Corrections to “Stochastic Barbalat’s Lemma and Its Applications”

Xin Yu and Zhaojing Wu

Abstract—The proof of Theorem 1 (stochastic Barbalat’s lemma) in the paper by Wu et al. is incorrect. This note provides a new statement of stochastic Barbalat’s lemma. In addition, some definitions, propositions, and proofs are given to replace those in the paper by Wu et al.

Index Terms—Stochastic Barbalat’s lemma, stochastic systems.

I. THE ERROR

There exists an error in the proof of [1, Theorem 1], which is pointed out as follows. In [1, eq. (17)]

\[ P \left\{ \phi(t) \leq \phi(T_{2\varepsilon}) \right\} \geq 1 - \varepsilon, \]

is incorrect. This is because \( T_{2\varepsilon} \) is a stopping time, which could be infinite on sample point \( \omega \not\in G \) (where \( G \) is defined below [1, eq. (14)]). If \( P[T_{2\varepsilon} < \infty] = 1 \) in [1], then (17) is correct. However, since the probability of \( \{T_{2\varepsilon} < \infty\} \) does not equal to 1, \( \phi(T_{2\varepsilon}) \) is undefined on \( \{T_{2\varepsilon} = \infty\} \) and [1, eq. (17)] is therefore incorrect.

To correct this error, we introduce a new definition of uniform continuity in probability, which is named as strongly uniform continuity in probability, and with its help, a new statement of stochastic Barbalat’s lemma is presented.

To maintain consistency of this manuscript with Wu’s work in [1], all notations are the same as in [1].

II. THE FIX

A. The Definitions and Proposition

We first introduce a new definition to replace Definition 3 in [1].

Definition 1: Stochastic process \( \phi(t) : R_+ \times \Omega \to R^n \) is strongly uniformly continuous in probability, if for any parameters \( \varepsilon, \delta > 0 \), there exists a constant \( \varepsilon' > 0 \) such that for any stopping time \( \tau \geq 0 \), the following holds:

\[ P \left\{ \{\tau < \infty\} \cap \left\{ \sup_{0 \leq s \leq \delta} |\phi(\tau + s) - \phi(\tau)| \geq \varepsilon' \right\} \right\} \leq \varepsilon. \]  

In the following remarks, we show that strongly uniform continuity in probability is well defined.

Remark 1: \( \phi(\tau) \) and \( \phi(\tau + \varepsilon) \) are well defined on \( \{\tau < \infty\} \).

Remark 2: If \( \phi(t) \) is strongly uniformly continuous in probability, then \( \phi(t) \) is uniformly continuous in probability (see Definition 3 in [1]).

The argument is as follows. If we let the stopping time, \( \tau \), in (1) to be any deterministic time \( t \in R_+ \) (deterministic constants are special stopping times, e.g., see Definition 2.30 of [2]), then \( \{\tau < \infty\} = \Omega \) and (1) degenerates to

\[ P \left\{ \sup_{0 \leq s \leq \delta} |\phi(t + s) - \phi(t)| \geq \varepsilon \right\} \leq \varepsilon, \quad \text{for any } t \in R_+. \]  

With the help of (2), the definition of uniform continuity in probability can be deduced easily.

Next, we give the definition of locally strongly uniform continuity in probability to replace [1, Def. 4].

Definition 2: Stochastic process \( \phi(t) : R_+ \times \Omega \to R^n \) is locally strongly uniformly continuous in probability, if for any parameters \( \varepsilon, \delta, \tau > 0 \), there exists a \( \delta' = \delta(\varepsilon, \tau) > 0 \), such that for any stopping time \( \tau > 0 \), the following inequality holds:

\[ P \left\{ \{\tau < \infty\} \cap \left\{ \sup_{0 \leq s \leq \delta'} |\phi(\tau + s) - \phi(\tau)| \geq \varepsilon \right\} \right\} \leq \varepsilon \]  

where \( \sigma_\tau = \inf\{t > 0 : |\phi(t)| > \tau\} \) with \( \inf\Omega = \infty \).

Remark 3: The locally strongly uniform continuity in probability is well defined, too.

For example, the frequently-used Winner process \( \omega(t) \) is locally strongly uniformly continuous in probability. In fact, for any \( \tau > 0 \) and any stopping time \( \tau > 0 \), define \( \sigma_\tau = \inf\{t > 0 : |\omega(t)| > \tau\} \). By Doob’s martingale inequality ([3, Theorem 3.2.4]), one has

\[ E \left\{ 1_{\sigma_\tau \wedge \tau < \infty} \sup_{0 \leq s \leq \delta} |\phi(\sigma_\tau \wedge (\tau + s)) - \phi(\sigma_\tau \wedge \tau)|^2 \right\} \]

\[ = E \left\{ 1_{\sigma_\tau \wedge \tau < \infty} \sup_{0 \leq s \leq \delta} \int_{\sigma_\tau \wedge \tau}^{\sigma_\tau \wedge (\tau + s)} \left( \begin{array}{c} \sigma_\tau \wedge (\tau + s) \\ \sigma_\tau \wedge \tau \\ \nabla \phi(t) \end{array} \right)^2 dt \right\} \]

\[ \leq 4E \left\{ 1_{\sigma_\tau \wedge \tau < \infty} \sup_{0 \leq s \leq \delta} |\phi(\sigma_\tau \wedge (\tau + s)) - \phi(\sigma_\tau \wedge \tau)| \right\} \leq 4t \]  

where \( 1_A \) denotes the indicator function of subset \( A \). Then for any \( \varepsilon > 0 \), using Chebyshev’s inequality, we have

\[ P \left\{ \{\sigma_\tau \wedge \tau < \infty\} \cap \left\{ \sup_{0 \leq s \leq \delta} |\phi(\sigma_\tau \wedge (\tau + s)) - w(\sigma_\tau \wedge \tau)| \geq \varepsilon \right\} \right\} \leq \frac{4t}{\varepsilon^2} \]  

So, for any \( \varepsilon, \tau > 0 \), one can choose \( \delta = \delta(\varepsilon, \tau) > 0 \) such that \( \delta \leq (\varepsilon^2 / 4) \), then

\[ P \left\{ \{\sigma_\tau \wedge \tau < \infty\} \cap \left\{ \sup_{0 \leq s \leq \delta} |\phi(\sigma_\tau \wedge (\tau + s)) - w(\sigma_\tau \wedge \tau)| \geq \varepsilon \right\} \right\} \leq \varepsilon. \]  

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Since \( \{ \tau < \infty \} \subseteq \{ \sigma, \land \tau < \infty \} \), we obtain the desired inequality
\[
P \left\{ \bigcap_{t \leq t_0} \left[ \left( \sup_{t \leq t_0} \left| w(\sigma, \land (\tau + s)) - w(\sigma, \land \tau) \right| \right) \geq \varepsilon \right] \right\} \leq \varepsilon
\] (7)
which shows that \( w(t) \) is locally strongly uniformly continuous in probability.

The following proposition being a replacement of Proposition 2 of [1], casts light on that locally strongly uniform continuity in probability and strong boundedness in probability imply strongly uniform continuity in probability.

**Proposition 1:** Stochastic process \( \phi(t) \) is strongly uniformly continuous in probability if \( \phi(t) \) is locally strongly uniformly continuous in probability and strongly bounded in probability.

**Proof:** Since \( \phi(t) \) is strongly bounded in probability, then for any \( \varepsilon > 0 \) there exists an \( r_1 = r_1(\varepsilon) > 0 \) such that
\[
P \left\{ \sup_{t \geq t_0} |\phi(t)| > r_1 \right\} \leq \frac{\varepsilon}{2}.
\] (8)
From the locally strongly uniform continuity of \( \phi(t) \), for any \( \varepsilon > 0 \) and above \( r_1, r_2 \), there exists a constant \( \delta = \delta(\varepsilon, r_1, r_2, \varepsilon) > 0 \), such that for any stopping time \( \tau \geq 0 \)
\[
P \left\{ \left( \tau < \infty \right) \cap \left( \sup_{t \leq t_0} \left| \phi(\sigma, \land (\tau + s)) - \phi(\sigma, \land \tau) \right| \geq \varepsilon \right) \right\} \leq \frac{\varepsilon}{2}.
\] (9)
where \( \sigma_{r_1} = \inf \{ t > 0 : |\phi(t)| > r_1 \} \) with \( \inf \emptyset = \infty \). Therefore, by (8) and (9), one has
\[
P \left\{ \left( \tau < \infty \right) \cap \left( \sup_{t \leq t_0} \left| \phi(\sigma, \land (\tau + s)) - \phi(\sigma, \land \tau) \right| \geq \varepsilon \right) \right\} \leq \varepsilon.
\] (10)
In fact, for any \( r_1 \geq r_2, \) as \( \{ \sup_{t \geq t_0} |\phi(t)| > r_1 \} \subseteq \{ \sup_{t \geq t_0} |\phi(t)| > r_2 \} \), \( \phi(t) \) being strongly bounded in probability (see [1, Def. 2]) equals to that
\[
\lim_{r \to \infty} P \left\{ \sup_{t \geq t_0} |\phi(t)| > r \right\} = 0.
\] (11)
Noting that
\[
P \left\{ \sup_{t \geq t_0} |\phi(t)| = \infty \right\} = P \left\{ \left( \lim_{n \to \infty} \sup_{t \geq t_0} |\phi(t)| > n \right) \right\} \leq \lim_{n \to \infty} P \left\{ \sup_{t \geq t_0} |\phi(t)| > n \right\} = \lim_{n \to \infty} P \left\{ \sup_{t \geq t_0} |\phi(t)| > r \right\} = 0.
\] (12)
one can prove that \( \phi(t) \) is strongly bounded in probability if and only if \( \sup_{t \geq t_0} |\phi(t)| < \infty \) a.s.

Some criterions on diffusion process being “strongly boundedness in probability” or almost sure boundedness” are presented in [1, 4, and 5].

**B. Stochastic Barbalat’s Lemma**

Now we give a new statement of stochastic Barbalat’s lemma to replace Theorem 1 in [1].

**Theorem 1 (Stochastic Barbalat’s Lemma):** If a continuous adapted process \( \phi(t) : R_+ \times \Omega \to R^n \) is strongly uniformly continuous in probability and \( E \int_0^\infty |\phi(t)| dt < \infty \), then \( \lim_{t \to \infty} \phi(t) = 0 \) a.s.

**Proof:** At first, we prove that \( |\phi(t)| \) is strongly uniformly continuous in probability. Noting the inequality \( |x| - |y| < |x - y| \), one has \( \{ \tau < \infty \} \cap \sup_{t \geq t_0} \{ |\phi(\sigma, \land (\tau + s)) - \phi(\sigma, \land \tau)| \geq \varepsilon \} \subseteq \{ \tau < \infty \} \cap \sup_{t \geq t_0} \{ |\phi(\sigma, \land (\tau + s)) - \phi(\tau)| \geq \varepsilon \} \) for any stopping times \( \tau \geq 0 \). So, \( |\phi(t)| \) is strongly uniformly continuous in probability if \( \phi(t) \) is strongly uniformly continuous in probability.

The sample space \( \Omega \) can be decomposed into the following three mutually exclusive events:
\[
A_1 = \{ \lim_{t \to \infty} \sup_{t \geq t_0} |\phi(t)| = 0 \},
\]
\[
A_2 = \{ \lim_{t \to \infty} \inf_{t \geq t_0} |\phi(t)| > 0 \},
\]
\[
A_3 = \{ \lim_{t \to \infty} \inf_{t \geq t_0} |\phi(t)| = 0 \text{ and } \lim_{t \to \infty} \sup_{t \geq t_0} |\phi(t)| > 0 \}.
\] (13)
To prove \( \lim_{t \to \infty} \phi(t) = 0 \) a.s., noting that \( \lim_{t \to \infty} \phi(t) = 0 \) a.s. is equivalent to \( \liminf_{t \to \infty} |\phi(t)| = 0 \) a.s., we only have to show that \( P\{A_2\} = P\{A_3\} = 0 \).

1) Since
\[
\int_0^\infty |\phi(t)| dt < \infty,
\] (14)
one has
\[
\int_0^\infty |\phi(t)| dt < \infty, \text{ a.s.},
\] (15)
then
\[
\liminf_{t \to \infty} |\phi(t)| = 0, \text{ a.s.}
\] (16)
Therefore, \( P\{A_2\} = 0 \) is obvious.

2) Now, we turn to proving that \( P\{A_3\} = 0 \) by contradiction. Suppose \( P\{A_3\} > 0 \), then there exist \( \varepsilon_1 > 0 \) and \( \varepsilon_2 > 0 \) such that
\[
P\{A_3\} > \varepsilon_0.
\] (17)
where
\[ A_0 = \{ |\phi(t)| \text{ crosses from below } \varepsilon_1 \text{ to above } 2\varepsilon_1 \text{ and back infinitely many times} \}. \tag{18} \]

Noting that \(|\phi(t)|\) is a continuous \(\mathcal{F}_t\) adapted process, we can define the following stopping times:
\[
T_{t_1}^1 = \inf \{ t \geq 0 : |\phi(t)| \in [0, \varepsilon_1] \},
\]
\[
T_{2k+1}^i = \inf \{ t \geq T_{2k-1}^i : |\phi(t)| \not\in [0, 2\varepsilon_1] \},
\]
\[
T_{2k+2}^i = \inf \{ t \geq T_{2k+1}^i : |\phi(t)| \not\in [0, \varepsilon_1] \}, \quad i = 2, 3, \ldots.
\]

For any \(i\), on the sample set \(A_0\), it is obvious that \(T_{t_1}^1, T_{2k+1}^i < \infty\), and by the continuity of \(|\phi(\cdot)|\), one has that \(T_{t_1}^1, T_{2k+1}^i \to \infty\), as \(i \to \infty\). From (16), one also knows that \(T_{t_1}^1 < \infty\) whenever \(T_{2k+1}^i < \infty\). Then by (14), one has
\[
\begin{align*}
\infty > E \int_0^\infty |\phi(t)| \, dt &
\geq \sum_{i=1}^{\infty} E \left\{ \mathbb{1}_{\{T_{2k+1}^i < \infty\}} \int_{T_{2k+1}^i}^{T_{2k+2}^i} |\phi(t)| \, dt \right\} \\
&> \sum_{i=1}^{\infty} \varepsilon_i E \left\{ \mathbb{1}_{\{T_{2k+1}^i < \infty\}} \left( T_{2k+1}^{i+1} - T_{2k+1}^i \right) \right\}.
\end{align*}
\tag{19}
\]

From the strongly uniform continuity in probability of \(|\phi(t)|\), for any \(\varepsilon > 0\), there is a constant \(\delta = \delta(\varepsilon, \varepsilon) > 0\) such that
\[
P \left\{ \{ T_{2k+1}^i < \infty \} \cap \left\{ \sup_{0 \leq s \leq \delta} |\phi(T_{2k+1}^i + s)| - |\phi(T_{2k+1}^i)| \geq \varepsilon \right\} \right\} \leq \varepsilon.
\tag{20}
\]

If taking \(\varepsilon = \varepsilon_1/\varepsilon, \varepsilon = \varepsilon_0/\varepsilon\), then there exists a constant \(\delta^* = \delta^*(\varepsilon_1/2, \varepsilon_0/\varepsilon) > 0\) such that
\[
P \left\{ \{ T_{2k+1}^i < \infty \} \cap \left\{ \sup_{0 \leq s \leq \delta^*} \left| \phi(T_{2k+1}^i + s) \right| - \phi(T_{2k+1}^i) \geq \frac{\varepsilon_1}{2} \right\} \right\} \leq \frac{\varepsilon_0}{2},
\tag{21}
\]

which together with
\[
P \left\{ \{ T_{2k+1}^i < \infty \} \geq P \{ A_0 \} \geq \varepsilon, \right\}
\tag{22}
\]
leads to
\[
P \left\{ \{ T_{2k+1}^i < \infty \} \cap \left\{ \sup_{0 \leq s \leq \delta^*} \left| \phi(T_{2k+1}^i + s) \right| - \phi(T_{2k+1}^i) \geq \frac{\varepsilon_1}{2} \right\} \right\}
- P \left\{ \{ T_{2k+1}^i < \infty \} \right\}
- P \left\{ \{ T_{2k+1}^i < \infty \} \right\}
\geq \frac{\varepsilon_0}{2}.
\tag{23}
\]
Then \(\{ T_{2k+1}^i < \infty \} \cap G_i\) is not zero probability set, where
\[
G_i = \left\{ \sup_{0 \leq s \leq \delta^*} \left| \phi(T_{2k+1}^i + s) \right| - \phi(T_{2k+1}^i) \leq \frac{\varepsilon_1}{2} \right\}.
\tag{24}
\]

On the sample set \(\{ T_{2k+1}^i < \infty \} \cap G_i\), the inequality
\[
T_{2k+1}^{i+1} - T_{2k+1}^i > \delta^*
\tag{25}
\]
always holds. Otherwise, if \(T_{2k+1}^{i+1} - T_{2k+1}^i < \delta^*\) for some \(\omega_0 \in \{ T_{2k+1}^i < \infty \} \cap G_i\), then
\[
\sup_{\varepsilon < \delta \leq \delta^*} \left| \phi \left( T_{2k+1}^i + s \right) \right| - \left| \phi \left( T_{2k+1}^i \right) \right| \geq \left| \phi \left( T_{2k+1}^{i+1} \right) \right| - \left| \phi \left( T_{2k+1}^i \right) \right| = \varepsilon_1
\tag{26}
\]
on \(\omega_0\). Since \(\omega_0 \in G_i\), (26) is a contradiction to the definition of \(G_i\).

So, from (19), (23) and (25), one has
\[
\infty > \int_0^\infty |\phi(t)| \, dt
\geq \sum_{i=1}^{\infty} \varepsilon_i E \left\{ \mathbb{1}_{\{T_{2k+1}^i < \infty\}} \left( T_{2k+1}^{i+1} - T_{2k+1}^i \right) \right\}
\geq \sum_{i=1}^{\infty} \varepsilon_i E \left\{ \mathbb{1}_{\{T_{2k+1}^i < \infty\} \cap G_i} \left( T_{2k+1}^{i+1} - T_{2k+1}^i \right) \right\}
\geq \sum_{i=1}^{\infty} \varepsilon_i \delta^* P \left\{ \{ T_{2k+1}^i < \infty \} \cap G_i \right\}
\geq \varepsilon_1 \delta^* \sum_{i=1}^{\infty} \frac{\varepsilon_0}{2}
= \infty
\tag{27}
\]
which leads to a contradiction. This yields that \(P \{ A_0 \} = 0\) and \(\lim_{t \to \infty} \phi(t) = 0\) a.s. \(\blacksquare\)

### III. THE PROOFS OF OTHER MAIN RESULTS

With the proposal of the new stochastic Barbalat’s lemma, some changes are required for the proofs of other main results of [1].

Consider the stochastic dynamic system
\[
dx = f(x, t) \, dt + g(x, t) \, dW(t), \quad x(0) = x_0 \in \mathbb{R}^n
\tag{28}
\]
where \(x \in \mathbb{R}^n\) is the state, \(W(t)\) is an \(m\)-dimensional standard Wiener process defined in \((\Omega, \mathcal{F}, \mathbb{P})\). The Borel measurable functions \(f: \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^n\) and \(g: \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^{n \times m}\) are piecewise continuous in \(t\) and locally bounded and locally Lipschitz continuous in \(x\) uniformly in \(t \in \mathbb{R}_+\).

System (28) has a unique strong solution on \([0, \sigma_\infty)\) with \(\sigma_\infty\) being the explosion time of \(x(t)\) (e.g., see [4, Lemma 1]).

Remark 5: The solution, \(x(t)\), of system (28) is well defined in \(t \in \mathbb{R}_+\) if it is strongly bounded in probability.

From Remark 4, we obtain that solution \(x(t)\) is strongly bounded in probability if and only if \(\sup_{t \geq 0} |x(t)| < \infty\) a.s. So, if \(x(t)\) is strongly bounded in probability, one has \(\sup_{t \geq 0} |x(t)| < \infty\) a.s. Then, the explosion time \(\sigma_\infty = \infty\) a.s., and solution \(x(t)\) is well defined in \(t \in \mathbb{R}_+\). \(\square\)

The following propositions are the replacements of [1, Propositions 4–6].

Proposition 2: If a stochastic process \(x(t)\) is locally strongly uniformly continuous in probability and strongly bounded in probability, and \(\beta(x)\) is a continuous function, then \(\beta(x(t))\) is strongly uniformly continuous in probability.
Proof: Since $x(t)$ is strongly bounded in probability, then for any $\epsilon > 0$ there exists an $r_1 = r_1(\epsilon) > 0$ such that

$$
P\left\{ \sup_{t \geq 0} |x(t)| > \frac{r_1}{2} \right\} \leq \frac{\epsilon}{2} \tag{29}$$

Since $\beta(x)$ is continuous, then $\beta(x)$ is uniformly continuous in the closed ball $B_{r_1/2} = \{ |x| \leq r_1/2 \}$. That is, for any $\epsilon > 0$, there exists a $\eta_1 = \eta_1(\epsilon, r_1)$ such that

$$
|\beta(x) - \beta(y)| < \frac{\epsilon}{2} \quad \text{for any } |x - y| < \eta_1, \quad x, y \in B_{r_1/2}. \tag{30}
$$

From the local strongly uniform continuity in probability of $x(t)$, for $\epsilon, \eta_1 = \eta_1(\epsilon, r_1), r_1 = r_1(\epsilon)$, given above, there exists a $\delta = \delta(\epsilon, \eta_1, r_1) = \delta(\epsilon, r_1) > 0$, such that for any stopping time $\tau \geq 0$, there holds

$$
P\left\{ \{ \tau < \infty \} \cap \left\{ \sup_{t \leq \tau} |x(\sigma_t \wedge (\tau + s)) - x(\sigma_t \wedge \tau)| \leq \eta_1 \right\} \right\} \leq \frac{\epsilon}{2} \tag{31}
$$

where $\sigma_t = \inf\{t \geq 0 : |x(t)| \geq r_1 \}$ with inf $\emptyset$ = $\infty$. Then, by (29)-(31), one has

$$
P\left\{ \{ \tau < \infty \} \cap \left\{ \sup_{t \leq \tau} |x(t)| \leq \frac{r_1}{2} \right\} \right\} \leq \frac{\epsilon}{2} \tag{32}
$$

So, $\beta(x(t))$ is strongly uniformly continuous in probability.

Proposition 3: The diffusion process $x(t)$ given by system (28) is locally strongly uniformly continuous in probability.

Proof: For any $r > 0$, since functions $f$ and $g$ are locally bounded in $x$ uniformly in $t \in R_+$, one can define two functions $\rho_1(r)$ and $\rho_2(r)$ as

$$
\rho_1(r) = \max_{|x| \leq r} \sup_{t \geq 0} |f(x, t)|, \quad \rho_2(r) = \max_{|x| \leq r} \sup_{t \geq 0} |g(x, t)|. \tag{33}
$$

Set $\sigma_r = \inf\{t \geq 0 : |x(t)| \geq r \}$, then $\sigma_r$ increases and tends to $\sigma_\infty$ as $r \to \infty$. For any stopping time $\tau \geq 0$, by Doob’s martingale inequality, we compute

$$
E\left\{ 1_{\{\sigma_r \wedge \tau < \infty\}} \sup_{0 \leq s \leq \tau} |f(\sigma_r \wedge (\tau + s)) - x(\sigma_r \wedge \tau)|^2 \right\} 
$$

$$
= E\left\{ 1_{\{\sigma_r \wedge \tau < \infty\}} \sup_{0 \leq s \leq \tau} \int_{\sigma_r \wedge \tau}^{\sigma_r \wedge (\tau + s)} f(x(t), t)dt + \int_{\sigma_r \wedge \tau}^{\sigma_r \wedge (\tau + s)} g(x(t), t)dW(t) \right\}^2 
$$

$$
\leq 2E \left\{ 1_{\{\sigma_r \wedge \tau < \infty\}} \sup_{0 \leq s \leq \tau} \int_{\sigma_r \wedge \tau}^{\sigma_r \wedge (\tau + s)} f(x(t), t)dt \right\}^2 
$$

$$
+ 2E \left\{ 1_{\{\sigma_r \wedge \tau < \infty\}} \sup_{0 \leq s \leq \tau} \int_{\sigma_r \wedge \tau}^{\sigma_r \wedge (\tau + s)} g(x(t), t)dW(t) \right\}^2 
$$

$$
\leq 2E \left\{ 1_{\{\sigma_r \wedge \tau < \infty\}} \int_{\sigma_r \wedge \tau}^{\sigma_r \wedge (\tau + h)} |f(x(t), t)|^2 dt \right\} 
$$

$$
+ 8E \left\{ 1_{\{\sigma_r \wedge \tau < \infty\}} \int_{\sigma_r \wedge \tau}^{\sigma_r \wedge (\tau + h)} |g(x(t), t)|^2 dt \right\} 
$$

$$
\leq 2h^2 \rho_1^2(r) + 8h \rho_2^2(r). \tag{34}
$$

For any $\epsilon > 0$, by Chebyshev’s inequality, it follows from (34) that

$$
P\left\{ \{ \sigma_r \wedge \tau < \infty\} \cap \left\{ \sup_{0 \leq s \leq \tau} |x(\sigma_r \wedge (\tau + s)) - x(\sigma_r \wedge \tau)| \geq \epsilon \right\} \right\} \leq \frac{2h^2 \rho_1^2(r) + 8h \rho_2^2(r)}{\epsilon^2}. \tag{35}
$$

Since $\{ \tau < \infty \} \subseteq \{ \sigma_r \wedge \tau < \infty \}$, then

$$
P\left\{ \{ \tau < \infty \} \cap \left\{ \sup_{0 \leq s \leq \tau} |x(\sigma_r \wedge (\tau + s)) - x(\sigma_r \wedge \tau)| \geq \epsilon \right\} \right\} \leq \frac{2h^2 \rho_1^2(r) + 8h \rho_2^2(r)}{\epsilon^2}. \tag{36}
$$

For $\epsilon, r > 0$ and any $\epsilon > 0$ mentioned above, one can find $h = h(\epsilon, r, \tau) > 0$ such that $(2h^2 \rho_1^2(r) + 8h \rho_2^2(r))/\epsilon^2 \leq \epsilon$. So, choosing $h = h(\epsilon, r, \tau)$, we have

$$
P\left\{ \{ \tau < \infty \} \cap \left\{ \sup_{0 \leq s \leq \tau} |x(\sigma_r \wedge (\tau + s)) - x(\sigma_r \wedge \tau)| \geq \epsilon \right\} \right\} \leq \epsilon. \tag{37}
$$

which shows that $x(t)$ is locally strongly uniformly continuous in probability.

Proposition 4: The diffusion process $x(t)$ given by system (28) is strongly uniformly continuous in probability if it is strongly bounded in probability.

Proof: It is obvious from Proposition 1 and Proposition 3 that $x(t)$ is strongly uniformly continuous in probability.

Now, with the help of Theorem 1 and above Propositions, Theorem 2 in [1] can be proved.

Theorem 2: If the solution process $x(t)$ of system (28) is strongly bounded in probability, and $E_s\int_{t+s}^{\infty} \beta(x(t), t)dt < \infty$, where $\beta(\cdot)$ is a continuous function, then $\lim_{r \to \infty} \beta(x(t)) = 0$, a.s.

Proof: It follows from Remark 5 that the solution process $x(t)$ is well defined in $t \in R_+$. Since $x(t)$ is a continuous adaptive process, so is $\beta(x(t))$. From $x(t)$ being strongly bounded in probability, one
knows that $\beta(x(t))$ is strongly uniformly continuous in probability by Propositions 2 and 3. Therefore, one has that $\lim_{t \to \infty} \beta(x(t)) = 0$, a.s. by the use of Theorem 1.

In a similar procedure, Theorems 3 and 4 of [1] (or Lemma 3 of [4], Lemma 1 of [5]) can also be proved by Theorem 1, which shows that some results on stochastic stability can follow from our stochastic Barbalat’s lemma.

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REFERENCES


