifold $M_u(\underline{x})$ of the internal dynamics at $x = \underline{x}$ and the final boundary condition $\eta(T)$ is on the local stable manifold $M_s(\overline{x})$ of the internal dynamics at $x = \overline{x}$, i.e.,

$$\begin{aligned} \eta(0) &\in M_u(\underline{x}) \\ \eta(T) &\in M_s(\overline{x}). \end{aligned} \tag{18}$$

Here, the local stable and unstable manifolds are defined as

$$\begin{aligned} M_u(\underline{x}) &= \Big\{ \eta \mid \mathbb{T}\left[\underline{\zeta}, \Phi[\eta, 0, t]\right] \to \underline{x} \text{ as } t \to -\infty \\ &\qquad \text{and } \mathbb{T}\left[\underline{\zeta}, \Phi[\eta, 0, t]\right] \in \mathbb{B} \text{ for } t \le 0 \Big\} \\ M_s(\overline{x}) &= \Big\{ \eta \mid \mathbb{T}\left[\overline{\zeta}, \Phi[\eta, T, t]\right] \to \overline{x} \text{ as } t \to \infty \\ &\qquad \text{and } \mathbb{T}\left[\overline{\zeta}, \Phi[\eta, T, t]\right] \in \mathbb{B} \text{ for } t \ge T \Big\} \end{aligned} \tag{19}$$

and the flow Φ (of the internal dynamics in Eq. (16)) represents the internal dynamics state at time t_2 with boundary condition η_1 at time t_1, i.e.,

$$\eta(t_2) = \Phi[\eta_1, t_1, t_2]. \tag{20}$$

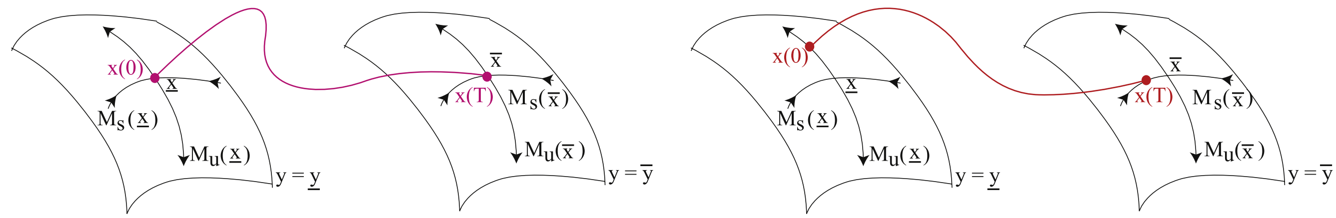
[II. *Bounded pre- and post-actuation condition*] The inverse input (in Eq. (15)), needed to maintain a constant output during pre- and post-actuation, should be bounded and satisfy the input constraint (in Eq. (2)), i.e.,

$$\begin{aligned} \|\underline{u}\{\Phi[\eta(0), 0, t]\}\|_\infty &\le U_{\max} \quad \forall\, t \le 0 \\ \|\overline{u}\{\Phi[\eta(T), T, t]\}\|_\infty &\le U_{\max} \quad \forall\, t \ge T. \end{aligned} \tag{21}$$

[III. *State transition condition*] There exists an input u that satisfies the input constraint (in Eq. (2)) and achieves the state transition

$$\text{from } x_{ref}(0) = \mathbb{T}\left[\underline{\zeta}, \eta(0)\right] \text{ to } x_{ref}(T) = \mathbb{T}\left[\overline{\zeta}, \eta(T)\right] \tag{22}$$

for some transition time T, while the state trajectory $x(t)$ remains in the area of interest ($x(t) \in \mathbb{B}$) during the transition time interval $\mathbb{I}_T$.



(a). SST: without pre- and post-actuation.   (b) OOT: with pre- and post-actuation.

Fig. 2. SST versus OOT. (a) Boundary states of the SST transition are chosen to be $x(0) = \underline{x}$ and $x(T) = \bar{x}$. (b) the OOT approach with pre- and post-actuation exploits the flexibility that the boundary states $(x(0), x(T))$ can be different from $\underline{x}$ and $\bar{x}$.

3.3. Flexibility in the internal state

The flexibility in the choice of the boundary values of the internal state in the OOT approach can reduce the transition time further when compared to the SST approach for invertible systems.

3.3.1. SST—without pre- and post-actuation

If pre- and post-actuation is not allowed, then the boundary value ψ of the internal state (in Eq. (17)) is constrained to

$$\psi = \psi_{eq} = \begin{bmatrix} \underline{\eta} \\ \bar{\eta} \end{bmatrix} \tag{23}$$

corresponding to the delimiting states $\underline{x}$, $\bar{x}$, i.e., (from Eq. (10))

$$\underline{x} = \mathbb{T}[\underline{\zeta}, \underline{\eta}]; \qquad \bar{x} = \mathbb{T}[\bar{\zeta}, \bar{\eta}]. \tag{24}$$

Then the standard state transition approach can be used to find the minimum time as in Definition 5.

3.3.2. OOT—with pre- and post-actuation

With pre- and post-actuation allowed, there is flexibility in the choice of the boundary value ψ of the internal dynamics as opposed to the case without pre- and post-actuation when the internal states are chosen to correspond to the boundary values (as in Eq. (23))—this is illustrated in Fig. 2. Pre- and post-actuation effectively increase the amount of time over which the input can be applied to the system. The availability of additional time to apply the input tends to reduce the transition time T, during which the output is changing.

3.3.3. SST cannot be faster than OOT

The choice of the internal state's boundary value as the equilibrium value (in the SST approach) is also a possible choice in the OOT approach. Therefore, the minimum time T^*_{SST} without pre- and post-actuation (SST) cannot be smaller than the minimum time T^*_{OOT} for output transition with pre- and post-actuation (OOT), i.e.,

$$T^*_{OOT} \leq T^*_{SST}. \tag{25}$$

3.4. Computational issues

The main computational issue in optimal output transition (OOT) is the same as in standard state transition (SST), i.e., to solve the time-optimal transition problem between two given boundary states $(x(0), x(T))$. Therefore, as in SST, analytical solutions will be challenging to obtain for OOT, and numerical approaches used in SST to find the time-optimal transition between two given boundary states $(x(0), x(T))$ (Bryson & Ho, 1975; Frank & Vassilis, 1995; Pao, 1996; Wing & Desoer, 1963) will be needed to solve the OOT problem for generic applications. However, for specific applications, analytical solutions can be found to the time-optimal state transition problem (Matthew & Raffaello, 2005; Mason et al., 2008). Moreover, feedback-based synthesis is possible for low-dimensional systems such as planar systems (Boscain & Piccoli, 2001; Sussmann, 1987). It is noted that analytical solutions to the state-transition problem (when available) can be used within the current OOT approach as well.

The difference between OOT and SST is the additional choice in the boundary states $(x(0), x(T))$ due to the flexibility in the choice of the internal state's boundary value ψ in Eq. (17). Thus, the new computational issue (in OOT when compared to SST) is to quantify the set of acceptable boundary states, which in turn, depends on the quantification of the set Ψ of acceptable boundary values ψ of the internal states—this is discussed below.

Quantifying the set of boundary values Ψ that satisfies the acceptability conditions I and II in Section 3.2.3 corresponds to finding the region of attraction of equilibrium points, which has been well-studied in the past, e.g., Chiang, Hirsch, and Wu (1988), Genesio, Tartaglia, and Vicino (1985) and Hassan (1991). In particular, the set of possible final boundary conditions $\eta(T) \in M_s(\bar{x})$ corresponds to the region of attraction of the final equilibrium point $\bar{\eta}$ in the internal dynamics with the input constraint in Eq. (21). Similarly, the the set of possible initial boundary conditions $\eta(0) \in M_u(\underline{x})$ corresponds to the region of attraction in the internal dynamics (backwards in time) of the initial equilibrium point $\underline{\eta}$ with the same input constraint in Eq. (21). It is noted that Lyapunov functions (when available) can be used to find bounded invariant regions from which all initial states converge to the equilibrium point and thereby, provide estimates of the region of attraction (Hassan, 1991). Such estimates of the region of attraction can be improved upon by using trajectory-reversing methods that simulate starting from initial states (in the estimated region of attraction) backwards in time to expand the estimate of the region of attraction, e.g. Genesio et al. (1985) and Hassan (1991).

An advantage of the OOT approach is that it can only reduce the transition time when compared to the SST approach because of the additional flexibility in the choice of the boundary values. In particular, if the numerical procedure includes the internal state's equilibrium values as a possible choice of the boundary values in the OOT approach, then the OOT solution cannot be worse than the SST solution as in Eq. (25). Therefore, even if the numerical procedure used cannot find the complete region of attraction and thereby results in a sub-optimal solution to the OOT problem, the resulting suboptimal OOT solution (which includes the SST as a possible solution) can only improve (and cannot be worse than) the performance when compared to the SST approach.

4. Example

The reduction in transition time with pre- and post-actuation is illustrated in this section with a simple two-state example, which yields analytical solutions to the minimum-time problem. The example has a sufficiently rich internal dynamics to illustrate the advantages of using pre- and post-actuation, without the typical computational complexity of minimum-time problems.

The example is used to also illustrate the non-uniqueness of solutions and the effect of limits on the available pre-actuation time.

4.1. Model of example system

Consider a two-state system with the following dynamics (as in Eq. (1))

$$\begin{aligned}\dot{x}_1(t) &= x_2^2(t) + u(t)\\ \dot{x}_2(t) &= u(t)\\ y(t) &= x_1(t)\end{aligned} \tag{26}$$

with a unity bound on the scalar input (as in Eq. (2))

$$|u(t)| \le 1 \quad \forall -\infty < t < \infty. \tag{27}$$

The goal is to transition the output

from $y(0) = \underline{y} = 0$ to $y(T) = \overline{y} = 1$

in minimum time T. The corresponding equilibrium delimiting states (in Eq. (6)) are

$$\underline{x} = \begin{bmatrix}\underline{y}\\ 0\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}, \qquad \overline{x} = \begin{bmatrix}\overline{y}\\ 0\end{bmatrix} = \begin{bmatrix}1\\ 0\end{bmatrix}. \tag{28}$$

4.2. Pre- and post-actuation input

A relationship between the input and the output can be found by differentiating the output (in Eq. (26)) to obtain (as in Eq. (11))

$$\frac{\mathrm{d}y}{\mathrm{d}t}(t) = \frac{\mathrm{d}x_1}{\mathrm{d}t}(t) = x_2^2(t) + u(t) \tag{29}$$

with the relative degree (in Remark 6) $r = r_1 = 1$. To maintain a constant output during pre- and post-actuation the left side of Eq. (29) has to be zero. Therefore, the inverse input during pre- and post-actuation (as in Eq. (15)) can be found from Eq. (29) as

$$u(t) = -x_2^2(t) \tag{30}$$

for all time t not in the transition-time interval $\mathbb{I}_T$.

4.3. Internal dynamics

The output $y = x_1$ becomes the state ζ in Eq. (13) and the internal state η can be chosen as the remaining state x_2. Therefore, the co-ordinate transformation ($\mathbb{T}$ in Eq. (10)) becomes the identity with

$$x(t) = \begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} = \begin{bmatrix}\zeta(t)\\ \eta(t)\end{bmatrix} \tag{31}$$

where the delimiting values can be found from Eq. (28). The internal dynamics (as in Eq. (16)) can be written using Eqs. (26), (30), (31) as

$$\dot{\eta}(t) = -\eta^2(t) \quad \forall\, t \le 0 \text{ and } t \ge T. \tag{32}$$

Note that time-derivative of the internal state $\dot{\eta}(t)$ is never positive and so the internal state cannot increase with time t with this scalar internal dynamics. Therefore, the global stable and unstable manifolds of the internal dynamics are ($M_s(\overline{x})$, $M_u(\underline{x})$ in Eq. (19)) are given by

$$\begin{aligned}M_u(\underline{x}) &= \{\eta \mid \eta \le 0\}\\ M_s(\overline{x}) &= \{\eta \mid \eta \ge 0\}.\end{aligned} \tag{33}$$

4.4. Solution to OOT

The minimum-time OOT problem is solved for the example. The potential boundary states (ψ in Eq. (17)) should satisfy the three conditions in Section 3.2.3. These conditions, on the example system, are studied below.

4.4.1. Delimiting condition on internal state

The application of the delimiting condition (in Eq. (18)) on the boundary values for the example system, yields, from Eqs. (18), (33),

$$\psi = \begin{bmatrix}\eta(0) \in M_u(\underline{x})\\ \eta(T) \in M_s(\overline{x})\end{bmatrix} = \begin{bmatrix}\eta(0) \le 0\\ \eta(T) \ge 0\end{bmatrix} \tag{34}$$

where the stable and unstable manifolds $M_s(\overline{x})$, $M_u(\underline{x})$ are defined in Eq. (33).

4.4.2. Bounded pre- and post-actuation condition

To ensure that the inverse input (in Eq. (30)) is bounded, the internal state $\eta = x_2$ needs to be bounded during pre- and post-actuation. The solution to the internal dynamics (as in Eq. (20)) with the inverse input is obtained using Eq. (32) as

$$\eta(t_2) = \Phi[\eta(t_1), t_1, t_2] = \frac{\eta(t_1)}{1 + \eta(t_1)\,[t_2 - t_1]} \tag{35}$$

which, from Eq. (34) yields

$$|\eta(t)| = \left|\frac{\eta(0)}{1 + \eta(0)\,[t]}\right| \le |\eta(0)| \quad \forall\, t \le 0 \tag{36}$$

$$|\eta(t)| = \left|\frac{\eta(T)}{1 + \eta(T)\,[t - T]}\right| \le |\eta(T)| \quad \forall\, t \ge T. \tag{37}$$

Therefore, the inverse input during pre- and post actuation (see Eq. (30)) is less than one, provided

$$-1 \le \eta(0) \le 0 \quad \text{and} \quad 0 \le \eta(T) \le 1 \tag{38}$$

because from Eqs. (36)–(38), for all time t during pre- and post-actuation,

$$|u(t)| = \left|-x_2^2(t)\right| = \left|-\eta^2(t)\right| \le 1. \tag{39}$$

4.4.3. State transition condition

The final condition is the existence of an input u that satisfies the input constraint (in Eq. (27)) and achieves the state transition (as in Eq. (22))

$$\text{from } x_{ref}(0) = \begin{bmatrix}0\\ \eta(0)\end{bmatrix} \text{ to } x_{ref}(T) = \begin{bmatrix}1\\ \eta(T)\end{bmatrix} \tag{40}$$

in a finite time T, which is a standard state transition problem. The minimum transition time $T(\psi)$, for a given boundary value ψ (in Eq. (34) that satisfies Eq. (38)), is found next.

4.5. State transition between boundary states

From standard optimal control theory (Bryson & Ho, 1975; Frank & Vassilis, 1995), the Hamiltonian H for the minimum-time problem (in Eq. (7)) is given by

$$\begin{aligned}H(t) &= 1 + \lambda'(t)f(t)\\ &= 1 + \lambda'(t)\begin{bmatrix}x_2^2(t) + u(t)\\ u(t)\end{bmatrix}\\ &= 1 + \lambda_1(t)\left[x_2^2(t) + u(t)\right] + \lambda_2(t)\,[u(t)]\\ &= 1 + \lambda_1(t)x_2^2(t) + [\lambda_1(t) + \lambda_2(t)]\,u(t)\\ &= 1 + \lambda_1(t)x_2^2(t) + \hat{\lambda}(t)u(t)\end{aligned} \tag{41}$$

that is minimized with the optimal input given by

$$\begin{aligned} u(t) &= 1 \quad \text{if } \hat{\lambda}(t) < 0 \\ u(t) &= -1 \quad \text{if } \hat{\lambda}(t) > 0, \end{aligned} \tag{42}$$

which implies that

$$\hat{\lambda}(t)u(t) < 0 \tag{43}$$

provided $\hat{\lambda}(t) \neq 0$. The Lagrange multiplier λ is given by

$$\begin{bmatrix} \frac{d\lambda_1}{dt}(t) \\ \frac{d\lambda_2}{dt}(t) \end{bmatrix} = -\frac{\partial H}{\partial x}(t) = \begin{bmatrix} 0 \\ -2x_2(t)\lambda_1(t) \end{bmatrix}. \tag{44}$$

Therefore, from Eq. (44),

$$\lambda_1(t) = \Lambda \tag{45}$$

is a constant and the modified Lagrange multiplier $\hat{\lambda}$ (also the switching variable that determines the value of the input from Eq. (42)) satisfies

$$\frac{d\hat{\lambda}}{dt}(t) = -2x_2(t)\Lambda. \tag{46}$$

The boundary conditions for the state x are as in Eq. (40) and $H(T) = 0$, which yields from Eq. (41)

$$\hat{\lambda}(T)u(T) = -1 - \Lambda\eta^2(T) = -1 - \Lambda x_2^2(T). \tag{47}$$

The goal is to solve for the state transition, i.e., to find the constant Λ (in Eq. (45)), the switching function $\hat{\lambda}$ (in Eq. (46)) such that the input (found from Eq. (43)) results in a state x trajectory (satisfying Eq. (26)) that meets the boundary condition in Eq. (40).

4.6. Standard state transition (SST) for example system

In this section, the boundary states are chosen to be the delimiting values (Eq. (28)) to solve the standard state transition (SST) problem, i.e.,

$$\eta(0) = x_2(0) = 0, \qquad \eta(0) = x_2(T) = 0 \tag{48}$$

in Eq. (40) resulting in

$$\begin{aligned} x_{ref}(0) &= \underline{x} = \begin{bmatrix} \underline{y} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ x_{ref}(T) &= \overline{x} = \begin{bmatrix} \overline{y} \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \end{aligned} \tag{49}$$

4.6.1. Last input switching time instant

The last time instant

$t_1 = T - \Delta_1$

(with $0 < t_1 < T$) when the input u switches sign (from $-U$ to U for $t > t_1$ where U is either $+1$ or -1) can be found by integrating Eqs. (26), (46) to yield

$$\begin{aligned} x_2(T) &= x_2(t_1) + U\Delta_1 \\ \hat{\lambda}(T) &= \hat{\lambda}(t_1) - [2\Lambda x_2(T)]\Delta_1 + \Lambda U\Delta_1^2. \end{aligned} \tag{50}$$

Since $x_2(T) = 0$ and $\hat{\lambda}(t_1) = 0$ at the switching instant t_1,

$$\begin{aligned} x_2(t_1) &= -U\Delta_1 \\ \Delta_1 &= \sqrt{\frac{\hat{\lambda}(T)}{\Lambda U}} = \sqrt{\frac{U\hat{\lambda}(T)}{\Lambda U^2}} = \sqrt{\frac{-1}{\Lambda}} \end{aligned} \tag{51}$$

where $\hat{\lambda}(T)u(T) = \hat{\lambda}(T)U = -1$ from Eq. (47) and $|U| = 1$. Therefore, the constant Λ should be negative, i.e., $\Lambda < 0$ for the existence of an input-switching instant t_1.

Remark 7. The state x_2 changes linearly with a constant input U (see Eq. (50)). Therefore, at least one input switch is needed to ensure that x_2 starts and ends at zero (as in Eq. (48)).

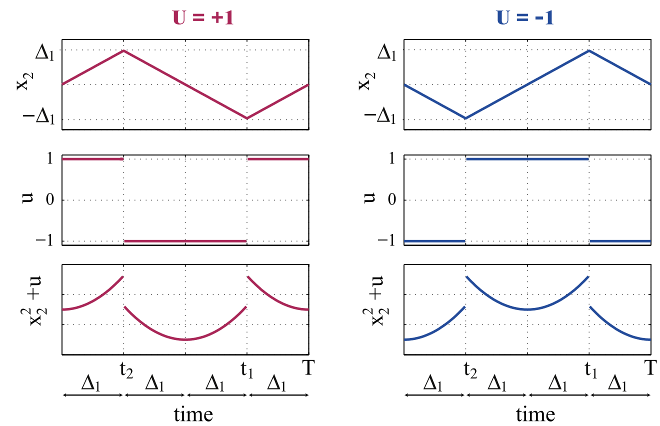


Fig. 3. State x_2, input u, and time derivative of state x_1, i.e., $dx_1/dt = x_2^2 + u$ (from Eq. (26)). The left column plots are for $U = +1$ and right column plots are for $U = -1$.

4.6.2. Second-to-last switching instant

The next (farthest from T) possible input-switch instant $t_2 = t_1 - \Delta_2$ (with $0 < t_2 < t_1 < T$) when the input u switches sign (from U to $-U$ for $t_2 < t < t_1$) can be found by solving Eqs. (26), (46) to yield

$$\begin{aligned} x_2(t_1) &= x_2(t_2) - U\Delta_2 \\ \hat{\lambda}(t_1) &= \hat{\lambda}(t_2) - [2\Lambda x_2(t_1)]\Delta_2 - U\Lambda\Delta_2^2 \end{aligned} \tag{52}$$

which is similar to Eq. (50) but with t_1, T, U, Δ_1 replaced by $t_2, t_1, -U, \Delta_2$, respectively. Since $\hat{\lambda}(t_1) = \hat{\lambda}(t_2) = 0$ at the switching instants t_1, t_2, and $x_2(t_1) = -U\Delta_1$, Eq. (52) yields

$$\begin{aligned} \Delta_2 &= -[2x_2(t_1)]/U = 2\Delta_1 \\ x_2(t_2) &= x_2(t_1) + U\Delta_2 = U\Delta_1. \end{aligned} \tag{53}$$

4.6.3. Other switching instants

Other possible input-switching instants

$t_k = t_{k-1} - \Delta_k$

$0 < t_k < t_{k-1} < \cdots t_2 < t_1 < T$

with $k = 3, 4, \ldots$ can be found by solving Eqs. (26), (46) to yield (similar to Eq. (53))

$$\begin{aligned} \Delta_k &= 2\Delta_1 \\ x_2(t_k) &= (-1)^k U\Delta_1 \end{aligned} \tag{54}$$

where $x_2(t)$ varies linearly between the switching instants (due to a constant input) as shown in Fig. 3.

4.6.4. Optimal SST transition

The state x_2 varies linearly between switching instants (from Eq. (26) with a constant input u between switching instants) as illustrated in Fig. 3. The state x_2 becomes zero at time instants $T - 2k\Delta_1$ at the middle of adjacent switching instants in Fig. 3, i.e.,

$$x_2(T - 2k\Delta_1) = 0 \quad \text{where } k = 1, 2, 3, \ldots. \tag{55}$$

Hence, to match the boundary condition $x_2(0) = 0$ (in Eq. (49)), the optimal time T_{SST}^* for the standard state transition is given by (from Eq. (55))

$$T_{SST}^* = 0 + 2k^*\Delta_1 \tag{56}$$

for some positive integer k^*. Additionally, the state x_1 must change to $\overline{y}$ in the time interval $[0, T_{SST}^*]$. The total change in x_1 is equal to k^* times the change in x_1 during the time interval $[(T - 2\Delta_1), T]$ (see the plot of dx_1/dt in Fig. 3), i.e.,

$$\begin{aligned}\bar{y} - \underline{y} &= 1 = x_1(T) - x_1(0) \\ &= k^* \left[\int_{T-2\Delta_1}^{T} \left(x_2^2(\tau) + u \right) d\tau \right] \\ &= k^* \left[\int_0^{\Delta_1} \left(\tau^2 - U \right) d\tau + \int_0^{\Delta_1} \left((\Delta_1 - \tau)^2 + U \right) d\tau \right] \\ &= k^* \left[\int_0^{\Delta_1} \tau^2 d\tau + \int_0^{\Delta_1} (\Delta_1 - \tau)^2 d\tau \right] \\ &= \frac{2}{3} k^* \Delta_1^3. \end{aligned} \tag{57}$$

The switching parameter Δ_1 can be found from Eq. (57) as

$$\Delta_1 = \left(1.5/k^*\right)^{1/3} \tag{58}$$

with the SST transition time T_{SST}^*, from Eq. (56),

$$T_{SST}^* = 2k^*\Delta_1 = 2\,(1.5)^{1/3}\,(k^*)^{2/3} \tag{59}$$

that is minimized with $k^* = 1$ (i.e., with one input switch), since k^* is a positive integer, to yield

$$T_{SST}^* = 2\,(1.5)^{1/3} \approx 2.2894 \tag{60}$$

$$\Delta_1 = (1.5)^{1/3} \approx 1.4471. \tag{61}$$

Thus, the time-optimal SST input to transition from $y = 0$ to $y = 1$ is

$$u(t) = \begin{cases} +U & \text{if } t \in [0, \Delta_1) \\ -U & \text{if } t \in (\Delta_1, 2\Delta_1] \\ 0 & \text{otherwise} \end{cases} \tag{62}$$

where U can be chosen as either $+1$ or -1.

Remark 8. The time-optimal SST input in Eq. (62) is not unique.

4.7. Optimal output transition (OOT) for example

For this example, the output transition time T is minimized if the time derivative $\dot{y}(t) = \dot{x}_1(t) = x_2^2(t) + u(t)$ is maximized in the time interval $[0, T]$ by choosing $u(t) = 1$. Then, the state dynamics (Eq. (26)) yields

$$x_2(t) = x_2(0) + t \tag{63}$$

and, using this expression for $x_2(t)$,

$$\begin{aligned}\bar{y} = 1 &= \int_0^T \left[x_2^2(t) + u(t) \right] dt \\ &= \left[[x_2(0) + T]^3 - x_2^3(0) \right]/3 + T. \end{aligned} \tag{64}$$

Therefore, the output transition time T for different initial conditions $x_2(0)$ can be found by solving

$$T^3 + 3T[1 + x_2^2(0)] + 3T^2 x_2(0) - 3 = 0 \tag{65}$$

and the optimal output transition time T_{OOT}^* can be found by minimizing T over different $x_2(0) = \eta(0)$ satisfying Eq. (38), i.e., $-1 \le x_2(0) \le 0$. The resulting values of T are shown in Fig. 4 as a function of $x_2(0)$. Additionally, to satisfy the condition $0 \le x_2(T) \le 1$ in Eq. (38), the transition time T must satisfy

$$T \ge -x_2(0) \tag{66}$$

from Eq. (63) when $t = T$—as shown in Fig. 4.

The minima of the transition time T occurs at two values of the initial state $x_2(0)$: (i) $x_2(0) = 0$; and (ii) $x_2(0) = -T$, which are discussed next.

4.7.1. Case (i) OOT with post-actuation

For the case $x_2(0) = 0$, Eq. (65) for the transition time T becomes

$$T^3 + 3T - 3 = 0. \tag{67}$$

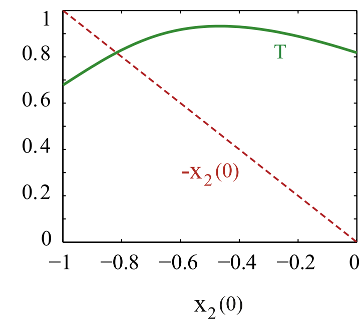


Fig. 4. Solution to minimum time T (from Eq. (65)) for different values of acceptable initial states $-1 \le x_2(0) \le 0$ (as in Eq. (38)) depicted by a solid line. Additional criteria that $T \ge -x_2(0)$ (in Eq. (66)) is indicated by the dashed line.

This results in a single real value for the optimal output transition (OOT) time T_{OOT}^* as

$$\begin{aligned}T_{OOT}^* &= \left[(3 + \sqrt{13})/2\right]^{1/3} - \left[(-3 + \sqrt{13})/2\right]^{1/3} \\ &\approx 0.8177, \end{aligned} \tag{68}$$

which is less than half the minimum transition time $T_{SST}^* = 2.2894$ achieved with the SST approach. Since the internal state is the equilibrium value ($x_2(0) = \eta(0) = 0$) at $t = 0$ there is no pre-actuation input in this case. During the output transition interval, with the input $u = 1$ and initial value $x_2(0) = 0$, the internal state $x_2 = \eta$ reaches

$$x_2(T_{OOT}^*) = T_{OOT}^*$$

at the end of the output transition at time $t = T_{OOT}^*$ from Eq. (26). After the output transition at time $t = T_{OOT}^*$, the internal state is found (from Eq. (35) with $t_2 = t$ and $t_1 = T_{OOT}^*$) as for all $t > T_{OOT}^*$,

$$\begin{aligned}x_2(t) &= \eta(t) \\ &= \frac{\eta(T_{OOT}^*)}{1 + \eta(T_{OOT}^*)\left[t - T_{OOT}^*\right]} \\ &= \frac{T_{OOT}^*}{1 + T_{OOT}^*\left[t - T_{OOT}^*\right]}. \end{aligned} \tag{69}$$

The corresponding post-actuation, inverse input (for $t > T_{OOT}^*$) is given by $u(t) = -x_2^2(t)$ from Eq. (30) with $x_2(t)$ from Eq. (69). Thus, the OOT input (with post-actuation only) is given by

$$u_{POST}^*(t) = \begin{cases} 0 & \text{if } t \le 0 \\ 1 & \text{if } 0 < t \le T_{OOT}^* \\ -\left[\dfrac{T_{OOT}^*}{1 + T_{OOT}^*\left(t - T_{OOT}^*\right)}\right]^2 & \text{if } T_{OOT}^* < t. \end{cases} \tag{70}$$

4.7.2. Case (ii) OOT with pre-actuation

For the case $x_2(0) = -T$, Eq. (65) for the transition time T becomes

$$T^3 + 3T\left[1 + T^2\right] + 3T^2[-T] - 3 = T^3 + 3T - 3 = 0, \tag{71}$$

which is the same as Eq. (67) and therefore, results in the same optimal output transition (OOT) time $T_{OOT}^* \approx 0.8177$ in Eq. (68). During the output transition interval, with the input $u = 1$ and initial value $x_2(0) = -T_{OOT}^*$, the internal state $x_2 = \eta$ reaches

$$x_2(T_{OOT}^*) = 0$$

at the end of the output transition at time $t = T_{OOT}^*$ from Eq. (26). Since the internal state is the equilibrium value ($x_2(T_{OOT}^*) = \eta(T_{OOT}^*) = 0$) at time $t = T_{OOT}^*$, there is no post-actuation input in this case.

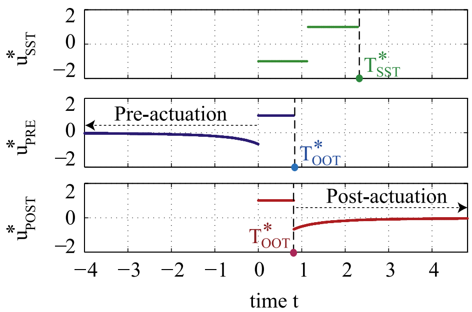


Fig. 5. Comparison of three optimal inputs: (top) optimal SST input without pre- and post-actuation (u_{SST} in Eq. (62) with $U = 1$); (middle) minimum-time OOT input with pre-actuation (u_{PRE} in Eq. (74)); and (bottom) minimum-time OOT input with post-actuation (u_{POST} in Eq. (70)).

Previous to the output transition at time $t = 0$, the internal state is found (from Eq. (35) with $t_2 = 0$ and $t_1 = t$) as

$$x_2(0) = \eta(0) = \frac{\eta(t)}{1+\eta(t)\,[0-t]} = \frac{x_2(t)}{1+x_2(t)\,[0-t]}, \qquad (72)$$

which can be rewritten to obtain

$$x_2(t) = \frac{x_2(0)}{1+tx_2(0)} = \frac{-T^*_{OOT}}{1-tT^*_{OOT}} \quad \forall\, t < 0. \qquad (73)$$

The corresponding pre-actuation, inverse input (for $t < 0$) is given by $u(t) = -x_2^2(t)$ from Eq. (30) with $x_2(t)$ from Eq. (73). Thus, the OOT input (with pre-actuation only) is given by

$$u^*_{PRE}(t) = \begin{cases} -\left[\dfrac{T^*_{OOT}}{tT^*_{OOT}-1}\right]^2 & \text{if } t \le 0 \\ 1 & \text{if } 0 < t \le T^*_{OOT} \\ 0 & \text{if } T^*_{OOT} < t. \end{cases} \qquad (74)$$

4.8. Comparison of SST and OOT results

All three inputs: (i) optimal SST input without pre- and post-actuation (u_{SST} in Eq. (62) with $U = 1$); (ii) minimum-time OOT input with post-actuation (u_{POST} in Eq. (70)); and (ii) minimum-time OOT input with pre-actuation (u_{PRE} in Eq. (74)), shown in Fig. 5, achieve the desired output transition from $y = 0$ to $y = 1$ and maintain the output constant outside the output-transition time interval $[0, T]$. However, the output-transition time T during which the output is changing is smaller with the use of pre- and post-actuation (i.e., the OOT approach) than the case without the use of pre- or post-actuation (SST approach). For the example nonlinear system, the use of pre- and post-actuation substantially reduced the output-transition time from $T = T^*_{SST} = 2.2894$ with the SST approach to $T = T^*_{OOT} = 0.8177$ with the OOT approach.

4.9. Amount of pre- and post-actuation time

Pre-actuation can be used only if the time at which the output-transition is needed is known ahead of time. If such preview information (of the desired output) is available, then the advantage of the approach here is that the pre-actuation input converges to the steady-state value. Therefore, the pre-actuation input can be truncated in time—e.g., a truncation time of 4 s is used in Fig. 5. The resulting error in the optimal input can be made as small as desired if a sufficiently-large preview time is available.

If sufficient preview time of the needed output-transition is not available, then it is possible to allow only post-actuation without pre-actuation by constraining the initial set of possible initial internal states to be the equilibrium value, i.e., $\eta(0) = \underline{\eta}$ in the boundary values ψ of the internal state in Eq. (17). As seen in the current example, post-actuation alone (without pre-actuation) can lead to a substantial reduction of the output transition time. However, in general, the amount of transition-time reduction using post-actuation alone when compared to the use of both pre- and post-actuation will depend on the specific system.

The current article does not place a restriction on the amount of pre- and post-actuation time. It is possible to re-formulate the optimization problem with relative weights on the amount of pre- and post-actuation time. The main point of the article would still be valid, i.e., the ability to use pre- and post-actuation could reduce (but never increase) the output-transition time when compared to the standard approaches without pre- and post-actuation. Such weightings of pre- and post-actuation has been studied for linear systems in Iamratanakul et al. (2008), and could be investigated further for nonlinear systems in future studies.

5. Conclusion

The time-optimal output transition problem was investigated for invertible nonlinear systems. It was shown that the use of pre- and post-actuation input outside the output-transition interval can reduce the output-transition time beyond the standard bang–bang-type inputs for optimal state transition that do not use pre- or post-actuation.

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