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The role of convexity in the adaptive control of rapidly time-varying systems

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ABSTRACT

In classical adaptive control the parameters are assumed to be fixed or slowly time-varying. In order to facilitate parameter estimation/tuning it is desirable to have the set of admissible parameters lie in a convex set; if this set is not convex, a common trick is to replace it with its convex hull, but the adaptive control problem is challenging if stabilizability of the set of admissible parameters is lost. However, such a convexity assumption is an artifact of the approach to the problem, rather than an inherent constraint, since most logic-based and supervisory approaches to the problem make no such requirement. On the other hand, here we show that losing stabilizability on the convex hull of the set of admissible parameters plays an important role in the adaptive control of rapidly time-varying systems.

1. Introduction

In classical parameter adaptive control, restricting the set of admissible parameters to a convex set is useful in carrying out parameter estimation/tuning, e.g. see [1]. Of course, if the set of admissible parameters is not convex, it is natural to replace it with its convex hull; however, this can create the problem of introducing uncontrollable or unobservable modes, which can create difficulty in proving that the associated adaptive controller is stabilizing. This has spurred a fair bit of effort to get around this problem, e.g. see [2,3], and [4]. These methods have been successful, and are effective in controlling plant models whose parameters are either fixed or slowly time-varying; this is also true of most logic-based and supervisory approaches to adaptive control, e.g. see [5,6].

Now let us turn to the adaptive control of rapidly time-varying systems. This problem is very difficult, and only limited results have been obtained, each of which requires fairly rich structure on the plant model:

(i) the form of the time-variations (or at least of the fast terms) is assumed to be known (e.g. see [7,8]);
(ii) the only uncertainty is a gain at the input, e.g. see [9];
(iii) the plant has stable zero dynamics (roughly speaking, this is the time-varying counterpart of minimum phase), e.g. see [10–15];
(iv) the plant has unstable zero dynamics but several stringent matching requirements must hold—see [16,17].

In this paper our goal is to ascertain performance limitations in the adaptive control of rapidly time-varying systems. To avoid imposing unnecessary structure on the set of admissible plant parameters (such as connectedness), we restrict our attention to that of jumps in the plant parameters. We demonstrate that, in two important cases, if the convex hull of the set of admissible parameters does not possess a weak notion of stabilizability, then regardless of the controller used, the performance must necessarily degrade rapidly as the time between parameter jumps decreases. This provides an inviolable bound on the achievable performance of any adaptive controller for such a rapidly time-varying uncertain system.

2. Mathematical preliminaries

Let $\mathbf{Z}$ denote the set of integers, $\mathbf{Z}^+$ represent the set of non-negative integers, $\mathbf{N}$ denote the set of natural numbers, $\mathbf{R}$ denote the set of real numbers, $\mathbf{R}^+$ represent the set of non-negative real numbers, and $\mathbf{C}$ represent the set of complex numbers. We will use the Euclidean norm to measure the size of a vector: for $x \in \mathbf{C}^n$, we define $\|x\| := (\sum_{i=1}^n |x_i|)^{1/2}$. The corresponding induced norm of a matrix $A \in \mathbf{C}^{m\times n}$ is defined in a usual manner: $\|A\| = \sup_{\|x\|\neq 0} \frac{\|Ax\|}{\|x\|}$. If $x \in \mathbf{C}^n$ we use x^T to denote the transpose and x^* to denote the complex conjugate transpose.

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For a given set $\mathcal{S} \subseteq \mathbf{R}^{m\times n}$, we let $PC(\mathcal{S})$ denote the set of all piecewise continuous functions $f : \mathbf{R}^+ \to \mathcal{S}$. To measure the size of $f \in PC(\mathcal{S})$, we define

$$\|f\|_\infty := \sup_{t\in\mathbf{R}^+} \|f(t)\|;\ [1]$$

we let $PC_\infty(S)$ denote the set of all $f \in PC(S)$ for which $\|f\|_\infty < \infty$. With $T_s > 0$, we let $PC_{con}(S, T_s)$ denote the set of all $f \in PC(S)$ which are piecewise constant with a minimum time of T_s between discontinuities. Last of all, we let $conv(S)$ denote the convex hull of S.

3. The setup

Here we will model the plant uncertainty as follows. For a suitable $l \in \mathbf{N}$ we start with a compact set $\Theta \subset \mathbf{R}^l$. With $A : \Theta \to \mathbf{R}^{n\times n}$, $B : \Theta \to \mathbf{R}^{n\times m}$ and $C : \Theta \to \mathbf{R}^{r\times n}$ continuous functions, and $\theta \in PC(\Theta)$, we consider the time-varying plant

$$\dot{x}(t) = A(\theta(t))x(t) + B(\theta(t))u(t), \qquad x(0) = x_0 \tag{1}$$

$$y(t) = C(\theta(t))x(t); \tag{2}$$

here $x(t) \in \mathbf{R}^n$ is the state, $y(t) \in \mathbf{R}^r$ is the measured output, and $u(t) \in \mathbf{R}^m$ is the control input, and we associate the plant with the triple $(A(\theta(\cdot)), B(\theta(\cdot)), C(\theta(\cdot)))$, or simply P. While the set Θ as well as the functions A, B, and C are known, the variable $\theta \in PC(\Theta)$ *is neither known nor measurable*. The control objective here is a form of stability, so it is reasonable to restrict θ to a subset of $PC(\Theta)$; we consider the case of time-variations which are simply jumps, with a minimum distance separating them: with $T_s > 0$, we consider the subset $PC_{con}(\Theta, T_s)$. Associated with this set of admissible θ's is the set of possible plant models:

$$\mathcal{P}(T_s) := \{(A(\theta(\cdot)), B(\theta(\cdot)), C(\theta(\cdot))) : \theta \in PC_{con}(\Theta, T_s)\}.$$

Remark 1. It turns out that the choice of C plays no role in the results which we prove here. However, we will allow a general form for $C(\theta)$ to emphasize the general applicability of the result.

Remark 2. In classical adaptive control many results are proven for the case of fixed parameters. The setup that we adopt here allows this—it corresponds to the case of $T_s = \infty$.

The control objective is to stabilize the system even though there are rapid variations in $\theta(t)$, which is an adaptive control problem. It is traditional in adaptive control to prove very weak notions of stability, often proving only that the system is well behaved asymptotically, with no uniformity over the admissible models in $\mathcal{P}(T_s)$. However, more recently, techniques such as the supervisory control method of Morse, e.g. see [5,6] (see the Concluding Remarks Section of the latter), and the periodic probing, estimation and control technique of the author, e.g. see [11,18,15,9,19,17], have been used to prove stronger uniform notions of stability, even when the parameters are varying, as illustrated in the following definition.

Definition 3. With $K : PC(\mathbf{R}^r) \to PC(\mathbf{R}^m)$ and $T_s > 0$, we say that the controller

$$u = K(y) \tag{3}$$

is **admissible** for $\mathcal{P}(T_s)$ if, for every $P \in \mathcal{P}(T_s)$, the closed-loop system is well-posed: for every $x_0 \in \mathbf{R}^m$ there are unique $u \in PC(\mathbf{R}^m)$ and $y \in PC(\mathbf{R}^r)$ which satisfy the plant model Eqs. (1)–(2) and the controller Eq. (3), in which case we let $\Phi(x_0, P)$ denote the map $x_0 \to \begin{bmatrix} x \\ u \end{bmatrix}$ from $\mathbf{R}^n \to PC(\mathbf{R}^n) \times PC(\mathbf{R}^m)$. If K is admissible for $\mathcal{P}(T_s)$ then we say that K **stabilizes** $\mathcal{P}(T_s)$ if

(i) $\Phi(0, P) = 0$ for every $P \in \mathcal{P}(T_s)$ and
(ii) the following quantity

$$\gamma(K, \mathcal{P}(T_s)) := \sup\left\{ \frac{\|\Phi(x_0, P)\|_\infty}{\|x_0\|} : x_0 \in \mathbf{R}^n \text{ is nonzero and } P \in \mathcal{P}(T_s)\right\}$$

is finite.

From Definition 3 we see that if K stabilizes $\mathcal{P}(T_s)$, then

$$\|\Phi(x_0, P)\|_\infty \le \gamma(K, \mathcal{P}(T_s))\|x_0\|$$

for every $x_0 \in \mathbf{R}^n$ and $P \in \mathcal{P}(T_s)$. Here the goal is to bound $\gamma(K, \mathcal{P}(T_s))$ in certain circumstances. As observed in Remark 1, $C(\theta(\cdot))$ plays no role in our result. To this end, we now define the convex hull of the admissible $(A(\cdot), B(\cdot))$ pairs: with μ playing the role of a dummy variable, we define

$$\mathcal{H} := \text{conv}\{(A(\mu), B(\mu)) : \mu \in \Theta\}. \tag{4}$$

The question at hand is: if there is a pair in $\mathcal{H}$ which loses stabilizability, what is the consequence on stabilizing the corresponding set $\mathcal{P}(T_s)$? Of course, if $T_s = \infty$, then we have the classical adaptive control setup of no time-variations, and there are general techniques such as supervisory control [5,6] as well as the periodic probing, estimation and control technique of [18] which yield stability. So the real concern is that this loss of stabilizability may impact the situation when $T_s < \infty$, measured in terms of a lower bound on $\gamma(K, \mathcal{P}(T_s))$. Here we will show, under suitable assumptions, that $\gamma(K, \mathcal{P}(T_s))$ must necessarily be large if T_s is small. We consider three situations:

- In Section 4, we consider the case of B being fixed, and we prove that if "weak stabilizability"[2] is lost then $\gamma(K, \mathcal{P}(T_s)) \to \infty$ as $T_s \to 0$.
- In Section 5, we assume that A is fixed and B is variable, and provide an example from the literature which demonstrates that no general result is provable.
- In Section 6, we consider the general case of allowable variations in both A and B, but consider a special controller structure associated with step tracking; in this situation a result similar to that of Section 4 can be proven.

4. The Case of time-variations in $A(\theta(\cdot))$ Only

In this case we assume that the only variation is in A, i.e. we assume that $B(\theta(\cdot))$ is constant, so we simply represent it by B. To proceed, we first define

$$\mathcal{A} := \{A(\mu) : \mu \in \Theta\} \subset \mathbf{R}^{n\times n}.$$

We now introduce a weak notion of stabilizability, which differs from the classical notion of stabilizability by not deeming eigenvalues on the imaginary to be in the "bad region ".

Definition 4. (A, B) is **weakly stabilizable** if

$$\text{rank}\begin{bmatrix} A - \lambda I & B \end{bmatrix} = n \tag{5}$$

for all $\lambda \in \mathbf{C}$ satisfying Re $\lambda > 0$; $\mathcal{H} \subset \mathbf{R}^{n\times n} \times \mathbf{R}^{n\times m}$ is **weakly stabilizable** if every pair $(A, B) \in \mathcal{H}$ is **weakly stabilizable**.

We now prove that if $(\text{conv}(\mathcal{A}), B)$ is **not** weakly stabilizable, then the closed-loop performance provided by a controller for $\mathcal{P}(T_s)$ is bounded below by a function of T_s. Before proceeding, define

$$\bar{a} := \sup_{\theta\in\Theta} \|A(\theta)\|, \qquad \bar{b} := \|B\|.$$

[1] Here we will be allowing sampled-data controllers, so we cannot use "ess sup" here.

[2] This is a slightly weaker version of the usual notion of stabilizability and will be defined shortly.

Theorem 5. *If* $(\mathrm{conv}(\mathcal{A}), B)$ *is not* ***weakly stabilizable*** *then for every* $\bar{T}_s > 0$, *there exists a constant* $\bar{\gamma} > 0$ *so that if* $T_s \in (0, \bar{T}_s)$ *and* K *stabilizes* $\mathcal{P}(T_s)$, *then*

$$\gamma(K, \mathcal{P}(T_s)) \geq \frac{\bar{\gamma}}{T_s}. \tag{6}$$

Remark 6. This theorem shows that if $\mathrm{conv}(\mathcal{A})$ has an element A for which (A, B) is not weakly stabilizable, then any controller which stabilizes $\mathcal{P}(T_s)$ provides performance which necessarily degrades rapidly as the switching time T_s tends to zero.

Remark 7. If T_s is large enough, then regardless of whether $(\mathrm{conv}(\mathcal{A}), B)$ is weakly stabilizable or not, as long as $(A(\mu), B(\mu))$ is controllable for every $\mu \in \Theta$ and $(C(\mu), A(\mu))$ is observable for every $\mu \in \Theta$, then the approach of [19] can be used to design a controller which stabilizes $\mathcal{P}(T_s)$. If T_s is small and $(\mathrm{conv}(\mathcal{A}), B)$ is not weakly stabilizable, then the existence of a controller K which stabilizes $\mathcal{P}(T_s)$ is an open problem; however, this Theorem shows that the performance would be so poor that the existence of a controller may be of little interest from an engineering point of view.

Example 8. Consider the case of $\Theta = \{\theta_1, \theta_2\}$ with $A(\theta_i) = A_i$ given by

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix},$$

with

$$B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

It is easy to verify that (A_i, B) is weakly stabilizable for $i = 1, 2$, but that $(\frac{1}{2}(A_1 + A_2), B)$ is **not** weakly stabilizable. The key idea in the proof of the theorem is that if you switch between A_1 and A_2 fast enough, spending half your time at each, then you can make the time varying plant look like

$$\dot{x} = \frac{1}{2}(A_1 + A_2)x + Bu,$$

with the accuracy of this model improving as the switching speed increases.

To prove the theorem we need the following preliminary result.

Lemma 9. *If* $A \in \mathbf{R}^{n\times n}$, $B \in \mathbf{R}^{n\times m}$, *and* $\lambda \in \mathbf{C}$ *satisfy*

$$\mathrm{rank}\begin{bmatrix} A - \lambda I & B \end{bmatrix} < n, \tag{7}$$

then there exist $v, w \in \mathbf{C}^n$ *with a norm of one which satisfy*

$$v^* \begin{bmatrix} A - \lambda I & B \end{bmatrix} = 0, \tag{8}$$

$$v^* w = 1, \tag{9}$$

$$v^* e^{At} B = 0, \quad t \geq 0, \tag{10}$$

and

$$v^* e^{At} = e^{\lambda t} v^*, \quad t \geq 0; \tag{11}$$

if $\lambda \in \mathbf{R}$ *then we can choose* $v, w \in \mathbf{R}^n$.

Proof. See the Appendix.

Proof of Theorem 5. Suppose that $(\mathrm{conv}(\mathcal{A}), B)$ in not **weakly stabilizable** and let $\bar{T}_s > 0$ be arbitrary. Choose $p \in \mathbf{N}$, $A_i \in \mathcal{A}$, $i = 1, \ldots, p$, and $c_i > 0$, $i = 1, \ldots, p$, satisfying

- $\sum_{i=1}^p c_i = 1$, and
- $\bar{A} := \sum_{i=1}^p c_i A_i$ is such that $(\bar{A}, B)$ is not weakly stabilizable.

Without loss of generality, we can assume that $c_1 \leq c_2 \leq \cdots \leq c_p$; define $\bar{h} := \frac{\bar{T}_s}{c_1}$.

Now we define a time-varying $\theta(\cdot)$ for which $A(\theta(\cdot))$ jumps among the A_i's. To this end, first choose $\theta_i \in \Theta$ satisfying

$$A(\theta_i) = A_i.$$

With $h \in (0, \bar{h})$ arbitrary, define $\theta^h(t)$ to be the discontinuous periodic function of period h described by

$$\theta^h(t) = \begin{cases} \theta_1 & t \in [0, c_1 h) \\ \theta_2 & t \in [c_1 h, (c_1 + c_2)h) \\ \vdots & \vdots \\ \theta_p & [(c_1 + \cdots c_{p-1})h, h). \end{cases}$$

It follows easily that

$$\frac{1}{h}\int_0^h A(\theta^h(\tau))\, d\tau = \bar{A} \tag{12}$$

and that the smallest time between discontinuities in $\theta_h(t)$ is $c_1 h$.

Next, suppose that the controller $u = K(y)$ stabilizes $\mathcal{P}(c_1 h)$; we will provide a lower bound on the performance $\gamma(K, \mathcal{P}(c_1 h))$. With $x_0 \in \mathbf{R}^n$ arbitrary, let $u = K(y)$ be applied to the plant (1)–(2) with $\theta = \theta^h$, yielding a closed-loop response given by

$$\begin{aligned} x((k+1)h) &= x(kh) + \int_{kh}^{(k+1)h} A(\theta^h(\tau))x(\tau)d\tau \\ &\quad + \int_{kh}^{(k+1)h} Bu(\tau)d\tau \\ &= x(kh) + \int_{kh}^{(k+1)h} A(\theta^h(\tau))x(kh)d\tau \\ &\quad + \int_{kh}^{(k+1)h} A(\theta^h(\tau))[x(\tau) - x(kh)]d\tau \\ &\quad + B\int_{kh}^{(k+1)h} u(\tau)d\tau \\ &= [I + h\bar{A}]x(kh) \\ &\quad + \underbrace{\int_{kh}^{(k+1)h} A(\theta^h(\tau))[x(\tau) - x(kh)]d\tau}_{=:f_1[k]} \\ &\quad + \underbrace{B\int_{kh}^{(k+1)h} u(\tau)d\tau}_{=:f_2[k]} \\ &= e^{\bar{A}h}x(kh) + f_1[k] + f_2[k] \\ &\quad + \underbrace{(I + \bar{A}h - e^{\bar{A}h})x(kh)}_{=:f_3[k]}. \end{aligned} \tag{13}$$

At this point we investigate each of $f_i[k]$, $i = 1, 2, 3$. Now it is easy to see that

$$\begin{aligned} \|\dot{x}(t)\| &\leq \|A(\theta^h(t))\| \times \|x(t)\| + \|B\| \times \|u(t)\| \\ &\leq (\bar{a} + \bar{b})\gamma(K, \mathcal{P}(c_1 h))\|x(0)\|, \quad t \geq 0; \end{aligned}$$

this means that

$$\|x(t) - x(kh)\| \leq h(\bar{a} + \bar{b})\gamma(K, \mathcal{P}(c_1 h))\|x(0)\|, \quad t \in [kh, (k+1)h],$$

so

$$\|f_1[k]\| \le \bar{a}(\bar{a}+\bar{b})h^2\gamma(K,\mathcal{P}(c_1h))\|x(0)\|,$$
$$t \in [kh,(k+1)h]. \tag{14}$$

Now define

$$u_k := \frac{1}{h}\int_{kh}^{(k+1)h} u(\tau)\,d\tau; \tag{15}$$

it is clear that

$$\|u_k\| \le \gamma(K,\mathcal{P}(c_1h))\|x(0)\| \tag{16}$$

and

$$f_2[k] = hBu_k. \tag{17}$$

Last of all, since

$$e^{\bar{A}h} - I - \bar{A}h = \sum_{i=2}^{\infty}\frac{\bar{A}^i h^i}{i!} = \frac{\bar{A}^2h^2}{2!}\sum_{j=0}^{\infty}\frac{2\bar{A}^j h^j}{(j+2)!},$$

we see that

$$\|e^{\bar{A}h} - I - \bar{A}h\| \le \frac{\bar{a}^2h^2}{2}e^{\bar{a}h},$$

which means that

$$\|f_3[k]\| \le \frac{\bar{a}^2h^2}{2}e^{\bar{a}h}\gamma(K,\mathcal{P}(c_1h))\|x(0)\|. \tag{18}$$

If we incorporate (15) and (17) into Eq. (13), we end up with

$$x((k+1)h) = e^{\bar{A}h}x(kh) + hBu_k + f_1[k] + f_3[k],$$

so if we solve this equation starting at zero, we end up with

$$x(kh) = e^{\bar{A}kh}x(0) + h\sum_{j=0}^{k-1}e^{\bar{A}(k-1-j)h}Bu_j$$
$$+ \sum_{j=0}^{k-1}e^{\bar{A}(k-1-j)h}[f_1[j]+f_3[j]], \quad k \ge 0. \tag{19}$$

At this point we analyze this equation and use the bounds on $\|f_1[k]\|$ and $\|f_3[k]\|$ given in (14) and (18) to construct the desired performance bound. To proceed, we examine two separate cases.

Case 1: There exists a real eigenvalue $\lambda > 0$ which satisfies

$$\operatorname{rank}\begin{bmatrix}\bar{A}-\lambda I & B\end{bmatrix} < n.$$

In this case it follows from Lemma 9 that there exist $v, w \in \mathbf{R}^n$ of unit norm which satisfy (8)–(11) (with A replaced by $\bar{A}$). We set $x(0) = w$ and consider the scaled output

$$\phi(t) := v^T x(t).$$

Then using (19) together with (9)–(11) (in the last two equations we replaced A with $\bar{A}$), it follows that

$$\begin{aligned}\phi(kT) &= v^Te^{\bar{A}kh}w + h\sum_{j=0}^{k-1}v^Te^{\bar{A}(k-1-j)h}Bu_j \\ &\quad + \sum_{j=0}^{k-1}v^Te^{\bar{A}(k-1-j)h}[f_1[j]+f_3[j]] \\ &= e^{\lambda kh}v^Tw + \sum_{j=0}^{k-1}e^{\lambda(k-1-j)h}v^T[f_1[j]+f_2[j]] \\ &= e^{\lambda kh} + \sum_{j=0}^{k-1}e^{\lambda(k-1-j)h}v^T[f_1(j)+f_2(j)].\end{aligned}$$

This immediately implies that

$$|\phi(kT)| \ge e^{\lambda kh} - e^{\lambda kh}\frac{\|v^T\|}{e^{\lambda h}-1}\max_{j\in\{0,1,\ldots,k-1\}}(\|f_1[j]\| + \|f_3[j]\|).$$

Using the bounds on $\|f_1[j]\|$ and $\|f_3[j]\|$ given in (14) and (18), respectively, as well as the fact that $\|w\| = \|v\| = 1$, we obtain

$$\begin{aligned}|\phi(kT)| &\ge e^{\lambda kh} - e^{\lambda kh}\frac{1}{e^{\lambda h}-1}\left[\bar{a}(\bar{a}+\bar{b})h^2 + \frac{\bar{a}^2}{2}h^2e^{\bar{a}h}\right] \\ &\qquad \times \gamma(K,\mathcal{P}(c_1h)) \\ &= e^{\lambda kh}\left\{1 - \frac{1}{e^{\lambda h}-1}\left[\bar{a}(\bar{a}+\bar{b})h^2 + \frac{\bar{a}^2}{2}h^2e^{\bar{a}h}\right]\right. \\ &\qquad \left.\times \gamma(K,\mathcal{P}(c_1h))\right\}.\end{aligned}$$

Now $\phi(kT)$ must be a bounded function of k since $\gamma(K,\mathcal{P}(c_1h))$ is finite, so the term on the RHS in "$\{\cdot\}$" must be less than or equal to zero, which provides a lower bound on $\gamma(K,\mathcal{P}(c_1h))$:

$$\gamma(K,\mathcal{P}(c_1h)) \ge \frac{e^{\lambda h}-1}{\bar{a}(\bar{a}+\bar{b})h^2 + \frac{\bar{a}^2}{2}h^2e^{\bar{a}h}}.$$

Define $f_1(h)$ by h times the RHS of the above inequality:

$$f_1(h) := \frac{e^{\lambda h}-1}{\bar{a}(\bar{a}+\bar{b})h + \frac{\bar{a}^2}{2}he^{\bar{a}h}}.$$

It is clear that

$$\lim_{h\to 0}f_1(h) = \frac{\lambda}{\bar{a}(\bar{a}+\bar{b}) + \frac{\bar{a}^2}{2}} > 0;$$

using the fact that f is continuous and positive for $h > 0$, it follows that

$$\gamma_1 := \inf_{h\in(0,\bar{h}]} f_1(h) > 0.$$

We conclude that

$$\gamma(K,\mathcal{P}(c_1h)) \ge \frac{\gamma_1}{h}, \quad h \in (0,\bar{h}].$$

Using a simple variable substitution we see that

$$\gamma(K,\mathcal{P}(T_s)) \ge \frac{\gamma_1c_1}{T_s}, \quad T_s \in (0,\bar{T}_s].$$

Since the quantity γ_1c_1 is independent of the choice of $T_s \in (0,\bar{T}_s]$, the desired result holds for this case.

Case 2: There exists a complex eigenvalue λ which satisfies $Re\ \lambda > 0$ and

$$\operatorname{rank}\begin{bmatrix}\bar{A}-\lambda I & B\end{bmatrix} < n.$$

Here we follow the same general approach of Case 1, but with some suitable modifications. In this case it follows from Lemma 9 that there exist $v, w \in \mathbf{C}^n$ of unit norm which satisfy (8)–(11) (with A replaced by $\bar{A}$). We first consider the situation in which $x(0) = Re(w)$, and label the corresponding state and control signal response to be $x^r(t)$ and $u^r(t)$, respectively; next of all, we consider the situation in which $x(0) = Im(w)$, and label the corresponding state and control signal response to be $x^i(t)$ and $u^i(t)$, respectively. If we now define $f_1^r[k], f_1^i[k], f_3^r[k], f_3^i[k], u_k^r$, and u_k^i in a natural way, then we end up with the natural counterparts of Eq. (19) derived above:

$$x^r(kh) = e^{\bar{A}kh}x^r(0) + h\sum_{j=0}^{k-1}e^{\bar{A}(k-1-j)h}Bu_j^r$$
$$+ \sum_{j=0}^{k-1}e^{\bar{A}(k-1-j)h}[f_1^r[j]+f_3^r[j]], \quad k \ge 0, \tag{20}$$

$$x^i(kh) = e^{\bar{A}kh}x^i(0) + h\sum_{j=0}^{k-1} e^{\bar{A}(k-1-j)h}Bu_j^i + \sum_{j=0}^{k-1} e^{\bar{A}(k-1-j)h}[f_1^i[j] + f_3^i[j]], \quad k \geq 0. \tag{21}$$

Now define

$$\phi(t) := v^*[x^r(t) + ix^i(t)],$$

$$f_1[k] := f_1^r[k] + if_1^i[k],$$

$$f_2[k] := f_2^r[k] + if_2^i[k],$$

$$u_k := u_k^r + iu_k^i.$$

Using (9)–(11) (in the latter two equations we replace A with $\bar{A}$), it follows that

$$\begin{aligned}\phi(kT) &= v^*e^{\bar{A}kh}w + \sum_{j=0}^{k-1} v^*e^{\bar{A}(k-1-j)h}Bu_j \\ &\quad + \sum_{j=0}^{k-1} v^*e^{\bar{A}(k-1-j)h}[f_1[j] + f_3[j]] \\ &= e^{\lambda kh} + \sum_{j=0}^{k-1} e^{\lambda(k-1-j)h}v^*[f_1[j] + f_2[j]].\end{aligned}$$

If λ_r denotes the real part of λ, it follows immediately that

$$|\phi(kT)| \geq e^{\lambda_r kh} - e^{\lambda_r kh}\frac{1}{e^{\lambda_r h} - 1} \times \max_{j\in\{0,1,\ldots,k-1\}}(\|f_1[j]\| + \|f_3[j]\|). \tag{22}$$

From (14) we have that

$$\|f_1^r[k]\| \leq \bar{a}(\bar{a} + \bar{b})h^2\gamma(K, \mathcal{P}(c_1h))\|x^r(0)\|,$$

and

$$\|f_1^i[k]\| \leq \bar{a}(\bar{a} + \bar{b})h^2\gamma(K, \mathcal{P}(c_1h))\|x^i(0)\|,$$

which means that

$$\begin{aligned}\|f_1[k]\| &\leq \|f_1^r[k]\| + \|f_1^i[k]\| \\ &\leq \bar{a}(\bar{a} + \bar{b})h^2\gamma(K, \mathcal{P}(c_1h))(\|x^r(0)\| + \|x^i(0)\|) \\ &\leq 2\bar{a}(\bar{a} + \bar{b})h^2\gamma(K, \mathcal{P}(c_1h))\|w\| \\ &\leq 2\bar{a}(\bar{a} + \bar{b})h^2\gamma(K, \mathcal{P}(c_1h)).\end{aligned}$$

Similarly,

$$\|f_3[k]\| \leq 2\frac{\bar{a}^2h^2}{2}e^{\bar{a}h}\gamma(K, \mathcal{P}(c_1h)).$$

Using the bounds on $\|f_1[j]\|$ and $\|f_3[j]\|$ in (22) yields

$$\begin{aligned}|\phi(kT)| &\geq e^{\lambda_r kh} - e^{\lambda_r kh}\frac{2}{e^{\lambda_r h} - 1}\left[\bar{a}(\bar{a} + \bar{b})h^2 + \frac{\bar{a}^2}{2}h^2e^{\bar{a}h}\right] \\ &\quad \times \gamma(K, \mathcal{P}(c_1h)) \\ &= e^{\lambda_r kh}\left\{1 - \frac{2}{e^{\lambda_r h} - 1}\left[\bar{a}(\bar{a} + \bar{b})h^2 + \frac{\bar{a}^2}{2}h^2e^{\bar{a}h}\right]\right. \\ &\quad \left.\times \gamma(K, \mathcal{P}(c_1h))\right\}.\end{aligned}$$

Now $\phi(kT)$ must be a bounded function of k since $\gamma(K, \mathcal{P}(c_1h))$ is finite, so the term on the RHS in "$\{\cdot\}$" must be less than or equal to zero, which provides a lower bound on $\gamma(K, \mathcal{P}(c_1h))$:

$$\gamma(K, \mathcal{P}(c_1h)) \geq \frac{1}{2}\frac{e^{\lambda_r h} - 1}{\bar{a}(\bar{a} + \bar{b})h^2 + \frac{\bar{a}^2}{2}h^2e^{\bar{a}h}}.$$

Proceeding as in Case 1, we see that there exists a constant γ_2 so that

$$\gamma(K, \mathcal{P}(T_s)) \geq \frac{\gamma_2}{T_s}, \quad T_s \in (0, \bar{T}_s].$$

Hence, the desired result holds in this case as well. □

Remark 10. Notice that this proof did not use the fact that only y could be measured and that θ and x could not be; furthermore, it did not require causality. Hence, this Theorem can be applied to a much larger class of controllers than the ones that we have considered here.

Remark 11. It turns out that the bound asserted to exist by Theorem 5 can be computed for simple cases. For instance, let us consider Example 8. Using the notation of the proof, we can set $c_1 = c_2 = \frac{1}{2}$, so that

$$\bar{A} = \begin{bmatrix} 1.5 & 0 \\ 0 & 1.5 \end{bmatrix}.$$

Turning to Lemma 9, we set $\lambda = 1.5$ and

$$v = w = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}.$$

In this case

$$\bar{a} = 2, \qquad \bar{b} = \sqrt{2}.$$

Following the Proof of Theorem 5, we have, for any stabilizing controller K, the following bound:

$$\begin{aligned}\gamma\left(K, \mathcal{P}\left(\frac{h}{2}\right)\right) &\geq \frac{e^{\lambda h} - 1}{\bar{a}(\bar{a} + \bar{b})h^2 + \frac{\bar{a}^2}{2}h^2e^{\bar{a}h}} \\ &= \frac{e^{1.5h} - 1}{2(2 + \sqrt{2})h^2 + 2h^2e^{2h}},\end{aligned}$$

so with $T_s = \frac{h}{2}$, or equivalently $h = 2T_s$, we have

$$\gamma(K, \mathcal{P}(T_s)) \geq \frac{e^{3T_s} - 1}{8(2 + \sqrt{2})T_s^2 + 8T_s^2e^{4T_s}}.$$

5. The case of time-variations in $B(\theta(\cdot))$ only

In this case we cannot, in general, prove a comparable result to that of the previous section. To see this, consider the plant

$$\dot{x}(t) = A_0x(t) + \theta(t)B_0u(t),$$
$$y(t) = C_0x(t)$$

with (A_0, B_0) controllable, (C_0, A_0) observable, and $\theta(\cdot) \in \mathcal{G}$ with $\mathcal{G}$ compact and not including zero. In the paper [9] the goal is that of achieving near optimal tracking of an exogenous reference signal in the presence of time-variations in $\theta(\cdot)$. However, if we set the exogenous input to zero then it is easy to see that the controller designed there would provide the kind of stability considered in this paper. The controller presented there is linear periodic of period T, which we label $K(T)$; with a bit of analysis one can prove that there exists a constant $\bar{\gamma}$ so that, for every $T_s > 0$, if we choose T sufficiently small then

$$\gamma(K(T), \mathcal{P}(T_s)) \leq \bar{\gamma}.$$

Hence, for this specific case we are unable to duplicate a result similar to that of the previous section, which means that no general result is proveable in the case of $A(\theta(\cdot))$ constant but $B(\theta(\cdot))$ varying.

6. The case of time-variations in both $A(\theta(\cdot))$ and $B(\theta(\cdot))$

Given the observation in the previous section, we clearly cannot prove a comparable result to Theorem 5 in the general case considered here. However, we can provide a bound on the closed-loop performance if a particular common controller configuration is adopted. More specifically, a classical tracking objective is step tracking, and a common trick[3] to convert a step tracking problem to a stabilization problem is to augment an integrator to the plant as follows:

$$\begin{bmatrix} \dot{x}(t) \\ \dot{u}(t) \end{bmatrix} = \underbrace{\begin{bmatrix} A(\theta(t)) & B(\theta(t)) \\ 0 & 0 \end{bmatrix}}_{=:A_{new}(\theta(t))} \underbrace{\begin{bmatrix} x(t) \\ u(t) \end{bmatrix}}_{x_{new}(t)} + \underbrace{\begin{bmatrix} 0 \\ I \end{bmatrix}}_{=:B_{new}} v(t), \tag{23}$$

$$y(t) = \underbrace{\begin{bmatrix} C & 0 \end{bmatrix}}_{=:C_{new}} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}; \tag{24}$$

we represent this plant by the triple $(A_{new}(\theta(\cdot)), B_{new}, C_{new})$, which we label P_{new}, and we define $\mathscr{P}_{new}(T_s)$ in the natural way. Here $v(t)$ plays the role of the new input, and the goal is to now find a controller to measure $y(t)$ and generate $v(t)$.[4] Indeed, this is the approach adopted by Morse in his ground-breaking work on supervisory control [5,6]. It turns out that the stabilizability of (A, B) is connected to that of (A_{new}, B_{new}):

Lemma 12. $(\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 \\ I \end{bmatrix})$ *is weakly stabilizable iff* (A, B) *is weakly stabilizable.*

Proof. This follows easily from the PBH test.

Notice that the model provided in (23)–(24) is of exactly the same form as given in Section 4, so the approach adopted there is applicable. With $\mathcal{H}$ given by (4), this leads to

Corollary 13. *If $\mathcal{H}$ is not **weakly stabilizable** then for every $\bar{T}_s > 0$, there exists a constant $\bar{\gamma} > 0$ so that if $T_s \in (0, \bar{T}_s)$ and $\mathcal{K}$ stabilizes $\mathscr{P}_{new}(T_s)$, then*

$$\gamma(K, \mathscr{P}_{new}(T_s)) \geq \frac{\bar{\gamma}}{T_s}. \tag{25}$$

Proof. This follows immediately from Theorem 5 and Lemma 12. □

Example 14. Here we examine the example of [6]: a system with a transfer function of

$$\frac{s - \frac{\theta+2}{6}}{s^2 + \theta s - \frac{2}{9}\theta(\theta+2)}$$

is considered; here $\theta \in [-1, 1]$. Of course, since we would like to allow time-variations in the parameter θ, we cannot use a transfer function model, though a state-space model will do. We will choose a form for which C is constant:

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ \frac{2}{9}\theta(t)(\theta(t)+2) & -\theta(t) \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ -\frac{7}{6}\theta(t) - \frac{1}{3} \end{bmatrix} u(t),$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t).$$

[3] This is not the only way to approach this problem—it is equally common to simply use unity feedback and force any LTI controller to have an integrator.

[4] It is common to introduce a reference signal r and let the measured output be $r - y$; however, here we are focussed on stability so we will not do so in our setup.

It is easy to see that $(A(\mu), B(\mu))$ controllable for all $\mu \in [-1, 1]$. Hence, with

$$A_{new}(\theta(t)) := \begin{bmatrix} 0 & 1 & 1 \\ \frac{2}{9}\theta(t)(\theta(t)+2) & -\theta(t) & -\frac{7}{6}\theta(t) - \frac{1}{3} \\ 0 & 0 & 0 \end{bmatrix},$$

$$B_{new}(\theta(t)) := \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

it follows from the classical PBH test that $(A_{new}(\mu), B_{new})$ is controllable for all $\mu \in [-1, 1]$. However, consider

$$\bar{A}_{new} = \frac{24}{49} A_{new}(-1) + \frac{25}{49} A_{new}(0.4) = \begin{bmatrix} 0 & 1 & 1 \\ 0 & \frac{14}{49} & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

It is easy to verify that the eigenvalue of $\bar{A}_{new}$ at $\frac{14}{49}$ is not controllable (w.r.t. B_{new}), i.e. $(\bar{A}_{new}, B_{new})$ is not weakly stabilizable. As for the case of Example 8 we can use the details of the proof of Theorem 5 to derive an explicit lower bound on $\gamma(K, \mathscr{P}_{new}(T_s))$ for any stabilizing K, yielding

$$\gamma(K, \mathscr{P}_{new}(T_s)) \geq \frac{e^{0.583T_s} - 1}{32.6T_s^2 + 11.4T_s^2 e^{4.78T_s}}.$$

Hence, while a suitably designed supervisory controller will stabilize the set of admissible LTI plants, its tolerance to time-variations is limited in the sense that the performance necessarily degrades as the frequency of jumps in the plant parameters increases.

Remark 15. This result demonstrates that if the goal is to control a system with rapidly time-varying parameters, then there are ramifications to placing an integrator at the plant input.

7. Summary and conclusions

Here we consider the problem of adaptively stabilizing a rapidly time-varying plant with jumps in the parameters. We demonstrate that, in two important cases, if the convex hull of the set of admissible parameters does not possess a weak notion of stabilizability, then regardless of the controller used, performance must necessarily degrade rapidly as the time between parameter jumps decreases. This provides an inviolable bound on the achievable performance of any adaptive controller for such a rapidly time-varying uncertain system.

In this paper the output parameter $C(\theta(\cdot))$ plays no role; the focus is on the loss of stabilizability. It is not at all clear how to prove a comparable result if detectability is lost.

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Appendix

Proof of Lemma 9. There clearly exists a non-zero $v \in \mathbf{C}^n$ satisfying (8); it follows immediately that we can choose v to be real if $\lambda \in \mathbf{R}$. Without loss of generality, we may as well assume that $\|v\| = 1$ (if it is not, then simply replace it with $\frac{1}{\|v\|}v$, and it is easy to check that it also enjoys the above properties). Furthermore, the first equation of (8) immediately implies that

$$v^* e^{At} = v^* e^{\lambda t}, \quad t \in \mathbf{R},$$

so (11) holds. If we combine this with the second equation of (8), we conclude that (10) holds. Last of all, if we set $w := \frac{1}{v^* v} v$, then it is easy to see that $\|w\| = 1$ and $v^* w = 1$, which means that (9) holds. □

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