Comments and Corrections

Corrections to "Stochastic Barbalat's Lemma and Its Applications"

Xin Yu and Zhaojing Wu

Abstract*—The proof of Theorem 1 (stochastic Barbalat's lemma) in the paper by Wu *et al.* is incorrect. This note provides a new statement of stochastic Barbalat's lemma. In addition, some definitions, propositions, and proofs are given to replace those in the paper by Wu *et al.

***Index Terms*—Stochastic Barbalat's lemma, stochastic systems.**

I. The Error

There exists an error in the proof of [1, Theorem 1], which is pointed out as follows. In [1, eq. (17)]

$$P\left\{\left|\phi(s)-\phi\left(T^i_{2\varepsilon_1}\right)\right|\leq\varepsilon\right\}\geq 1-\epsilon,$$

is incorrect. This is because $T^i_{2\varepsilon_1}$ is a stopping time, which could be infinite on sample point $\omega\notin G$ (where G is defined below [1, eq. (14)]). If $P\{T^i_{2\varepsilon_1}<\infty\}=1$ in [1], then (17) is correct. However, since the probability of $\{T^i_{2\varepsilon_1}<\infty\}$ does not equal to 1, $\phi(T^i_{2\varepsilon_1})$ is undefined on $\{T^i_{2\varepsilon_1}=\infty\}$ and [1, eq. (17)] is therefore incorrect.

To correct this error, we introduce a new definition of uniform continuity in probability, which is named as strongly uniform continuity in probability, and with its help, a new statement of stochastic Barbalat's lemma is presented.

To maintain consistency of this manuscript with Wu's work in [1], all notations are the same as in [1].

II. The Fix

A. The Definitions and Proposition

We first introduce a new definition to replace Definition 3 in [1].

Definition 1: Stochastic process $\phi(t):R_+\times\Omega\to R^n$ is strongly uniformly continuous in probability, if for any parameters $\epsilon,\varepsilon>0$, there exists a constant $\delta=\delta(\epsilon,\varepsilon)>0$ such that for any stopping time $\tau\geq 0$, the following condition holds:

$$P\left\{\{\tau<\infty\}\cap\left\{\sup_{0\leq s\leq\delta}|\phi(\tau+s)-\phi(\tau)|\geq\varepsilon\right\}\right\}\leq\epsilon. \quad (1)$$

In the following remarks, we show that strongly uniform continuity in probability is well defined.

Remark 1: $\phi(\tau)$ and $\phi(\tau+s)$ are well defined on $\{\tau<\infty\}$. □

Remark 2: If $\phi(t)$ is strongly uniformly continuous in probability, then $\phi(t)$ is uniformly continuous in probability (see Definition 3 in [1]).

The argument is as follows. If we let the stopping time, τ, in (1) to be any deterministic time $t\in R_+$ (deterministic constants are special stopping times, e.g., see Definition 2.30 of [2]), then $\{t<\infty\}=\Omega$ and (1) degenerates to

$$P\left\{\sup_{0\leq s\leq\delta}|\phi(t+s)-\phi(t)|\geq\varepsilon\right\}\leq\epsilon,\quad\text{for any } t\in R_+. \quad (2)$$

With the help of (2), the definition of uniform continuity in probability can be deduced easily. □

Next, we give the definition of locally strongly uniform continuity in probability to replace [1, Def. 4].

Definition 2: Stochastic process $\phi(t):R_+\times\Omega\to R^n$ is locally strongly uniformly continuous in probability, if for any parameters $\epsilon,\varepsilon,r>0$, there exists a $\delta=\delta(\epsilon,\varepsilon,r)>0$, such that for any stopping time $\tau\geq 0$, the following inequality holds:

$$P\left\{\{\tau<\infty\}\cap\left\{\sup_{0\leq s\leq\delta}|\phi(\sigma_r\wedge(\tau+s))-\phi(\sigma_r\wedge\tau)|\geq\varepsilon\right\}\right\}\leq\epsilon \quad (3)$$

where $\sigma_r=\inf\{t\geq 0:|\phi(t)|\geq r\}$ with $\inf\emptyset=\infty$.

Remark 3: The locally strongly uniform continuity in probability is well defined, too.

For example, the frequently-used Winner process $w(t)$ is locally strongly uniformly continuous in probability. In fact, for any $r>0$ and any stopping time $\tau\geq 0$, define $\sigma_r=\inf\{t\geq 0:|w(t)|\geq r\}$. By Doob's martingale inequality ([3, Theorem 3.2.4]), one has

$$\begin{aligned}&E\left\{1_{\{\sigma_r\wedge\tau<\infty\}}\sup_{0\leq s\leq t}|w(\sigma_r\wedge(\tau+s))-w(\sigma_r\wedge\tau)|^2\right\}\\&=E\left\{1_{\{\sigma_r\wedge\tau<\infty\}}\sup_{0\leq s\leq t}\left|\int_{\sigma_r\wedge\tau}^{\sigma_r\wedge(\tau+s)}dw\right|^2\right\}\\&\leq 4E\left\{1_{\{\sigma_r\wedge\tau<\infty\}}\int_{\sigma_r\wedge\tau}^{\sigma_r\wedge(\tau+t)}ds\right\}\\&\leq 4t\end{aligned} \quad (4)$$

where 1_A denotes the indicator function of subset A. Then for any $\varepsilon>0$, using Chebyshev's inequality, we have

$$P\left\{\{\sigma_r\wedge\tau<\infty\}\cap\left\{\sup_{0\leq s\leq t}|w(\sigma_r\wedge(\tau+s))-w(\sigma_r\wedge\tau)|\geq\varepsilon\right\}\right\}\leq\frac{4t}{\varepsilon^2}. \quad (5)$$

So, for any $\varepsilon,\epsilon>0$, one can choose $\delta=\delta(\epsilon,\varepsilon)>0$ such that $\delta\leq(\varepsilon^2\epsilon/4)$, then

$$P\left\{\{\sigma_r\wedge\tau<\infty\}\cap\left\{\sup_{0\leq s\leq\delta}|w(\sigma_r\wedge(\tau+s))-w(\sigma_r\wedge\tau)|\geq\varepsilon\right\}\right\}\leq\epsilon. \quad (6)$$

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X. Yu is with the School of Electrical and Information Engineering, Jiangsu University, Zhenjiang, Jiangsu Province 212013, China (e-mail: yuxin218@163.com; xyu@ujs.edu.cn).

Z. Wu is with School of Mathematics and Informational Science, Yantai University, Yantai, Shandong Province 264005, China (e-mail: wuzhaojing00@188.com).

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Since $\{\tau < \infty\} \subseteq \{\sigma_r \wedge \tau < \infty\}$, we obtain the desired inequality

$$P\left\{\{\tau<\infty\}\cap\left\{\sup_{0\le s\le\delta}|w(\sigma_r\wedge(\tau+s))-w(\sigma_r\wedge\tau)|\ge\varepsilon\right\}\right\}\le\epsilon \quad (7)$$

which shows that $w(t)$ is locally strongly uniformly continuous in probability. □

The following proposition being a replacement of Proposition 2 of [1], casts light on that locally strongly uniform continuity in probability and strong boundedness in probability imply strongly uniform continuity in probability.

Proposition 1: Stochastic process $\phi(t)$ is strongly uniformly continuous in probability if $\phi(t)$ is locally strongly uniformly continuous in probability and strongly bounded in probability.

Proof: Since $\phi(t)$ is strongly bounded in probability, then for any $\epsilon > 0$ there exists an $r_1 = r_1(\epsilon) > 0$ such that

$$P\left\{\sup_{t\ge0}|\phi(t)| > \frac{r_1}{2}\right\} \le \frac{\epsilon}{2}. \quad (8)$$

From the locally strongly uniform continuity of $\phi(t)$, for any $\varepsilon > 0$ and above ϵ, r_1, there exists a constant $\delta = \delta(\varepsilon, \epsilon, r_1(\epsilon)) = \delta(\varepsilon, \epsilon) > 0$, such that for any stopping time $\tau \ge 0$

$$P\left\{\{\tau<\infty\}\cap\left\{\sup_{0\le s\le\delta}|\phi(\sigma_{r_1}\wedge(\tau+s))-\phi(\sigma_{r_1}\wedge\tau)|\ge\varepsilon\right\}\right\}\le\frac{\epsilon}{2} \quad (9)$$

where $\sigma_{r_1} = \inf\{t \ge 0 : |\phi(t)| \ge r_1\}$ with $\inf\emptyset = \infty$. Therefore, by (8) and (9), one has

$$\begin{aligned}
&P\left\{\{\tau<\infty\}\cap\left\{\sup_{0\le s\le\delta}|\phi(\tau+s)-\phi(\tau)|\ge\varepsilon\right\}\right\}\\
&= P\left\{\{\tau<\infty\}\cap\left\{\sup_{t\ge0}|\phi(t)|\le\frac{r_1}{2}\right\}\right.\\
&\qquad\left.\cap\left\{\sup_{0\le s\le\delta}|\phi(\tau+s)-\phi(\tau)|\ge\varepsilon\right\}\right\}\\
&\quad + P\left\{\{\tau<\infty\}\cap\left\{\sup_{t\ge0}|\phi(t)|>\frac{r_1}{2}\right\}\right.\\
&\qquad\left.\cap\left\{\sup_{0\le s\le\delta}|\phi(\tau+s)-\phi(\tau)|\ge\varepsilon\right\}\right\}\\
&\le P\{\{\tau<\infty\}\cap\{\sigma_{r_1}=\infty\}\\
&\qquad\left.\cap\left\{\sup_{0\le s\le\delta}|\phi(\tau+s)-\phi(\tau)|\ge\varepsilon\right\}\right\}\\
&\quad + P\left\{\sup_{t\ge0}|\phi(t)|>\frac{r_1}{2}\right\}\\
&= P\{\{\tau<\infty\}\cap\{\sigma_{r_1}=\infty\}\\
&\qquad\left.\cap\left\{\sup_{0\le s\le\delta}|\phi(\sigma_{r_1}\wedge(\tau+s))-\phi(\sigma_{r_1}\wedge\tau)|\ge\varepsilon\right\}\right\}\\
&\quad + P\left\{\sup_{t\ge0}|\phi(t)|>\frac{r_1}{2}\right\}\\
&\le P\left\{\{\tau<\infty\}\cap\left\{\sup_{0\le s\le\delta}|\phi(\sigma_{r_1}\wedge(\tau+s))\right.\right.\\
&\qquad\qquad -\phi(\sigma_{r_1}\wedge\tau)|\ge\varepsilon\}\}+\frac{\epsilon}{2}\\
&\le \epsilon.
\end{aligned} \quad (10)$$

So, $\phi(t)$ is strongly uniformly continuous in probability. ■

We reuse the definition of strongly boundedness in probability in [1], and the following remark shows that the definition of strongly boundedness in probability actually equals to the definition of almost sure boundedness given by [4].

Remark 4: Stochastic process $\phi(t)$ is strongly bounded in probability, if and only if $\sup_{t\ge0}|\phi(t)| < \infty$ *a.s.*.

In fact, for any $r_1 \ge r_2$, as $\{\sup_{t\ge0}|\phi(t)| > r_1\} \subseteq \{\sup_{t\ge0}|\phi(t)| > r_2\}$, $\phi(t)$ being strongly bounded in probability (see [1, Def. 2]) equals to that

$$\lim_{r\to\infty} P\left\{\sup_{t\ge0}|\phi(t)| > r\right\} = 0. \quad (11)$$

Noting that

$$\begin{aligned}
P\left\{\sup_{t\ge0}|\phi(t)|=\infty\right\} &= P\left\{\cap_{n=1}^{\infty}\left\{\sup_{t\ge0}|\phi(t)|>n\right\}\right\}\\
&= \lim_{n\to\infty} P\left\{\sup_{t\ge0}|\phi(t)|>n\right\}\\
&= \lim_{r\to\infty} P\left\{\sup_{t\ge0}|\phi(t)|>r\right\}
\end{aligned} \quad (12)$$

one can prove that $\phi(t)$ is strongly bounded in probability if and only if $\sup_{t\ge0}|\phi(t)| < \infty$ *a.s.* □

Some criterions on diffusion process being "strongly boundedness in probability" or "almost sure boundedness" are presented in [1], [4], and [5].

B. Stochastic Barbalat's Lemma

Now we give a new statement of stochastic Barbalat's lemma to replace Theorem 1 in [1].

Theorem 1 (Stochastic Barbalat's Lemma): If a continuous adapted process $\phi(t) : R_+ \times \Omega \to R^n$ is strongly uniformly continuous in probability and $E\int_0^\infty |\phi(t)|dt < \infty$, then $\lim_{t\to\infty}\phi(t) = 0$ *a.s.*

Proof: At first, we prove that $|\phi(t)|$ is strongly uniformly continuous in probability. Noting that the inequality $||x| - |y|| \le |x - y|$, one has $\{\{\tau < \infty\} \cap \sup_{0\le s\le\delta}||\phi(\tau+s)| - |\phi(\tau)|| \ge \varepsilon\} \subseteq \{\{\tau < \infty\} \cap \sup_{0\le s\le\delta}|\phi(\tau+s) - \phi(\tau)| \ge \varepsilon\}$ for any stopping times $\tau \ge 0$. So, $|\phi(t)|$ is strongly uniformly continuous in probability if $\phi(t)$ is strongly uniformly continuous in probability.

The sample space Ω can be decomposed into the following three mutually exclusive events:

$$\begin{aligned}
A_1 &= \left\{\limsup_{t\to\infty}|\phi(t)| = 0\right\}\\
A_2 &= \left\{\liminf_{t\to\infty}|\phi(t)| > 0\right\}\\
A_3 &= \left\{\liminf_{t\to\infty}|\phi(t)| = 0 \text{ and } \limsup_{t\to\infty}|\phi(t)| > 0\right\}.
\end{aligned} \quad (13)$$

To prove $\lim_{t\to\infty}\phi(t) = 0$ *a.s.*, noting that $\lim_{t\to\infty}\phi(t) = 0$ *a.s.* is equivalent to $\lim_{t\to\infty}|\phi(t)| = 0$ *a.s.*, we only have to show that $P\{A_2\} = P\{A_3\} = 0$.

1) Since

$$E\int_0^\infty |\phi(t)|\,dt < \infty, \quad (14)$$

one has

$$\int_0^\infty |\phi(t)|\,dt < \infty, \quad a.s., \quad (15)$$

then

$$\liminf_{t\to\infty}|\phi(t)| = 0, \quad a.s. \quad (16)$$

Therefore, $P\{A_2\} = 0$ is obvious.

2) Now, we turn to proving that $P\{A_3\} = 0$ by contradiction. Suppose $P\{A_3\} > 0$, then there exist $\varepsilon_1 > 0$ and $\epsilon_0 > 0$ such that

$$P\{A_0\} \ge \epsilon_0 \quad (17)$$

where

$$A_0 = \{|\phi(t)| \text{ crosses from below } \varepsilon_1 \text{ to above } 2\varepsilon_1 \text{ and back infinitely many times}\}. \quad (18)$$

Noting that $|\phi(t)|$ is a continuous $\mathcal{F}_t$ adapted process, we can define the following stopping times:

$$\begin{aligned} T_{\varepsilon_1}^1 &= \inf\{t \ge 0 : |\phi(t)| \in [0, \varepsilon_1]\}, \\ T_{2\varepsilon_1}^1 &= \inf\left\{t \ge T_{\varepsilon_1}^1 : |\phi(t)| \notin [0, 2\varepsilon_1]\right\} \\ T_{\varepsilon_1}^i &= \inf\left\{t \ge T_{2\varepsilon_1}^{i-1} : |\phi(t)| \in [0, \varepsilon_1]\right\} \\ T_{2\varepsilon_1}^i &= \inf\left\{t \ge T_{\varepsilon_1}^i : |\phi(t)| \notin [0, 2\varepsilon_1]\right\}, \quad i = 2, 3, \cdots. \end{aligned}$$

For any i, on the sample set A_0, it is obvious that $T_{\varepsilon_1}^i, T_{2\varepsilon_1}^i < \infty$, and by the continuity of $|\phi(\cdot)|$, one has that $T_{\varepsilon_1}^i, T_{2\varepsilon_1}^i \to \infty$, as $i \to \infty$. From (16), one also knows that $T_{\varepsilon_1}^{i+1} < \infty$ whenever $T_{2\varepsilon_1}^i < \infty$. Then by (14), one has

$$\begin{aligned} \infty > E\int_0^\infty |\phi(t)|\, dt &\ge \sum_{i=1}^{\infty} E\left\{1_{\left\{T_{2\varepsilon_1}^i < \infty\right\}} \int_{T_{2\varepsilon_1}^i}^{T_{\varepsilon_1}^{i+1}} |\phi(t)|\, dt\right\} \\ &\ge \sum_{i=1}^{\infty} \varepsilon_1 E\left\{1_{\left\{T_{2\varepsilon_1}^i < \infty\right\}} \left(T_{\varepsilon_1}^{i+1} - T_{2\varepsilon_1}^i\right)\right\}. \end{aligned} \quad (19)$$

From the strongly uniform continuity in probability of $|\phi(t)|$, for any $\varepsilon, \epsilon > 0$, there is a constant $\delta = \delta(\varepsilon, \epsilon) > 0$ such that

$$P\left\{\left\{T_{2\varepsilon_1}^i < \infty\right\} \cap \left\{\sup_{0 \le s \le \delta} \left|\left|\phi\left(T_{2\varepsilon_1}^i + s\right)\right| - \left|\phi\left(T_{2\varepsilon_1}^i\right)\right|\right| \ge \varepsilon\right\}\right\} \le \epsilon. \quad (20)$$

If taking $\varepsilon = \varepsilon_1/2$, $\epsilon = \epsilon_0/2$, then there exists a constant $\delta^* = \delta^*(\varepsilon_1/2, \epsilon_0/2) > 0$ such that

$$P\left\{\left\{T_{2\varepsilon_1}^i < \infty\right\} \cap \left\{\sup_{0 \le s \le \delta^*} \left|\left|\phi\left(T_{2\varepsilon_1}^i + s\right)\right| - \left|\phi\left(T_{2\varepsilon_1}^i\right)\right|\right| \ge \frac{\varepsilon_1}{2}\right\}\right\} \le \frac{\epsilon_0}{2} \quad (21)$$

which together with

$$P\left\{T_{2\varepsilon_1}^i < \infty\right\} \ge P\{A_0\} \ge \epsilon_0, \quad (22)$$

leads to

$$\begin{aligned} &P\left\{\left\{T_{2\varepsilon_1}^i < \infty\right\} \cap \left\{\sup_{0 \le s \le \delta^*} \left|\left|\phi\left(T_{2\varepsilon_1}^i + s\right)\right| - \left|\phi\left(T_{2\varepsilon_1}^i\right)\right|\right| < \frac{\varepsilon_1}{2}\right\}\right\} \\ &= P\left\{T_{2\varepsilon_1}^i < \infty\right\} \\ &\quad - P\left\{\left\{T_{2\varepsilon_1}^i < \infty\right\} \cap \left\{\sup_{0 \le s \le \delta^*} \left|\left|\phi\left(T_{2\varepsilon_1}^i + s\right)\right| - \left|\phi\left(T_{2\varepsilon_1}^i\right)\right|\right| \ge \frac{\varepsilon_1}{2}\right\}\right\} \\ &\ge \frac{\epsilon_0}{2}. \end{aligned} \quad (23)$$

Then $\{T_{2\varepsilon_1}^i < \infty\} \cap G_i$ is not zero probability set, where

$$G_i = \left\{\sup_{0 \le s \le \delta^*} \left\|\phi\left(T_{2\varepsilon_1}^i + s\right)| - |\phi\left(T_{2\varepsilon_1}^i\right)\right\| < \frac{\varepsilon_1}{2}\right\}. \quad (24)$$

On the sample set $\{T_{2\varepsilon_1}^i < \infty\} \cap G_i$, the inequality

$$T_{\varepsilon_1}^{i+1} - T_{2\varepsilon_1}^i > \delta^* \quad (25)$$

always holds. Otherwise, if $T_{\varepsilon_1}^{i+1} - T_{2\varepsilon_1}^i \le \delta^*$ for some $\omega_0 \in \{T_{2\varepsilon_1}^i < \infty\} \cap G_i$, then

$$\sup_{0 \le s \le \delta^*} \left\|\phi\left(T_{2\varepsilon_1}^i + s\right)| - |\phi\left(T_{2\varepsilon_1}^i\right)\right\| \ge \left\|\phi\left(T_{\varepsilon_1}^{i+1}\right)| - |\phi\left(T_{2\varepsilon_1}^i\right)\right\| = \varepsilon_1 \quad (26)$$

on ω_0. Since $\omega_0 \in G_i$, (26) is a contradiction to the definition of G_i.

So, from (19), (23) and (25), one has

$$\begin{aligned} \infty > E\int_0^\infty |\phi(t)|\, dt &\ge \sum_{i=1}^{\infty} \varepsilon_1 E\left\{1_{\left\{T_{2\varepsilon_1}^i < \infty\right\}} \left(T_{\varepsilon_1}^{i+1} - T_{2\varepsilon_1}^i\right)\right\} \\ &\ge \sum_{i=1}^{\infty} \varepsilon_1 E\left\{1_{\left\{\left\{T_{2\varepsilon_1}^i < \infty\right\} \cap G_i\right\}} \left(T_{\varepsilon_1}^{i+1} - T_{2\varepsilon_1}^i\right)\right\} \\ &\ge \sum_{i=1}^{\infty} \varepsilon_1 \delta^* P\left\{\left\{T_{2\varepsilon_1}^i < \infty\right\} \cap G_i\right\} \\ &\ge \varepsilon_1 \delta^* \sum_{i=1}^{\infty} \frac{\epsilon_0}{2} \\ &= \infty \end{aligned} \quad (27)$$

which leads to a contradiction. This yields that $P\{A_3\} = 0$ and $\lim_{t\to\infty} \phi(t) = 0$ *a.s.* ■

III. The Proofs of Other Main Results

With the proposal of the new stochastic Barbalat's lemma, some changes are required for the proofs of other main results of [1].

Consider the stochastic dynamic system

$$dx = f(x,t)dt + g(x,t)dW(t), \quad x(0) = x_0 \in R^n \quad (28)$$

where $x \in R^n$ is the state, $W(t)$ is an m-dimensional standard Wiener process defined in $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$. The Borel measurable functions $f : R^n \times R_+ \to R^n$ and $g : R^n \times R_+ \to R^{n\times m}$ are piecewise continuous in t and locally bounded and locally Lipschitz continuous in x uniformly in $t \in R_+$.

System (28) has a unique strong solution on $[0, \sigma_\infty)$ with σ_∞ being the explosion time of $x(t)$ (e.g., see [4, Lemma 1]).

Remark 5: The solution, $x(t)$, of system (28) is well defined in $t \in R_+$ if it is strongly bounded in probability.

From Remark 4, we obtain that solution $x(t)$ is strongly bounded in probability if and only if $\sup_{t\ge0} |x(t)| < \infty$ *a.s.* So, if $x(t)$ is strongly bounded in probability, one has $\sup_{t\ge0} |x(t)| < \infty$ *a.s.*. Then, the explosion time $\sigma_\infty = \infty$ *a.s.*, and solution $x(t)$ is well defined in $t \in R_+$. □

The following propositions are the replacements of [1, Propositions 4–6].

Proposition 2: If a stochastic process $x(t)$ is locally strongly uniformly continuous in probability and strongly bounded in probability, and $\beta(x)$ is a continuous function, then $\beta(x(t))$ is strongly uniformly continuous in probability.

Proof: Since $x(t)$ is strongly bounded in probability, then for any $\epsilon > 0$ there exists an $r_1 = r_1(\epsilon) > 0$ such that

$$P\left\{\sup_{t\geq 0}|x(t)| > \frac{r_1}{2}\right\} \leq \frac{\epsilon}{2}. \tag{29}$$

Since $\beta(x)$ is continuous, then $\beta(x)$ is uniformly continuous in the closed ball $B_{r_1/2} = \{|x| \leq r_1/2\}$. That is, for any $\varepsilon > 0$, there exists an $\eta_1 = \eta_1(\varepsilon) > 0$ such that

$$\begin{aligned}&|\beta(x) - \beta(y)| < \frac{\varepsilon}{2},\\ &\text{for any } |x-y| < \eta_1, \quad x, y \in B_{\frac{r_1}{2}}.\end{aligned} \tag{30}$$

From the local strongly uniform continuity in probability of $x(t)$, for $\epsilon, \eta_1 = \eta_1(\varepsilon), r_1 = r_1(\epsilon)$ given above, there exists a $\delta = \delta(\epsilon, \eta_1, r_1) = \delta(\epsilon, \varepsilon) > 0$, such that for any stopping time $\tau \geq 0$, there holds

$$P\left\{\{\tau<\infty\}\cap\left\{\sup_{0\leq s\leq\delta}|x(\sigma_{r_1}\wedge(\tau+s))-x(\sigma_{r_1}\wedge\tau)|\geq\eta_1\right\}\right\}\leq\frac{\epsilon}{2} \tag{31}$$

where $\sigma_{r_1} = \inf\{t \geq 0 : |x(t)| \geq r_1\}$ with $\inf\emptyset = \infty$. Then, by (29)–(31), one has

$$\begin{aligned}
&P\left\{\{\tau < \infty\} \cap \left\{\sup_{0\leq s\leq\delta}|\beta(x(\tau+s)) - \beta(x(\tau))| \geq \varepsilon\right\}\right\}\\
&= P\left\{\{\tau < \infty\} \cap \left\{\sup_{t\geq 0}|x(t)| \leq \frac{r_1}{2}\right\}\right.\\
&\qquad\left.\cap\left\{\sup_{0\leq s\leq\delta}|\beta(x(\tau+s)) - \beta(x(\tau))| \geq \varepsilon\right\}\right\}\\
&\quad + P\left\{\{\tau < \infty\} \cap \left\{\sup_{t\geq 0}|x(t)| > \frac{r_1}{2}\right\}\right.\\
&\qquad\left.\cap\left\{\sup_{0\leq s\leq\delta}|\beta(x(\tau+s)) - \beta(x(\tau))| \geq \varepsilon\right\}\right\}\\
&\leq P\left\{\{\tau < \infty\} \cap \left\{\sup_{t\geq 0}|x(t)| \leq \frac{r_1}{2}\right\}\right.\\
&\qquad\left.\cap\left\{\sup_{0\leq s\leq\delta}|x(\tau+s) - x(\tau)| \geq \eta_1\right\}\right\}\\
&\quad + P\left\{\sup_{t\geq 0}|x(t)| > \frac{r_1}{2}\right\}\\
&\leq P\{\{\tau < \infty\} \cap \{\sigma_{r_1} = \infty\}\\
&\qquad\left.\cap\left\{\sup_{0\leq s\leq\delta}|x(\tau+s) - x(\tau)| \geq \eta_1\right\}\right\} + \frac{\epsilon}{2}\\
&= P\{\{\tau < \infty\} \cap \{\sigma_{r_1} = \infty\}\\
&\qquad\left.\cap\left\{\sup_{0\leq s\leq\delta}|x(\sigma_{r_1}\wedge(\tau+s)) - x(\sigma_{r_1}\wedge\tau)| \geq \eta_1\right\}\right\} + \frac{\epsilon}{2}\\
&\leq P\left\{\{\tau < \infty\} \cap \left\{\sup_{0\leq s\leq\delta}|x(\sigma_{r_1}\wedge(\tau+s))\right.\right.\\
&\qquad\qquad -x(\sigma_{r_1}\wedge\tau)| \geq \eta_1\}\} + \frac{\epsilon}{2}\\
&\leq \epsilon.
\end{aligned} \tag{32}$$

So, $\beta(x(t))$ is strongly uniformly continuous in probability. ■

Proposition 3: The diffusion process $x(t)$ given by system (28) is locally strongly uniformly continuous in probability.

Proof: For any $r > 0$, since functions f and g are locally bounded in x uniformly in $t \in R_+$, one can define two functions $\rho_1(r)$ and $\rho_2(r)$ as

$$\begin{aligned}\rho_1(r) &= \max_{|x|\leq r}\sup_{t\geq 0}|f(x,t)|,\\ \rho_2(r) &= \max_{|x|\leq r}\sup_{t\geq 0}|g(x,t)|.\end{aligned} \tag{33}$$

Set $\sigma_r = \inf\{t \geq 0 : |x(t)| \geq r\}$, then σ_r increases and tends to σ_∞ *a.s.* as $r \to \infty$. For any stoping time $\tau \geq 0$, by Doob's martingale inequality, we compute

$$\begin{aligned}
&E\left\{1_{\{\sigma_r\wedge\tau<\infty\}}\sup_{0\leq s\leq h}|x(\sigma_r\wedge(\tau+s)) - x(\sigma_r\wedge\tau)|^2\right\}\\
&= E\left\{1_{\{\sigma_r\wedge\tau<\infty\}}\sup_{0\leq s\leq h}\right.\\
&\qquad\left.\times\left|\int_{\sigma_r\wedge\tau}^{\sigma_r\wedge(\tau+s)} f(x(t),t)dt + \int_{\sigma_r\wedge\tau}^{\sigma_r\wedge(\tau+s)} g(x(t),t)dW(t)\right|^2\right\}\\
&\leq 2E\left\{1_{\{\sigma_r\wedge\tau<\infty\}}\sup_{0\leq s\leq h}\left|\int_{\sigma_r\wedge\tau}^{\sigma_r\wedge(\tau+s)} f(x(t),t)\,dt\right|^2\right\}\\
&\quad + 2E\left\{1_{\{\sigma_r\wedge\tau<\infty\}}\sup_{0\leq s\leq h}\left|\int_{\sigma_r\wedge\tau}^{\sigma_r\wedge(\tau+s)} g(x(t),t)\,dW(t)\right|^2\right\}\\
&\leq 2hE\left\{1_{\{\sigma_r\wedge\tau<\infty\}}\int_{\sigma_r\wedge\tau}^{\sigma_r\wedge(\tau+h)}|f(x(t),t)|^2\,dt\right\}\\
&\quad + 8E\left\{1_{\{\sigma_r\wedge\tau<\infty\}}\int_{\sigma_r\wedge\tau}^{\sigma_r\wedge(\tau+h)}|g(x(t),t)|^2\,dt\right\}\\
&\leq 2h^2\rho_1^2(r) + 8h\rho_2^2(r).
\end{aligned} \tag{34}$$

For any $\varepsilon > 0$, by Chebyshev's inequality, it follows from (34) that

$$\begin{aligned}&P\left\{\{\sigma_r\wedge\tau<\infty\}\cap\left\{\sup_{0\leq s\leq h}|x(\sigma_r\wedge(\tau+s))-x(\sigma_r\wedge\tau)|\geq\varepsilon\right\}\right\}\\ &\qquad\leq\frac{2h^2\rho_1^2(r)+8h\rho_2^2(r)}{\varepsilon^2}.\end{aligned} \tag{35}$$

Since $\{\tau < \infty\} \subseteq \{\sigma_r \wedge \tau < \infty\}$, then

$$\begin{aligned}&P\left\{\{\tau<\infty\}\cap\left\{\sup_{0\leq s\leq h}|x(\sigma_r\wedge(\tau+s))-x(\sigma_r\wedge\tau)|\geq\varepsilon\right\}\right\}\\ &\qquad\leq\frac{2h^2\rho_1^2(r)+8h\rho_2^2(r)}{\varepsilon^2}.\end{aligned} \tag{36}$$

For $\varepsilon, r > 0$ and any $\epsilon > 0$ mentioned above, one can find $h = h(\epsilon, \varepsilon, r) > 0$ such that $(2h^2\rho_1^2(r) + 8h\rho_2^2(r))/\varepsilon^2 \leq \epsilon$. So, choosing $\delta = h(\epsilon, \varepsilon, r)$, we have

$$P\left\{\{\tau<\infty\}\cap\left\{\sup_{0\leq s\leq\delta}|x(\sigma_r\wedge(\tau+s))-x(\sigma_r\wedge\tau)|\geq\varepsilon\right\}\right\}\leq\epsilon \tag{37}$$

which shows that $x(t)$ is locally strongly uniformly continuous in probability. ■

Proposition 4: The diffusion process $x(t)$ given by system (28) is strongly uniformly continuous in probability if it is strongly bounded in probability.

Proof: It is obvious from Proposition 1 and Proposition 3 that $x(t)$ is strongly uniformly continuous in probability. ■

Now, with the help of Theorem 1 and above Propositions, Theorem 2 in [1] can be proved.

Theorem 2: If the solution process $x(t)$ of system (28) is strongly bounded in probability, and $E\int_0^\infty|\beta(x(t))|dt < \infty$, where $\beta(\cdot)$ is a continuous function, then $\lim_{t\to\infty}\beta(x(t)) = 0$, *a.s.*

Proof: It follows from Remark 5 that the solution process $x(t)$ is well defined in $t \in R_+$. Since $x(t)$ is a continuous adaptive process, so is $\beta(x(t))$. From $x(t)$ being strongly bounded in probability, one knows that $\beta(x(t))$ is strongly uniformly continuous in probability by Propositions 2 and 3. Therefore, one has that $\lim_{t\to\infty}\beta(x(t)) = 0$, *a.s.* by the use of Theorem 1. ■

In a similar procedure, Theorems 3 and 4 of [1] (or Lemma 3 of [4], Lemma 1 of [5]) can also be proved by Theorem 1, which shows that some results on stochastic stability can follow from our stochastic Barbalat's lemma.

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